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On Solitons of Standing Wave Solutions for the Cubic-Quartic Nonlinear Schrödinger equation

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Abstract

We investigate the standing wave solutions of the form $A(x,t) = \varphi(x)e^{-i\Omega t}$ for $\Omega > 0$ for the one-dimensional Schrödinger equation with a quartic term of the form $iA_t + PA_{xx} + Q|A|^2A + \varepsilon|A|^3A = 0$ where $i = \sqrt{-1}$, and $\varepsilon > 0$ is a fixed coefficient. We show that there are both bright and dark solitons for small $\varepsilon > 0$ in the case that $P < 0$ and $Q < 0$ and in the case that $P > 0$ and $Q < 0$ by using phase portrait analysis.

Keywords. Solitons, Standing Wave Solutions, the nonlinear Schrödinger equation

1 Introduction

In general, the Schrödinger equation governs the envelope of group waves, which propagate in the water and the plasma, etc. Moreover, the nonlinear Schrödinger equation governs the non-linearity of the envelope. The fact that the solution for the nonlinear Schrödinger equation can be a soliton is known and of interest [Zakharov72]. Many studies of group waves have been carried out in the water wave area and some other area as well. For example, in the fiber-optic communication system, the GVD (Group Velocity Dispersion), in which problem the launched pulse may spread outside its timing window due to dispersion, limits the transmission data rate caused by the pulse overlapping between adjacent timing windows. Nonlinear refraction of SPM (Self-Phase Modulation) can also limit the system performance by causing spectral broadening of the optical pulse. Those effects are also described by the linear or nonlinear Schrödinger equation and analyzed to achieve the optimal system performance (see [Agrawal97]).

We study the following the cubic-quartic nonlinear Schrödinger equation

(CQNLS) $$iA_t + PA_{xx} + Q|A|^2A + \varepsilon|A|^3A = 0.$$

There exist some studies concerned the CNLS with higher-perturbed terms from physical view points [Nohara04] [Nohara05b]. The CNLS with quintic nonlinear terms has been formulated in the vibration of elastic plates with cubic characteristics of spring [Nohara05a]. Moreover, a sufficient condition is suggested to have soliton solutions for the Schrödinger equation with the perturbed term of the general degree $(n+1)$ [Nohara07] and the approximation formula of the perturbed soliton solution is shown [Nohara06].

The aim of this paper is to seek solitons of the standing wave solutions for the CQNLS by using phase portrait analysis. The standing wave solutions are represented by

$$A(x,t) = \varphi(x)e^{-i\Omega t}.$$

By virtue of the above form of the solution, the CQNLS is reduced to the second order ordinary differential equation. We investigate homoclinic or heteroclinic orbits of the ODE, that correspond to the envelopes of solitons.

This paper is organised as follows. In Section 2, we formulate our target equation, CQNLS, and state theorems on solitons (Theorem 2.1). In Sections 3-6, we analyse the standing wave solutions as the proof of Theorem 2.1 for $P < 0$ and $Q < 0$ (Sec. 3), for $P > 0$ and $Q < 0$ (Sec. 4), for $P < 0$ and $Q > 0$ (Sec. 5), and for $P > 0$ and $Q > 0$ (Sec. 6).

2 Target equation and main theorems

Our target equation is a one-dimensional Schrödinger equation with the quartic nonlinearity of the following form,

$$\begin{cases} iA_t + PA_{xx} + Q|A|^2A + \varepsilon|A|^3A = 0 & \text{for } x \in \mathbb{R},\ t > 0, \\ A|_{t=0} = A_0(x). \end{cases} \tag{2.1}$$

where, $i = \sqrt{-1}$, P and Q are known real numbers, $\varepsilon > 0$ is a fixed coefficient, and $A = A(x,t)$; $\mathbb{R} \times \mathbb{R}_+ \ \rightarrow \ \mathbb{C}$ is an unknown function.

In this paper, we are concerned with solitons of the standing wave solutions $A = A(x,t)$ in the following form.

$$A(x,t) = \varphi(x)\mathrm{e}^{-i\Omega t}, \tag{2.2}$$

where, $\varphi(x) \in C^2(\mathbb{R};\mathbb{R})$ represents an envelope of solutions, and the angular frequency $\Omega \in \mathbb{R}$ is a non-zero fixed constant.

Substituting the function A of the form (2.2) into the equation (2.1) and dividing by $\mathrm{e}^{-i\Omega t}$, we have the envelope equation

$$\varphi'' = -\frac{\Omega}{P}\,\varphi\,[1 + \frac{1}{\Omega}(Q|\varphi|^2 + \varepsilon|\varphi|^3))]. \tag{2.3}$$

We call its solution φ, an envelope solution. The characteristics of soliton can be mainly stated by using a separatorix of the orbit and integrability of the envelope.

Fact 1. (Classification of Solitons) *Solutions $A = A(x,t)$ of the equation* (2.1) *with the representation* (2.2) *include a bright, gray and dark soliton. We denote*

$\varphi_c(x) = \varphi(x) - c$ *for a constant* c.

(A) *A bright and gray soliton have the following characteristics* (1), (2) *and* (3).

(1) *The phase portrait* (φ, φ') *of a bright and gray soliton constructs a homoclinic orbit.*

(2)[1] *The envelope* $\varphi = \varphi(x)$ *is integrable modulo constants, that is, there exists a constant* c *such that* $\int_{-\infty}^{\infty} |\varphi_c(x)|\, dx < \infty$.

A bright soliton can be distinguished from a gray soliton by (B) *and* (C).

(B) *A bright soliton has the following characteristics* (4) *or* (5).

(4) $\varphi''(x_{b+}) < 0$, *where* x_{b+} *is defined by the equality* $\varphi_c(x_{b+}) = \max_x \varphi_c(x)$, *for* $\varphi_c(x) > 0$.

(5) $\varphi''(x_{b-}) > 0$, *where* x_{b-} *is defined by the equality* $\varphi_c(x_{b-}) = \min_x \varphi_c(x)$, *for* $\varphi_c(x) < 0$.

(C) *A gray soliton has the following characteristics* (6) *or* (7).

(6) $\varphi''(x_{g-}) > 0$, *where* x_{g-} *is defined by the equality* $\varphi_c(x_{g-}) = \min_x \varphi_c(x)$, *for* $\varphi(x) > 0$.

(7) $\varphi''(x_{g+}) < 0$, *where* x_{g+} *is defined by the equality* $\varphi_c(x_{g+}) = \max_x \varphi_c(x)$, *for* $\varphi(x) < 0$.

(D) *A dark soliton has the following characteristics* (8), (9) *and* (10).

(8) *The phase portrait* (φ, φ') *of a dark soliton constructs a heteroclinic orbit.*

(9) *The envelope* $\varphi = \varphi(x)$ *is non-integrable in the sense that there does not exist any constant* c *such that* $\int_{-\infty}^{\infty} |\varphi_c(x)|\, dx < \infty$.

(10) *The envelope* $\varphi = \varphi(x)$ *is bounded.*

On envelope solutions φ for the CNLS without the quartic term (i.e., $\varepsilon = 0$), it is known that there are

(i) bright solitons (sech solutions) for $P < 0$ and $Q < 0$

(ii) dark solitons (tanh solutions) for $P > 0$ and $Q < 0$.

It is also known that the equation (2.1) has no soliton solution for P and Q except for the above two cases (e.g. [Watanabe85]).

Now we state the theorem in the following.

Theorem 2.1. (Solitons for the CQNLS) *Assume that* $\Omega > 0$ *and* $\varepsilon > 0$. *Consider the equation* (2.1).

(1) *Let* $P < 0$ *and* $Q < 0$.

i) *If* $\varepsilon^2 < \dfrac{25}{216}\dfrac{(-Q)^3}{\Omega}$, *then there exist both bright and dark solitons simultaneously.*

ii) *If* $\varepsilon^2 = \dfrac{25}{216}\dfrac{(-Q)^3}{\Omega}$, *then there exist only dark solitons.*

iii) *If* $\dfrac{25}{216}\dfrac{(-Q)^3}{\Omega} < \varepsilon^2 < \dfrac{4}{27}\dfrac{(-Q)^3}{\Omega}$, *then there exist only gray solitons.*

iv) *If* $\varepsilon^2 \geq \dfrac{4}{27}\dfrac{(-Q)^3}{\Omega}$, *then there exist no soliton.*

(2) *Let* $P > 0$ *and* $Q < 0$.

[1]Note that there exists a constant c such that $\lim_{x\to\pm\infty} \varphi_c(x) = \lim_{x\to\pm\infty} \varphi_c'(x) = 0$.

i) *If* $\varepsilon^2 < \frac{4}{27}\frac{(-Q)^3}{\Omega}$, *then there exist both bright and dark solitons simultaneously.*

ii) *If* $\varepsilon^2 \geq \frac{4}{27}\frac{(-Q)^3}{\Omega}$, *then there exist no soliton.*

(3) *Let* $P < 0$ *and* $Q > 0$. *There exist no soliton for all* $\varepsilon > 0$.

(4) *Let* $P > 0$ *and* $Q > 0$. *There exist no soliton for all* $\varepsilon > 0$.

3 The case $P < 0$, $Q < 0$

Putting

$$-a^2 := \frac{Q}{\Omega},\ b^2 := -\frac{\Omega}{P},\ c^2 := \frac{\varepsilon}{\Omega}, \tag{3.1}$$

we can express (2.3) as

$$\varphi'' = b^2\varphi\,(1 - a^2|\varphi|^2 + c^2|\varphi|^3)\ (=:\ f(|\varphi|)), \tag{3.2}$$

that is,

$$(+):\ \varphi'' = b^2\varphi f_+(\varphi)\ \ (\varphi > 0)\ \text{ and }\ (-):\ \varphi'' = b^2\varphi f_-(\varphi)\ \ (\varphi < 0), \tag{3.3}$$

where $f_+(x)(:= f(x)) = 1 - a^2x^2 + c^2x^3$ and $f_-(x)(:= f(-x)) = 1 - a^2x^2 - c^2x^3$.

Hereafter, we let a represent the positive square root of $a^2 = -\frac{Q}{\Omega}$ according to (3.1), that is, $a := \sqrt{-\frac{Q}{\Omega}}$. Similarly, we represent $b := \sqrt{-\frac{\Omega}{P}}$ and $c := \sqrt{\frac{\varepsilon}{\Omega}}$.

3.1 Phase portrait analysis for $\varphi \geq 0$

We rewrite the equation (3.3)$-(+)$ in the dynamical system

$$\begin{cases} \varphi' = \eta, \\ \eta' = b^2\varphi(1 - a^2\varphi^2 + c^2\varphi^3)\ (= b^2\varphi f_+(\varphi)), \end{cases} \tag{3.4}$$

whose fixed points in the area $\varphi \geq 0$ are

$$(\varphi, \eta) = (0,0),\ (\alpha, 0),\ (\beta, 0) \qquad (0 < \alpha < \beta),$$

where, α and β are the positive solutions of $f_+(x) = 0$ if

$$0 < c^4 < \frac{4}{27}a^6, \tag{3.5}$$

which is equivalent to $f_+(\frac{2a^2}{3c^2}) < 0$.

Since the Jacobian matrix of the dynamical system (3.4) becomes

$$J = J_{(\varphi,\varphi')} = \begin{pmatrix} 0 & 1 \\ K_+(\varphi) & 0 \end{pmatrix}, \qquad \text{where } K_+(\varphi) = b^2(f_+(\varphi) + \varphi f_+'(\varphi)),$$

we obtain the eigenvalues of J and specify the fixed points in the area $\{(\varphi,\eta)\ ;\ \varphi \geq 0\}$ as

$$(0,0)\ :\ \text{saddle}, \qquad (\alpha,0)\ :\ \text{center}, \qquad (\beta,0)\ :\ \text{saddle}. \tag{3.6}$$

In fact, the fixed point $(0,0)$ is a saddle point since the characteristic equation $\lambda^2 = K_+(0) = b^2(f_+(0)+0f_+'(0)) = b^2$ gives the real and opposite signed eigenvalues $\lambda = \pm b = \pm\sqrt{-\frac{\Omega}{P}}$. For $(\alpha,0)$ we can get $\lambda^2 = K_+(\alpha) = b^2(f_+(\alpha)+\alpha f_+'(\alpha)) < 0$ since $f_+(\alpha) = 0$, $f_+'(\alpha) < 0$ by observing the figure of f_+. Hence, the eigenvalues become imaginary $\lambda = \pm i\sqrt{-K_+(\alpha)}$, that implies a center. Similarly, for $(\beta,0)$, it follows that $\lambda^2 = K_+(\beta) = b^2(f_+(\beta)+\beta f_+'(\beta)) > 0$ from $f_+(\beta) = 0$, $f_+'(\beta) > 0$. Then, the eigenvalues become $\lambda = \pm\sqrt{K_+(\beta)}$, hence, the fixed point $(\beta,0)$ is saddle.

3.2 Phase portrait analysis for $\varphi < 0$

The equation (3.3)$-(-)$ is rewritten in the dynamical system

$$\begin{cases} \varphi' = \eta, \\ \eta' = b^2\varphi(1 - a^2\varphi^2 - c^2\varphi^3)\ (= b^2\varphi f_-(\varphi)). \end{cases} \tag{3.7}$$

Noting that $f_-(-\varphi) = f_+(\varphi)$, the fixed points in the area $\varphi < 0$ are

$$(\varphi,\eta) = (-\beta,0),\ (-\alpha,0) \qquad (0 < \alpha < \beta).$$

The Jacobian matrix is given as

$$J = J_{(\varphi,\varphi')} = \begin{pmatrix} 0 & 1 \\ K_-(\varphi) & 0 \end{pmatrix}, \qquad \text{where } K_-(\varphi) = b^2(f_-(\varphi) + \varphi f_-'(\varphi)),$$

hence, the fixed points in the area $\varphi < 0$ are

$$(-\beta,0)\ :\ \text{saddle} \quad \text{and} \quad (-\alpha,0)\ :\ \text{center}. \tag{3.8}$$

In fact, for $(-\beta,0)$ the characteristic equation becomes $\lambda^2 = K_-(-\beta) = b^2(f_-(-\beta) - \beta f_-'(-\beta)) > 0$ since $f_-(-\beta) = 0$, $f_-'(-\beta) < 0$ by observing the figure of f_-. Hence, the eigenvalues are $\lambda = \pm\sqrt{K_-(-\beta)}$. Similarly, for $(-\alpha,0)$, we see $\lambda^2 = K_-(-\alpha) = b^2(f_-(-\alpha) - \alpha f_-'(-\alpha)) < 0$ since $f_-(-\alpha) = 0$, $f_-'(-\alpha) > 0$. Hence, $\lambda = \pm i\sqrt{-K_-(-\alpha)}$. Thus we obtain the phase portraits for $\varphi \geq 0$ ((3.3)$-(+)$) and for $\varphi < 0$ ((3.3)$-(-)$), respectively. In order to complete phase portrait for the equation (3.2) we proceed to the following "unification" argument. In the "unification" argument, we calculate potential energy of the dynamical systems (3.4) and (3.7) for deciding which saddle points among $(-\beta,0)$, $(0,0)$, and $(\beta,0)$ should be connected each other in order to unify the phase portraits along the border $\{(\varphi,\eta)\ ;\ \varphi = 0\}$.

Precisely, one has choice whether (a) to connect $(-\beta,0)$ and $(\beta,0)$ as two heteroclinic orbits, and make two looped homoclinic orbits that starts and terminates at $(0,0)$, or (b) to connect $(-\beta,0)$ and $(0,0)$, $(0,0)$ and $(\beta,0)$, respectively, as two heteroclinic orbits, or (c) to make two looped homoclinic orbits that start and terminate at $(-\beta, 0)$ and $(\beta, 0)$, respectively. We will discuss the choice by observing potential energy in the next subsection. In the following table, we summarize the choice of connection, where $\leftarrow$, $\rightarrow$, and $\rightleftharpoons$ denote heteroclinic orbits, $\hookleftarrow$ and $\hookrightarrow$ represent homoclinic orbits.

Table 1: Choice of connection among the saddle points $(-\beta,0)$, $(0,0)$, and $(\beta,0)$ at the case $P<0$, $Q<0$; $c\neq 0$.

Cases	(a)	(b)		(c)
Connection	$(-\beta,0)\rightleftharpoons(\beta,0)$	$(-\beta,0)\rightarrow(0,0)$	$(0,0)\rightarrow(\beta,0)$	$(-\beta,0)\hookleftarrow$
	$\hookrightarrow(0,0)\hookleftarrow$	$(-\beta,0)\leftarrow(0,0)$	$(0,0)\leftarrow(\beta,0)$	$\hookrightarrow(\beta,0)$
Restriction for potential V_+	$V_+(\beta)>0$	$V_+(\beta)=0$		$V_+(\beta)<0$

3.3 Unification of phase portraits $\varphi \geq 0$ and $\varphi < 0$

In this subsection we consider the conditions for the parameter c^2 (or ε) for cases (a), (b), and (c), in the above subsection. To this end, integrating both sides of (3.3)$-$(+) with respect to φ after multiplying φ' implies that

$$\frac{1}{2}(\varphi')^2-\frac{b^2}{2}\varphi^2+\frac{a^2b^2}{4}\varphi^4-\frac{b^2c^2}{5}\varphi^5=E_1 \quad \text{for a constant } E_1\in\mathbb{R}, \tag{3.9}$$

or equivalently,

$$\frac{1}{2}(\varphi')^2+V_+(\varphi)=E_1. \tag{3.10}$$

Here, we denote by $V_+(\varphi)$ the potential energy defined by $V_+(\varphi):=-b^2\varphi^2v_+(\varphi)$, where $v_+(\varphi):=\frac{1}{2}-\frac{a^2}{4}\varphi^2+\frac{c^2}{5}\varphi^3$.

First, for the case (a), i.e., the realization of two hetero-clinic orbits between $(-\beta,0)$ and $(\beta,0)$, the potential energy at their points, $V_+(-\beta)$ and $V_+(\beta)$ should be the same value, and the value should be greater than $V_+(0)$, that is, $V_+(\beta)>0\ (=V_+(0))$. It is easy to see the condition is equivalent to $0<\varepsilon^2<\frac{25}{216}\frac{(-Q)^3}{\Omega}$ corresponds to the case (1) i) in Theorem 2.1. Note that there is also the potential energy $V_-(\varphi)$ in the area $\varphi<0$, however, by the symmetry $V_-(\varphi)=V_+(-\varphi)$, $V_-(-\beta)\ (=V_+(\beta))>0$ is fulfilled under the same condition.

Secondly, for the case (b) we impose the condition $\varepsilon^2=\frac{25}{216}\frac{(-Q)^3}{\Omega}$, which is equivalent to $V_+(\beta)=0\ (=V_+(0))$, and corresponds to the case (1) ii) in Theorem 2.1.

Thirdly, for the case (c) we impose the condition $\varepsilon^2>\frac{25}{216}\frac{(-Q)^3}{\Omega}$ or $V_+(\beta)<0\ (=V_+(0))$. Combining with the condition (3.5) for the existence of the fixed points $(-\beta,0)$ and $(\beta,0)$, we have $\frac{25}{216}\frac{(-Q)^3}{\Omega}<\varepsilon^2<\frac{4}{27}\frac{(-Q)^3}{\Omega}$, for the case (1) iii) in Theorem 2.1 .

Finally, for the case (a), we discuss connectivity of the line of the heteroclinic orbits from (to) $(-\beta, 0)$ to (from) $(\beta, 0)$ across the φ' axis in the phase portrait. Since the line of the heteroclinic orbit in the area $\varphi \geq 0$ ends at $(\beta, 0)$, we substitute $\varphi(+\infty) = \beta$ and $\varphi'(+\infty) = 0$ into (3.9) to have

$$E_1 = -b^2(\frac{1}{2}\beta^2 - \frac{a^2}{4}\beta^4 + \frac{c^2}{5}\beta^5). \tag{3.11}$$

On the other hand, from (3.3)−(−) we obtain for $\varphi < 0$ that

$$\frac{1}{2}(\varphi')^2 - \frac{b^2}{2}\varphi^2 + \frac{a^2b^2}{4}\varphi^4 + \frac{b^2c^2}{5}\varphi^5 = E_2 \qquad \text{for a constant } E_2 \in \mathbb{R}. \tag{3.12}$$

Then, substituting $\varphi(-\infty) = -\beta$ and $\varphi'(-\infty) = 0$ into the above expression, (3.11) becomes

$$E_1 = -b^2\left(\frac{1}{2}(\beta)^2 - \frac{a^2}{4}(\beta)^4 + \frac{c^2}{5}(\beta)^5\right) = -b^2\left(\frac{1}{2}(-\beta)^2 - \frac{a^2}{4}(-\beta)^4 - \frac{c^2}{5}(-\beta)^5\right) = E_2.$$

Taking the limit $\varphi \to 0$ in (3.9) and (3.12), we have $(\varphi')^2 \to 2E_1$. and $(\varphi')^2 \to 2E_2$. Since $E_1 = E_2$, we thus have

$$\lim_{\substack{\varphi \to +0 \\ \text{along } (3.3)-(+)}} \varphi' = \lim_{\substack{\varphi \to -0 \\ \text{along } (3.3)-(-)}} \varphi', \tag{3.13}$$

which ensures the connectivity of the heteroclinic orbits from (to) the fixed point $(-\beta, 0)$ to (from) $(\beta, 0)$.

Summarizing the above analysis, we can obtain the phase portrait and its potential shown as Figure 1, 2 and 3. Tha fact that bright, gray and dark solitons in Theorem 2.1(1) satisfy Fact 1 is easily found.

4 The case $P > 0$, $Q < 0$

Putting $\hat{b}^2 := \frac{\Omega}{P}$, it follows from (2.3) that

$$\varphi'' = -\hat{b}^2\varphi\,(1 - a^2|\varphi|^2 + c^2|\varphi|^3). \tag{4.1}$$

Hereafter, we represent $\hat{b} := \sqrt{\frac{\Omega}{P}} > 0$.

We consider the phase portrait for two areas $\varphi \geq 0$ and $\varphi < 0$. For $\varphi \geq 0$ we obtain

$$\begin{cases} \varphi' = \eta, \\ \eta' = -\hat{b}^2\varphi(1 - a^2\varphi^2 + c^2\varphi^3) \; (= \hat{b}^2\varphi f_+(\varphi)), \end{cases} \tag{4.2}$$

whose fixed points are given as

$$(\varphi, \eta) = (0,0),\ (\alpha, 0),\ (\beta, 0) \qquad (0 < \alpha < \beta),$$

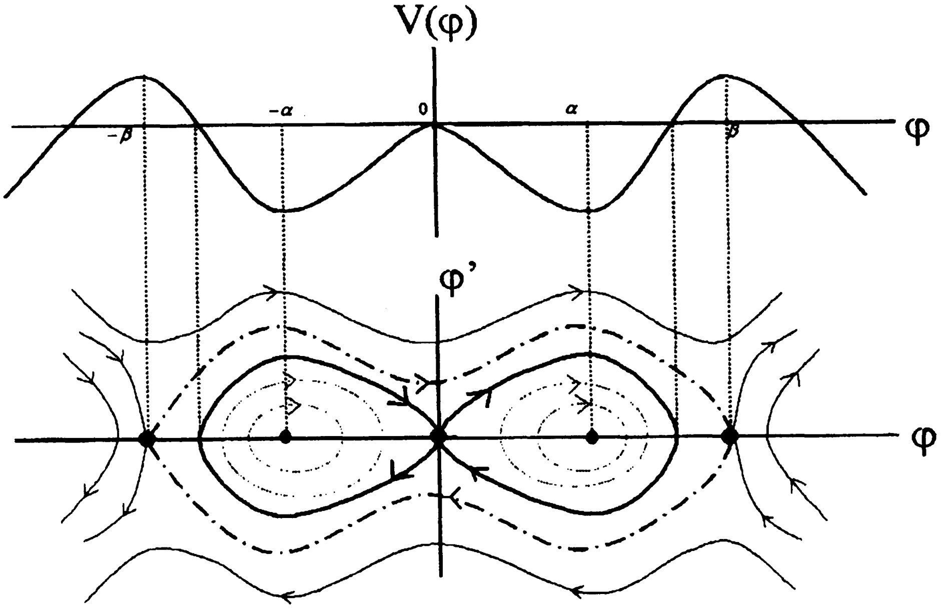

Figure 1: Phase portrait and Potential: the case of $P < 0, Q < 0, V_+(\beta) > 0$.
a thick solid line: a homoclinic orbit, a dotted line: a periodic orbit, a dash-dotted line: a heteroclinic orbit, a thin solid line: a blow-up orbit

where, α and β are the positive solutions of $f_+(x) = 0$ under the assumption

$$0 < c^4 < \frac{4}{27}a^6. \tag{4.3}$$

The fixed points are classified as

$$(0,0)\ :\ \text{center},\qquad (\alpha,0)\ :\ \text{saddle},\qquad (\beta,0)\ :\ \text{center}.$$

Similarly, for the area $\varphi < 0$, the fixed points of the system

$$\begin{cases} \varphi' = \eta, \\ \eta' = -b^2\varphi(1 - a^2\varphi^2 - c^2\varphi^3)\ (= b^2\varphi f_-(\varphi)) \end{cases} \tag{4.4}$$

are given as

$$(\varphi,\eta) = (-\beta,0),\ (-\alpha,0) \qquad (0 < \alpha < \beta),$$

and the fixed points are classified as

$$(-\beta,0)\ :\ \text{center},\qquad (-\alpha,0)\ :\ \text{saddle}.$$

Thus, the similar unification argument to the one in the previous section realizes the phase portrait in Figure 4 with its potential which shows the existence of both bright and dark solitons. Tha fact that bright and dark solitons in Theorem 2.1(2) satisfy Fact 1 is easily found.

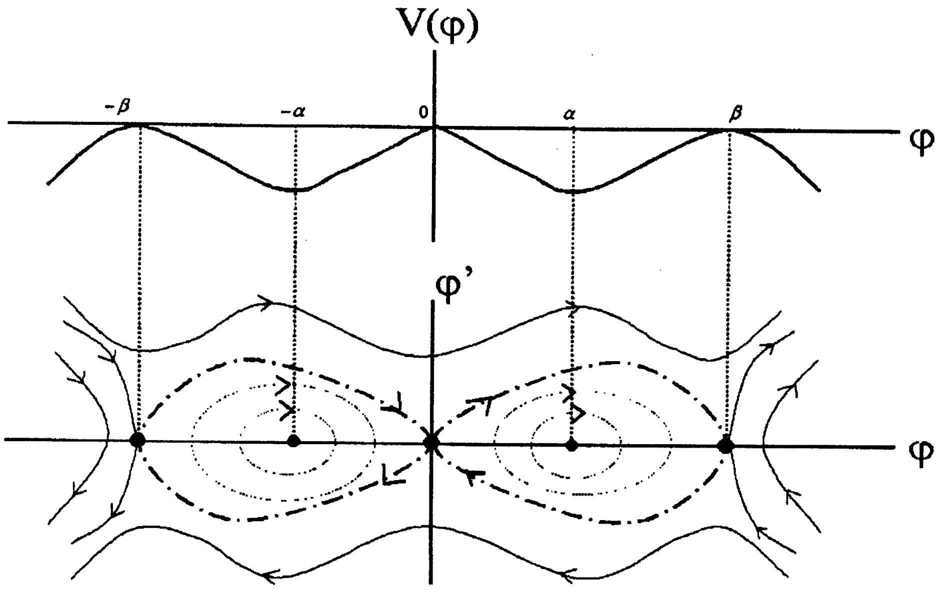

Figure 2: Phase portrait and Potential: the case of $P < 0, Q < 0, V_+(\beta) = 0$.
a dotted line: a periodic orbit, a dash-dotted line: a heteroclinic orbit, a thin solid line: a blow-up orbit

The potential is derived from (4.2) as

$$V_+(\varphi) := b^2\varphi^2 v_+(\varphi), \quad \text{where } v_+(\varphi) = \frac{1}{2} - \frac{a^2}{4}\varphi^2 + \frac{c^2}{5}\varphi^3.$$

For realizing the orbits of the bright solitons, one must impose the condition $V_+(\alpha) > V_+(\beta)$. However, this condition for the potential energy V_+ is automatically fulfilled without imposing any smallness assumption on the coefficient ε except the condition (4.3). In fact, $V_+'(\varphi) = -b^2\varphi f_+(\varphi) > 0$ for the interval $\alpha < \varphi < \beta$ implies $V_+(\alpha) > V_+(\beta)$.

5 The case $P < 0,\ Q > 0$

In this case, there are no soliton for both $c = 0$ and $c \neq 0$. The equation (2.3) can be given as

$$\varphi'' = b^2\varphi\ (1 + \hat{a}^2|\varphi|^2 + c^2|\varphi|^3),$$

for $\hat{a}^2 := \frac{Q}{\Omega}$, hence, the second derivative is always positive for all $\varphi > 0$, which allows only φ such that $\varphi(x) \to \infty$ at $|x| < \infty$.

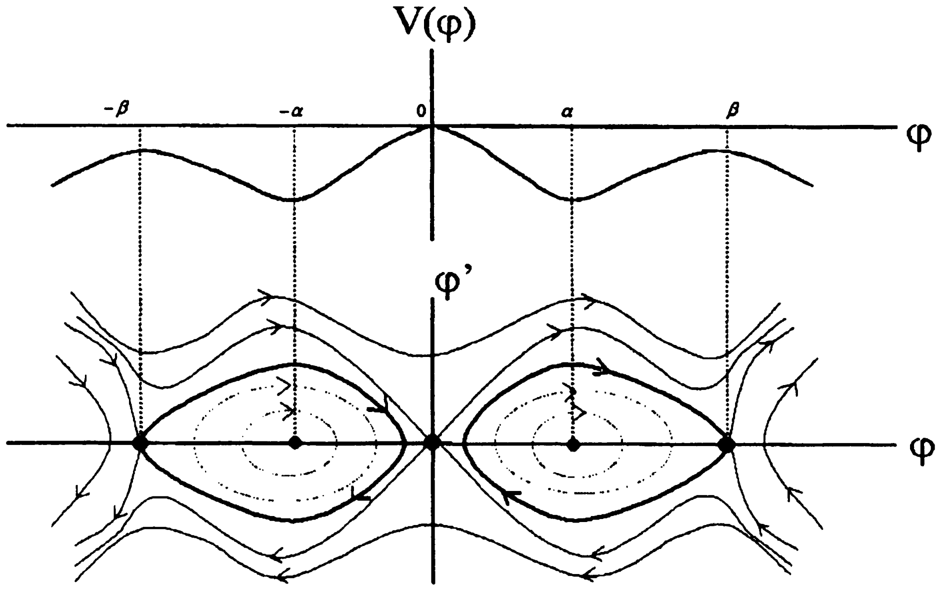

Figure 3: Phase portrait and Potential: the case of $P < 0, Q < 0, V_+(\beta) < 0$.
a thick solid line: a homoclinic orbit, a dotted line: a periodic orbit, a thin solid line: a blow-up orbit

6 The case $P > 0$, $Q > 0$

The equation (2.3) for this case becomes

$$\varphi'' = -\hat{b}^2\varphi\,(1 + \hat{a}^2|\varphi|^2 + c^2|\varphi|^3),$$

which implies that the second derivative is always negative for all $\varphi > 0$. For $c = 0$, it can be easily seen that there is only one fixed point $(0, 0)$ for the above equation. Moreover, the fixed point $(0, 0)$ is not a saddle point, but a center point. Hence, there is no soliton for the case $c = 0$.

For $c \neq 0$, the phase portrait is exactly the same as the case $c = 0$. This is because the zero point of the polynomial $1 + a^2\varphi^2 + c^2\varphi^3$ locates in $\varphi < 0$, while, the zero point of the polynomial $1 + a^2\varphi^2 - c^2\varphi^3$ is in $\varphi > 0$, hence, the formula $1 + a^2|\varphi|^2 + c^2|\varphi|^3$ does not have zero points. Hence, there is the only one fixed point $(0, 0)$, i.e., a center. Thus, we can conclude that there is no soliton for both cases $c = 0$ and $c \neq 0$.

7 Conclusion

We have considered the one-dimensional cubic-quartic nonlinear Schrödinger equation (CQNLS). Solitons of the standing wave solutions have been investigated by phase portrait analysis. For two cases ($P < 0$ and $Q < 0$) and ($P > 0$ and $Q < 0$), it is observed that there are both bright and dark solitons simultaneously when the coefficient of the quartic term is small.

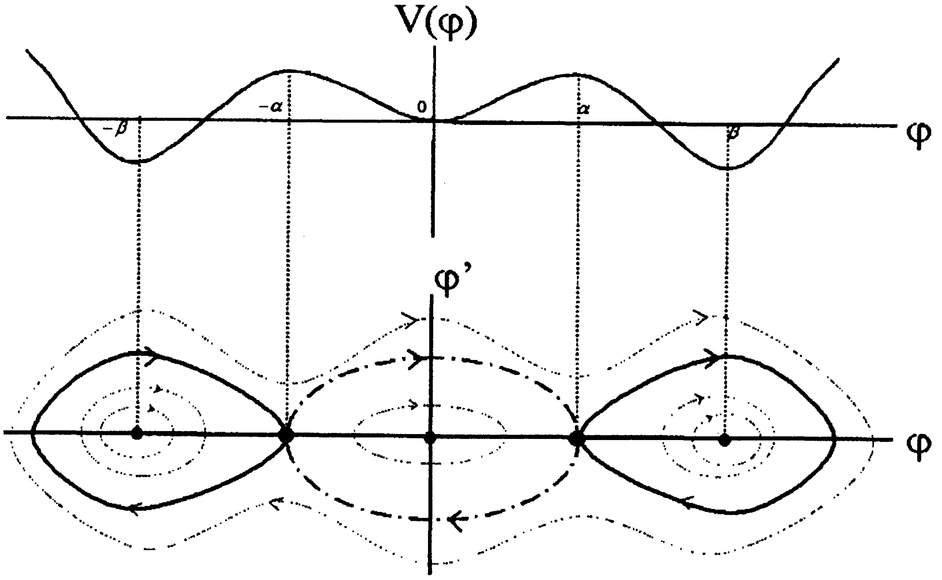

Figure 4: Phase portrait and Potential: the case of $P > 0, Q < 0$.
a thick solid line: a homoclinic orbit, a dotted line: a periodic orbit, a dash-dotted line: a heteroclinic orbit, a thin solid line: a blow-up orbit

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