RECIPROCAL SUMS OF ELEMENTS IN A BINARY RECURRENCE SEQUENCE*

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Abstract

A brief survey of identities about reciprocal sums of products of elements in a binary recurrence sequence is presented. The emphasis is on elements satisfying a second order linear recurrence relation with not necessarily constant coefficients, such as the Fibonacci or the Lucas polynomials.

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1   Binary sequences with constant coefficients

By a reciprocal sum, we refer to a series of the form $\sum_n 1/E_n$, where E_n is an expression involving elements satisfying a linear recurrence relation of fixed order. There have appeared a number of such sums of elements satisfying a second order linear recurrence relation with constant coefficients, with the sequences of Fibonacci and Lucas numbers being investigated most.

Let us recall some facts about second order recurrences with constant coefficients. For fixed $P, Q(\neq 0) \in \mathbb{R}$, denote by $L(P, Q)$ the set of sequences $\{R_n\}_{n \in \mathbb{Z}}$ of real numbers satisfying a second order linear recurrence relation of the form

$$R_{n+1} = PR_n - QR_{n-1} \qquad (n \in \mathbb{Z}). \tag{1.1}$$

Let α, β be the roots of its characteristic equation

$$x^2 - Px + Q = 0 \text{ with } |\alpha| \leq |\beta|, \ \Delta = P^2 - 4Q.$$

The case when $P = 1$ and $Q = -1$ is historically of most interest. In this case, with initial values $F_0 = 0$, $F_1 = 1$, the unique solution sequence to (1.1) is the sequence $\{F_n\}$ of *Fibonacci numbers*, while if the initial values are $L_0 = 2$, $L_1 = P$, the unique solution sequence of (1.1) is the sequence $\{L_n\}$ of *Lucas numbers*.

A general discussion about reciprocal sums of the two types

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k_1} F_{n+k_2} \cdots F_{n+k_r}} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+k_1} F_{n+k_2} \cdots F_{n+k_r}}$$

up to 1969 was given in [8]; see also [23]. There it was shown that the so-called second degree summation, i.e., the denominator contains a product of two Fibonacci numbers, can be completely settled. The case of alternating series yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+k}} = \frac{1}{F_k} \left(\frac{k}{r} - \sum_{j=1}^{k} \frac{F_{j-1}}{F_j} \right) \quad (k \in \mathbb{N}), \tag{1.2}$$

and for the non-alternating case, $\sum_{n=1}^{\infty} 1/F_n F_{n+k}$ has explicit values when k is even, while for odd k it can be reduced to the form

$$a + b \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} \quad \text{with } a, b \in \mathbb{Q}.$$

Moreover, the third degree summation can be put in the form

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+s} F_{n+t}} = c + d \sum_{n=1}^{\infty} \frac{1}{F_{n+2} F_{n+1} F_n} \quad (s, t \in \mathbb{N}),$$

with the constants c and d being determined for certain given s and t. In general, all summations of the shape

$$\sum_{n=1}^{\infty} \frac{1}{\prod_{i=1}^{r} F_{n+k_i}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=1}^{r} F_{n+k_i}}$$

can be written under the form

$$a + b \sum_{n=1}^{\infty} \frac{1}{F_{n+r-1} F_{n+r-2} \cdots F_n} \quad \text{and} \quad c + d \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_{n+r-1} F_{n+r-2} \cdots F_n},$$

respectively, where $a, b, c, d \in \mathbb{Q}$. Identities of these kinds have regularly been posed as problems in the literature, such as in [42].

A reciprocal sum of Fibonacci numbers with subscripts 2^n was proposed in 1974 as a problem by D.A. Millin in [33], and was subsequently solved by I.J. Good in [15] as

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7 - \sqrt{5}}{2}. \tag{1.3}$$

Since then, many proofs have been offered, [19]. In 1976, Hoggart and Bicknell [20] gave the following more general formula (see also [7]), when $k \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n k}} = \begin{cases} \dfrac{2L_k - F_{2k}\sqrt{5} + 5F_k^2}{2F_{2k}} & \text{if } k \text{ is odd} \\[2mm] \dfrac{2 - F_k\sqrt{5} + L_k}{2F_k} & \text{if } k \text{ is even}. \end{cases} \tag{1.4}$$

In 1976, using the Lambert series expansion, Bruckman and Good [9] evaluated several reciprocal sums including

$$\sum_{n=0}^{\infty} \frac{L_{k \cdot 3^n}}{F_{k \cdot 3^{n+1}}} = \frac{(\sqrt{5} - 1)^k}{2^k F_k}, \quad \sum_{n=0}^{\infty} \frac{F_{k \cdot 3^n}}{L_{k \cdot 3^{n+1}}} = \frac{(\sqrt{5} - 1)^k}{2^k \sqrt{5} L_k}, \quad \sum_{n=0}^{\infty} \frac{(-1)^{s(n)} F_{s(n+1) - s(n)}}{F_{s(n)} F_{s(n+1)}} = \frac{1}{F_{s(0)}} \left(\frac{1 - \sqrt{5}}{2} \right)^{s(0)},$$

where $s(n)$ is a positive integer-valued function with $s(n) \to \infty$ as $n \to \infty$.

In 1977, Greig ([17], [18]) summed a finite reciprocal series of the form $\sum_n 1/F_{k \cdot 2^n}$, which after passing to limit yields (1.4) as well as other reciprocal sums expressed as a combination of smaller sums. In the same year, Bruckman, [10], obtain closed form expressions of certain reciprocal sums in terms of complete elliptic integrals of the first and second kind.

In 1984, Popov [37] showed, among other things, that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2nm+2k}F_{2(n+1)m+2k}} = \frac{1}{2F_{2m}}\left(\frac{L_{2k}}{F_{2k}} - \sqrt{5}\right),$$

where m is odd and $-(m-1) \le 2k \le m-1$. He later extended this result in [38] with $\ell, m \in \mathbf{N}$ to

$$\sum_{n=0}^{\infty} \frac{Q^{nm}}{U_{\ell+nm}U_{\ell+(n+1)m}} = \frac{\alpha^{-\ell}}{U_\ell\, u_m} \tag{1.5}$$

and

$$\sum_{n=0}^{\infty} \frac{Q^{nm}}{V_{\ell+nm}V_{\ell+(n+1)m}} = \frac{\alpha^{1-\ell}}{\alpha^2 - Q} \cdot \frac{1}{U_m\, V_\ell}, \tag{1.6}$$

where

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n. \tag{1.7}$$

In 1990, Andfe-Jeannin [3] gave explicit formulae for the reciprocal sums of elements satisfying (1.1) with $Q = -1$ in terms of the values of Lambert series, namely, when k is an odd positive integer,

$$\sum_{n=1}^{\infty} \frac{1}{U_{kn}U_{k(n+1)}} = \frac{2(\alpha - \beta)}{U_k}\left\{L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})\right\} + \frac{\beta^k}{U_k^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{V_{kn}V_{k(n+1)}} = \frac{2}{(\alpha - \beta)U_k}\left\{L(\beta^{2k}) - 2L(\beta^{8k})\right\} + \frac{\beta^k}{(\alpha - \beta)U_kV_k},$$

where U_n, V_n are as defined in (1.7) and

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}.$$

In 1994, Good, [16] discovered a symmetry property of alternating reciprocal sums of elements G_n satisfying (1.1) with $Q = -1$ of the form

$$\frac{\alpha^k - \beta^k}{\alpha - \beta}\sum_{n=1}^{m}\frac{(-1)^n}{G_nG_{n+k}} = \frac{\alpha^m - \beta^m}{\alpha - \beta}\sum_{n=1}^{k}\frac{(-1)^n}{G_nG_{n+m}},$$

where all $G_1, \ldots, G_{m+k}$ are assumed nonzero.

In 1995, Melham and Shanon [31] computed the sums in (1.5), (1.6) for elements U_n, V_n satisfying the recurrence (1.1) with $Q = -1$ and initial values

$$U_0 = 0, \ U_1 = 1; \quad V_0 = 2, \ V_1 = P.$$

Among the established reciprocal sums are, when $k \in \mathbf{N}$,

$$\sum_{n=0}^{\infty} \frac{1}{U_{k\cdot 2^n}} = \begin{cases} \dfrac{1 - U_{k-1}}{U_k} + \dfrac{1}{\alpha} & \text{if } P > 2 \\[2mm] \dfrac{1 - U_{k-1}}{U_k} + \dfrac{1}{\beta} & \text{if } P < -2 \end{cases} \tag{1.8}$$

$$\sum_{n=1}^{\infty} \frac{1}{U_{kn}U_{k(n+1)}} = \frac{1}{\alpha^k U_k^2}, \quad \sum_{n=1}^{\infty} \frac{1}{V_{kn}V_{k(n+1)}} = \frac{1}{2(\alpha - \beta)U_k}.$$

Melham furthered his investigations in two further papers [29] and [30] obtaining more identities some of which generalize those of Andfe-Jeannin [3] such as

$$\sum_{i=1}^{n} \frac{L_k^i F_{k(i-1)} + (-1)^{k(i+1)} F_k}{\left(F_k L_k^{i+1} - F_{k(i+2)}\right)\left(F_k L_k^{i+2} - F_{k(i+3)}\right)} = \frac{F_k L_k^{n+1} - F_{k(n+2)}}{F_k \left(F_k L_k^{n+2} - F_{k(n+3)}\right)} \quad (k, n \in \mathbb{N})$$

$$\sum_{n=1}^{\infty} \frac{1}{U_{kn} U_{k(n+m)}} = \frac{2(\alpha - \beta)}{U_{km}} \left(L(\beta^{2k}) - 2L(\beta^{4k}) + 2L(\beta^{8k})\right) - \frac{1}{U_{km}} \sum_{n=1}^{m} \frac{1}{\alpha^{kn} U_{kn}} \quad (k, m \text{ odd})$$

$$\sum_{n=1}^{\infty} \frac{1}{V_{kn} V_{k(n+m)}} = \frac{1}{(\alpha - \beta) U_{km}} \left(2L(\beta^{2k}) - 4L(\beta^{8k}) - \sum_{n=1}^{m} \frac{1}{\alpha^{kn} V_{kn}}\right) \quad (k, m \text{ odd}).$$

In 1999, Rabinowitz [39] employed a partial fraction decomposition to derive a reduction algorithm which enabled him to obtain various finite reciprocal sums such as, when $k \in \mathbb{N}$,

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_n F_{n+k}} = \frac{1}{F_k} \sum_{j=1}^{k} \left(\frac{F_{j-1}}{F_j} - \frac{F_{j+N-1}}{F_{j+N}}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k}} = \begin{cases} \dfrac{1}{F_k} \left(\displaystyle\sum_{i=1}^{\infty} \frac{1}{F_i F_{i+1}} - \displaystyle\sum_{i=1}^{[k/2]} \frac{1}{F_{2i} F_{2i+1}}\right) & \text{if } k \text{ is odd} \\[3ex] \dfrac{1}{F_k} \displaystyle\sum_{i=1}^{k/2} \frac{1}{F_{2i-1} F_{2i}} & \text{if } k \text{ is even} \end{cases}$$

$$\sum_{n=1}^{N} \frac{1}{H_{n+b} H_{n+b+2}} = \frac{1}{H_{b+1} H_{b+2}} - \frac{1}{H_{N+b+1} H_{N+b+2}} \quad (b \in \mathbb{N} \cup \{0\}),$$

where H_n satisfies the recurrence (1.1) with $P = 1, Q = -1$.

In 2001, Hu, Sun and Liu, [21], extended the results of second degree summation to the most general situation, which we now elaborate. Let $\{w_n\} \in L(P, Q)$ and let f be a function such that $f(n) \in \mathbb{Z}$ and $w_{f(n)} \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$.

1) If $m \in \mathbb{N}$, then

$$\sum_{n=0}^{m-1} \frac{Q^{f(n)} U_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \frac{Q^{f(0)} U_{f(m)-f(0)}}{w_{f(0)} w_{f(m)}}.$$

2) Assume that $P, Q \in \mathbb{R} \setminus \{0\}$ and $P^2 - 4Q \geq 0$. If $\lim_{n \to \infty} f(n) = +\infty$ and $w_1 \neq \alpha w_0$, then

$$\sum_{n=0}^{\infty} \frac{Q^{f(n)} U_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \frac{\alpha^{f(0)}}{(w_1 - \alpha w_0) w_{f(0)}},$$

where $\Delta f(n) = f(n+1) - f(n)$. As pointed out by Hu, Sun and Liu, the above results contain almost all previously known second degree reciprocal sums identities.

Two other aspects about infinite reciprocal sums are their irrationality and transcendence. It was proved by André-Jeanin, [1], using Apery's method that $\sum_{n \geq 1} 1/F_n$ and $\sum_{n \geq 1} 1/L_n$ are irrational; an alternative proof using q-exponential and q-logarithm is due to Duverney, [13]. Badea [4] (see also [2], [5], [28]) proved that $\sum_{n \geq 1} 1/F_{2^n+1}$ and $\sum_{n \geq 1} 1/L_{2^n+1}$ are irrational, while $\sum_{n \geq 1} 1/F_{2^n}$ and $\sum_{n \geq 1} 1/L_{2^n}$ are clearly irrational from their explicit shapes in (1.3), (1.8). Tachiya, [41], proved results about irrationality of certain Lambert series with applications to reciprocal sums of the form $\sum_n 1/R_{an+b}$. Quantitative results about irrationality measure have appeared, e.g. in [27]. Regarding transcendence and algebraic independence, Mignotte, [32], showed that $\sum_{n \geq 1} (2 + (-1)^n)/F_{2^n}$ is transcendental. Bundchuuh and Pethö, [11], used Schmidt's simultaneous approximation theorem and Mahler's method to deduce transcendence results about reciprocal sums of the form $\sum_n B_n/R_{f(n)}$, where the numerators B_n do not grow too fast and the denominators $R_{f(n)}$ satisfy (1.1) with exponential growth. Nishioka, [34], using Mahler's method proved algebraic independence

of numbers of the form $\sum_n a_n/R_{g(n)}$, where $g(n) \approx d^n$, $d \geq 2$. Algebraic independence results along this line have been of much progress, see e.g. [14], [24]. Nyblom ([35], [36]) using Roth's theorem proved transcendence of numbers with reciprocal sums $\sum_n 1/R_{n!}$, $\sum_n 1/2^n F_{n!}$, $\sum_n 1/2^n L_{n!}$ as special cases.

2 Binary sequences with non-constant coefficients

A natural question arisen from the investigation of reciprocal sums of elements satisfying a binary recurrecne with constant coefficients is whether the identities previously discovered continue to hold when the coefficients in the binary recurrence are not necessarily constant. It is shown in [25] that the answer is generally positive. In principle, this is achieved by extending results anologous to the two main assertions of Hu, Sun and Liu mentioned in the last part of the previous section. In the rest of this paper, a discussion along this line is given with emphasis being placed on deriving reciprocal sums for Fibonacci and Lucas polynomials.

Let $A := \{a_n\}_{n \in \mathbb{Z}}$ and $B := \{b_n\}_{n \in \mathbb{Z}}$ be two sequences of elements in a field and assume that $a_n \neq 0$ for all $n \in \mathbb{Z}$. Let $\mathcal{L}(A, B)$ be the set of all second order recurrence sequences $\{W_n\}_{n \in \mathbb{Z}}$ satisfying

$$W_{n+2} = b_{n+2}W_{n+1} + a_{n+2}W_n \qquad (n \in \mathbb{Z}). \tag{2.1}$$

For a fixed $m \in \mathbb{Z}$, define

$$C_{m,0} = 0, \quad C_{m,1} = 1, \quad C_{m,n} = b_{m+n}C_{m,n-1} + a_{m+n}C_{m,n-2} \quad (n \in \mathbb{Z}) \tag{2.2}$$

$$\delta_n = \begin{cases} a_2 a_3 \cdots a_{n+1} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ (a_1 a_0 a_{-1} \cdots a_{n+2})^{-1} & \text{if } n < 0. \end{cases} \tag{2.3}$$

Recently, in [25], the following general results about reciprocal sums of elements satisfying (2.1) were proved.

Theorem 1. *Let* $\{A_n\}$, $\{B_n\} \in \mathcal{L}(A, B)$, $D = A_1 B_0 - A_0 B_1$ *and let* $\{C_{m,n}\}$, δ_n *be defined as above. Let* $f : \mathbb{N} \cup \{0\} \to \mathbb{Z}$. *Assume that* $B_{f(k)} \neq 0$ *for all* $k \in \mathbb{N} \cup \{0\}$.
I. If $t \in \mathbb{N}$, *then*

$$\sum_{k=0}^{t-1} \frac{(-1)^{f(k)} \delta_{f(k)} \, D \, C_{f(k), \, f(k+1)-f(k)}}{B_{f(k)} B_{f(k+1)}} = \frac{(-1)^{f(0)} \delta_{f(0)} \, D \, C_{f(0), \, f(t)-f(0)}}{B_{f(0)} B_{f(t)}} = \frac{A_{f(t)}}{B_{f(t)}} - \frac{A_{f(0)}}{B_{f(0)}}.$$

II. If $\lim_{k \to \infty} f(k) = +\infty$ *and* $\lim_{n \to \infty} A_n/B_n = \xi$ *(with respect to a suitable topology of the field), then*

$$\sum_{k=0}^{\infty} \frac{(-1)^{f(k)} \delta_{f(k)} \, D \, C_{f(k), \, f(k+1)-f(k)}}{B_{f(k)} B_{f(k+1)}} = \xi - \frac{A_{f(0)}}{B_{f(0)}}.$$

If we take, in Theorem 1,

$$a_n = -Q, \quad b_n = P, \quad B_n = w_n, \quad C_{m,n} = A_n = U_n \quad (n \in \mathbb{Z}),$$

we simply recover the results of Hu, Sun and Liu mentioned earlier.

Special cases of Theorem 1 are the Fibonacci polynomials $F_n(x)$, and the Lucas polynomials $L_n(x)$, which satisfy the recurrence (2.1) with $b_n = x \neq 0$, $a_n = 1$, i.e.,

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad L_{n+2}(x) = xL_{n+1}(x) + L_n(x) \quad (n \in \mathbb{Z}),$$

and respective initial values

$$F_0(x) = 0, \quad F_1(x) = 1; \quad L_0(x) = 2, \quad L_1(x) = x.$$

Putting

$$\alpha(x) = \frac{x + \sqrt{x^2 + 4}}{2}, \quad \beta(x) = \frac{x - \sqrt{x^2 + 4}}{2},$$

it is well-known that

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)}, \quad L_n(x) = \alpha(x)^n + \beta(x)^n.$$

Applying Theorem 1 to these two particular cases of Fibonacci and Lucas polynomials, we get

Proposition 1. *Let* $f : \mathbb{N} \cup \{0\} \to \mathbb{Z}$. *Assume that* $F_{f(n)}(x)$ *and* $L_{f(n)}(x)$ *are nonzero for all* $n \in \mathbb{N} \cup \{0\}$.
I. If $t \in \mathbb{N}$, *then*

1) $$\sum_{k=0}^{t-1} \frac{(-1)^{f(k)+1} F_{f(k+1)-f(k)}(x)}{F_{f(k)}(x) F_{f(k+1)}(x)} = \frac{(-1)^{f(0)+1} F_{f(t)-f(0)}(x)}{F_{f(0)}(x) F_{f(t)}(x)} = \frac{F_{f(t)+1}(x)}{F_{f(t)}(x)} - \frac{F_{f(0)+1}(x)}{F_{f(0)}(x)}$$

2) $$\sum_{k=0}^{t-1} \frac{(-1)^{f(k)} (x^2 + 4) F_{f(k+1)-f(k)}(x)}{L_{f(k)}(x) L_{f(k+1)}(x)} = \frac{(-1)^{f(0)} (x^2 + 4) F_{f(t)-f(0)}(x)}{L_{f(0)}(x) L_{f(t)}(x)}$$
$$= \frac{L_{f(t)+1}(x)}{L_{f(t)}(x)} - \frac{L_{f(0)+1}(x)}{L_{f(0)}(x)}.$$

II. If $\lim_{n \to \infty} f(n) = \infty$, *then*

1) $$\sum_{k=0}^{\infty} \frac{(-1)^{f(k)+1} F_{f(k+1)-f(k)}(x)}{F_{f(k)}(x) F_{f(k+1)}(x)} = \Omega(x) - \frac{F_{f(0)+1}(x)}{F_{f(0)}(x)},$$

2) $$\sum_{k=0}^{\infty} \frac{(-1)^{f(k)} (x^2 + 4) F_{f(k+1)-f(k)}(x)}{L_{f(k)}(x) L_{f(k+1)}(x)} = \Omega(x) - \frac{L_{f(0)+1}(x)}{L_{f(0)}(x)},$$

where $\Omega(x) = \begin{cases} \alpha(x) & \text{when } x > 0 \\ \beta(x) & \text{when } x < 0. \end{cases}$

Proof. I. For the case of Fibonacci polynomials, in Theorem 1 take

$$A_n = F_{n+1}(x), \quad B_n = F_n(x), \quad C_{m,n} = F_n(x) \quad \text{with} \quad \delta_{f(k)} = 1, \quad D = -1.$$

For the case of Lucas polynomials, in Theorem 1 take

$$A_n = L_{n+1}(x), \quad B_n = L_n(x), \quad C_{m,n} = F_n(x) \quad \text{with} \quad \delta_{f(k)} = 1, \quad D = x^2 + 4.$$

II. The assertions follow from Part I observing that

$$\lim_{n \to \infty} \frac{F_{n+1}(x)}{F_n(x)} = \lim_{n \to \infty} \frac{L_{n+1}(x)}{L_n(x)} = \begin{cases} \alpha(x) & \text{if } x > 0 \\ \beta(x) & \text{if } x < 0. \end{cases}$$

□

Applying Proposition 1, we deduce a number of identities about infinite reciprocal sums generalizing those of Fibonacci and Lucas numbers.

Corollary 1. *Let* $a, j, t, k, r \in \mathbb{N}$ *with* k *even,* r *odd,* $s \in \mathbb{N} \cup \{0\}$ *and let* $\Omega(x)$ *be as defined in Proposition 1. Then*

1) $$\sum_{n=0}^{\infty} \frac{(-1)^{tn}}{F_{tn+j}(x) F_{t(n+1)+j}(x)} = \frac{(-1)^{j+1}}{F_t(x)} \left(\Omega(x) - \frac{F_{j+1}(x)}{F_j(x)} \right).$$

$$2)\quad \sum_{n=1}^{\infty} \frac{(-1)^{[(n-1)/t]t}}{F_n(x)F_{n+t}(x)} = \frac{1}{F_t(x)} \sum_{j=1}^{t}(-1)^j \frac{F_{j+1}(x)}{F_j(x)} + \frac{1-(-1)^t}{2F_t(x)}\,\Omega(x).$$

$$3)\quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1+[(n-1)/t]t}}{F_n(x)F_{n+t}(x)} = \frac{1}{F_t(x)}\left(t\Omega(x) - \sum_{j=1}^{t}\frac{F_{j+1}(x)}{F_j(x)}\right).$$

$$4)\quad \sum_{n=0}^{\infty} \frac{(-1)^{at^n+s+1}F_{(t-1)at^n}(x)}{F_{at^n+s}(x)F_{at^{n+1}+s}(x)} = \Omega(x) - \frac{F_{a+s+1}(x)}{F_{a+s}(x)}.$$

$$5)\quad \sum_{n=1}^{\infty} \frac{1}{F_{2^n k}(x)} = \frac{\Omega(x)^2 + 1}{\Omega(x)\left(\Omega(x)^{2k}-1\right)}.$$

$$6)\quad \sum_{n=0}^{\infty} \frac{(-1)^{tn}}{L_{tn+j}(x)L_{t(n+1)+j}(x)} = \frac{(-1)^j}{(x^2+4)F_t(x)}\left(\Omega(x) - \frac{L_{j+1}(x)}{L_j(x)}\right).$$

$$7)\quad \sum_{n=1}^{\infty} \frac{(-1)^{[(n-1)/t]t}}{L_n(x)L_{n+t}(x)} = \frac{-1}{(x^2+4)F_t(x)} \sum_{j=1}^{t}(-1)^j \frac{L_{j+1}(x)}{L_j(x)} + \frac{1-(-1)^t}{2(x^2+4)F_t(x)}\,\Omega(x).$$

$$8)\quad \sum_{n=1}^{\infty} \frac{(-1)^{n+[(n-1)/t]t}}{L_n(x)L_{n+t}(x)} = \frac{1}{(x^2+4)F_t(x)}\left(t\,\Omega(x) - \sum_{j=1}^{t}\frac{L_{j+1}(x)}{L_j(x)}\right).$$

$$9)\quad \sum_{n=0}^{\infty} \frac{(-1)^{at^n+s}F_{(t-1)at^n}(x)}{L_{at^n+s}(x)L_{at^{n+1}+s}(x)} = \frac{1}{(x^2+4)}\left(\Omega(x) - \frac{L_{a+s+1}(x)}{L_{a+s}(x)}\right).$$

$$10)\quad 2\sum_{n=1}^{\infty} \frac{1}{\alpha(x)^{rn}F_{rn}(x)} = \frac{1}{\alpha(x)^r F_r(x)} + F_r(x)\sum_{n=1}^{\infty}\frac{1}{F_{rn}(x)F_{r(n+1)}(x)}.$$

$$11)\quad 2\sum_{n=1}^{\infty} \frac{1}{\alpha(x)^{rn}L_{rn}(x)} = \frac{1}{\alpha(x)^r L_r(x)} + F_r(x)\sum_{n=1}^{\infty}\frac{\alpha(x)-\beta(x)}{L_{rn}(x)L_{r(n+1)}(x)}.$$

$$12)\quad \sum_{n=0}^{\infty} \frac{(-1)^{tn}}{L_{t(2n+1)+2j}(x) - (-1)^{tn+j}L_t(x)} = \frac{(-1)^{j+1}}{(x^2+4)F_t(x)}\left(\Omega(x) - \frac{F_{j+1}(x)}{F_j(x)}\right).$$

$$13)\quad \sum_{n=0}^{\infty} \frac{(-1)^{tn}}{L_{t(2n+1)+2j}(x) + (-1)^{tn+j}L_t(x)} = \frac{(-1)^j}{(x^2+4)F_t(x)}\left(\Omega(x) - \frac{L_{j+1}(x)}{L_j(x)}\right).$$

$$14)\quad \sum_{n=0}^{\infty} \frac{1}{F_{t(2n+1)+j}(x)^2 - (-1)^j F_t(x)^2} = \frac{(-1)^{j+1}}{F_{2t}(x)}\left(\Omega(x) - \frac{F_{j+1}(x)}{F_j(x)}\right).$$

$$15)\quad \sum_{n=0}^{\infty} \frac{x^2+4}{L_{t(2n+1)+j}(x)^2 - (-1)^j L_t(x)^2} = \frac{(-1)^{j+1}}{F_{2t}(x)}\left(\Omega(x) - \frac{F_{j+1}(x)}{F_j(x)}\right).$$

$$16)\quad \sum_{n=0}^{\infty} \frac{1}{F_{(2n+2)r+2j}(x) + F_r(x)} = \frac{1}{L_r(x)F_{2j}(x)} + \frac{F_r(x)}{F_{2r}(x)}\left(\Omega(x) - \frac{F_{2j+1}(x)}{F_{2j}(x)}\right).$$

$$\sum_{n=0}^{\infty} \frac{1}{F_{(2n+2)r+2j}(x) - F_r(x)} = \frac{1}{L_r(x)F_{2j}(x)} - \frac{F_r(x)}{F_{2r}(x)}\left(\Omega(x) - \frac{F_{2j+1}(x)}{F_{2j}(x)}\right).$$

$$17) \sum_{n=0}^{\infty} \frac{x^2+4}{L_{(2n+1)k+2j}(x) + L_k(x)} = \frac{1}{F_k(x)F_{2j}(x)} + \frac{L_k(x)}{F_{2k}(x)}\left(\Omega(x) - \frac{F_{2j+1}(x)}{F_{2j}(x)}\right).$$

$$\sum_{n=0}^{\infty} \frac{x^2+4}{L_{(2n+1)k+2j}(x) - L_k(x)} = \frac{1}{F_k(x)F_{2j}(x)} - \frac{L_k(x)}{F_{2k}(x)}\left(\Omega(x) - \frac{F_{2j+1}(x)}{F_{2j}(x)}\right).$$

Proof. For parts 1) and 6), put $f(n) = tn + j$ in Proposition 1 part II 1) and 2), respectively.

Parts 2) and 7) follow from summing the identities in parts 1) and 6), respectively, over j from 1 to t and rearranging.

Part 3) follows from multiplying the identity in part 1) by $(-1)^{j+1}$, summing over j from 1 to t and rearranging.

For parts 4) and 9), put $f(n) = at^n + s$ in Proposition 1 part II 1) and 2), respectively.

Part 5) follows easily by putting $t = 2$, $s = 0$ in the identity of part 4) and simplifying. This is a particularly pleasing identity with each denominator in the left hand sum containing only one term of exponential index; this is the only such case derivable from part 4).

Part 8) follows from multiplying the identity in part 6) by $(-1)^j$, summing over j from 1 to t and rearranging.

Part 10) follows from dividing through by $\alpha(x)^{r(n+1)}F_{rn}(x)F_{r(n+1)}(x)$ in the identity

$$\alpha(x)^r F_{r(n+1)}(x) + F_{rn}(x) = \alpha(x)^{r(n+1)}F_r(x) \quad (n \in N),$$

and summing over n.

Part 11) follows in the same manner as part 10) but using instead the identity

$$\alpha(x)^r L_{r(n+1)}(x) + L_{rn}(x) = \{\alpha(x) - \beta(x)\}\alpha(x)^{r(n+1)}F_r(x).$$

For part 12), we start with the identity

$$\frac{x^2+4}{L_{t+2a}(x) - (-1)^a L_t(x)} = \frac{1}{F_a(x)F_{a+t}(x)}.$$

Putting $a = j, j+t, \ldots, j+Nt$, and multiplying by $(-1)^{0t}, (-1)^{1t}, \ldots, (-1)^{Nt}$, respecteively, we get

$$\frac{(-1)^{0t}(x^2+4)}{L_{t+2j}(x) - (-1)^j L_t(x)} = \frac{(-1)^{0t}}{F_j(x)F_{j+t}(x)}, \frac{(-1)^{1t}(x^2+4)}{L_{3t+2j}(x) - (-1)^{j+t}L_t(x)} = \frac{(-1)^{1t}}{F_{j+t}(x)F_{j+2t}(x)}, \ldots,$$

$$\frac{(-1)^{Nt}(x^2+4)}{L_{t(2N+1)+2j}(x) - (-1)^{j+Nt}L_t(x)} = \frac{(-1)^{Nt}}{F_{j+Nt}(x)F_{j+(N+1)t}(x)}.$$

Summing these expressions and letting $N \to \infty$, we get

$$(x^2+4)\sum_{n=0}^{\infty} \frac{(-1)^{tn}}{L_{t(2n+1)+2j}(x) - (-1)^{tn+j}L_t(x)} = \sum_{n=0}^{\infty} \frac{(-1)^{nt}}{F_{j+nt}(x)F_{j+(n+1)t}}$$

and the result follows from replacing the right hand expression by the result of part 1).

Part 13) follows along the same line of proof as in part 12) but starting instead with the identity

$$\frac{1}{L_{t+2a}(x) + (-1)^a L_t(x)} = \frac{1}{L_a(x)L_{a+t}},$$

and using the result of part 6) at the end.

For part 14), we start with the identity

$$\frac{1}{F_{a+t}(x)^2 - (-1)^a F_t(x)^2} = \frac{1}{F_a(x)F_{a+2t}(x)}.$$

Putting $a = j, j + 2t, \ldots, j + 2Nt$, we get

$$\frac{1}{F_{j+t}(x)^2 - (-1)^j F_t(x)^2} = \frac{1}{F_j(x)F_{j+2t}(x)}, \quad \frac{1}{F_{j+3t}(x)^2 - (-1)^j F_t(x)^2} = \frac{1}{F_{j+2t}(x)F_{j+4t}(x)}, \ldots,$$

$$\frac{1}{F_{j+(2N+1)t}(x)^2 - (-1)^j F_t(x)^2} = \frac{1}{F_{j+2Nt}(x)F_{j+2(N+1)t}(x)}.$$

Summing these expressions and letting $N \to \infty$, we get

$$\sum_{n=0}^{\infty} \frac{1}{F_{t(2n+1)+j}(x)^2 - (-1)^j F_t(x)^2} = \sum_{n=0}^{\infty} \frac{1}{F_{2tn+j}(x)F_{2t(n+1)+j}(x)},$$

and the result follows from the result of part 1) with $2t$ in place of t.

Part 15) follows along the same line of proof as in part 14) but starting instead with the identity

$$\frac{x^2 + 4}{L_{a+t}(x)^2 - (-1)^a L_t(x)^2} = \frac{1}{F_a(x)F_{a+2t}(x)}.$$

To prove the first identity of part 16), we start with the identity

$$F_n(x)^2 - (-1)^{n-r} F_r(x)^2 = F_{n-r}(x)F_{n+r}(x).$$

Replacing n by $(2n + 1)r + 2j$ and rearranging, we get

$$\frac{1}{F_{(2n+1)r+2j}(x) + F_r(x)} = \frac{F_{(2n+1)r+2j}(x) - F_r(x)}{F_{2nr+2j}(x)F_{(2n+2)r+2j}(x)}. \tag{2.4}$$

Next, from the identity

$$L_r(x)F_{(2n+1)r+2j}(x) = F_{(2n+2)r+2j}(x) + (-1)^r F_{2nr+2j}(x),$$

using the fact that r is odd, we get

$$\frac{F_{(2n+1)r+2j}(x)}{F_{2nr+2j}(x)F_{(2n+2)r+2j}(x)} = \frac{1}{L_r(x)} \left(\frac{1}{F_{2nr+2j}(x)} - \frac{1}{F_{(2n+2)r+2j}(x)} \right). \tag{2.5}$$

Summing (2.4) over n from 0 to N and using (2.5) to simplify the first term on the right side which becomes a telescoping sum, we get

$$\sum_{n=0}^{N} \frac{1}{F_{(2n+1)r+2j}(x) + F_r(x)} = \frac{1}{L_r(x)} \left(\frac{1}{F_{2j}(x)} - \frac{1}{F_{(2N+2)r+2j}(x)} \right) - \sum_{n=0}^{N} \frac{F_r(x)}{F_{2nr+2j}(x)F_{(2n+2)r+2j}(x)}.$$

The result now follows by taking $N \to \infty$, making use of the result in part 1) and the fact that $1/F_N(x) \to 0$ $(N \to \infty)$. The second identity is proved similarly.

To prove the first identity of part 17), we start with the identity

$$L_n(x)^2 - (-1)^{n-k} L_k(x)^2 = (x^2 + 4)F_{n-k}(x)F_{n+k}(x).$$

Replacing n by $(2n + 1)k + 2j$ and rearranging, we get

$$\frac{x^2 + 4}{L_{(2n+1)k+2j}(x) + L_k(x)} = \frac{L_{(2n+1)k+2j}(x) - L_k(x)}{F_{2nk+2j}(x)F_{(2n+2)k+2j}(x)}. \tag{2.6}$$

Next, from the identity

$$F_k(x)L_{(2n+1)k+2j}(x) = F_{(2n+2)k+2j}(x) + (-1)^{k+1} F_{2nk+2j}(x),$$

using the fact that k is even, we get

$$\frac{L_{(2n+1)k+2j}(x)}{F_{2nk+2j}(x)F_{(2n+2)k+2j}(x)} = \frac{1}{F_k(x)}\left(\frac{1}{F_{2nk+2j}(x)} - \frac{1}{F_{(2n+2)k+2j}(x)}\right). \tag{2.7}$$

Summing (2.6) over n from 0 to N and using (2.7) to simplify the first term on the right side which becomes a telescoping sum, we get

$$\sum_{n=0}^{N}\frac{x^2+4}{L_{(2n+1)k+2j}(x)+L_k(x)} = \frac{1}{F_k(x)}\left(\frac{1}{F_{2j}(x)} - \frac{1}{F_{(2N+2)k+2j}(x)}\right) - \sum_{n=0}^{N}\frac{L_k(x)}{F_{2nk+2j}(x)F_{(2n+2)k+2j}(x)}.$$

The result now follows by taking $N \to \infty$, making use of the result in part 1) and the fact that $1/F_N(x) \to 0$ $(N \to \infty)$. The second identity is proved analogously. $\square$

Some of these and similar identities have already appeared in [6], [37] and [38]. Next, we prove some identities about *finite* reciprocal sums of Fibonacci and Lucas polynomials, cf. [3], [16].

Corollary 2. *Let* $t, a, m \in \mathbb{N}$, $a \geq 2$. *Then*

1) $$\sum_{n=1}^{2t}(-1)^n\frac{F_{[(n+1)/2]a+(1+(-1)^n)/2}(x)}{F_{[(n+1)/2]a+(1+(-1)^n)/2-1}(x)} = \sum_{n=1}^{t}\frac{(-1)^{an}}{F_{an-1}(x)F_{an}(x)};$$

2) $$\sum_{n=1}^{2t}(-1)^n\frac{F_{[(n+1)/2]a-(3-(-1)^n)/2}(x)}{F_{[(n+1)/2]a-(3-(-1)^n)/2+3}(x)} = \sum_{n=1}^{t}\frac{(-1)^{an}(x^2+1)}{F_{an+1}(x)F_{an+2}(x)}.$$

3) *For constants* λ, ν, *define* $G_n(x) := \lambda F_n(x) + \nu L_n(x)$ $(n \in \mathbb{N})$. *Then*

$$F_m(x)\sum_{n=1}^{t}\frac{(-1)^n}{G_n(x)G_{n+m}(x)} = F_t(x)\sum_{n=1}^{m}\frac{(-1)^n}{G_n(x)G_{n+t}(x)}. \tag{2.8}$$

Proof. For part 1), using the analogue of Cassini's formula (Theorem 53 on p. 74 of [23]), which is easily checked by induction,

$$F_{n-1}(x)F_{n+1}(x) - F_n(x)^2 = (-1)^n \quad (n \in \mathbb{N}),$$

we have

$$\sum_{n=1}^{2t}(-1)^n\frac{F_{[(n+1)/2]a+(1+(-1)^n)/2}(x)}{F_{[(n+1)/2]a+(1+(-1)^n)/2-1}(x)} = \sum_{n=1}^{t}\left(\frac{F_{an+1}(x)}{F_{an}(x)} - \frac{F_{an}(x)}{F_{an-1}(x)}\right) = \sum_{n=1}^{t}\frac{(-1)^{an}}{F_{an}(x)F_{an-1}(x)}.$$

For part 2), using the identity

$$F_{n-1}(x)F_{n+1}(x) - F_{n-2}(x)F_{n+2}(x) = (-1)^n(x^2+1) \quad (n \in \mathbb{N}),$$

we have

$$\sum_{n=1}^{2t}(-1)^n\frac{F_{[(n+1)/2]a-(3-(-1)^n)/2}(x)}{F_{[(n+1)/2]a-(3-(-1)^n)/2+3}(x)} = \sum_{n=1}^{t}\left(\frac{F_{an-1}(x)}{F_{an+2}(x)} - \frac{F_{an-2}(x)}{F_{an+1}(x)}\right) = \sum_{n=1}^{t}\frac{(-1)^{an}(x^2+1)}{F_{an+1}(x)F_{an+2}(x)}.$$

To prove part 3), we start from the two identities, when $m, n \in \mathbb{N}$,

$$F_{m+1}(x)F_{m+n}(x) - F_m(x)F_{m+n+1}(x) = (-1)^m F_n(x)$$
$$F_{m+1}(x)L_{m+n}(x) - F_m(x)L_{m+n+1}(x) = (-1)^m L_n(x)$$

which yield $F_{m+1}(x)G_{m+n}(x) - F_m(x)G_{m+n+1}(x) = (-1)^m G_n(x)$, and so

$$\frac{1}{G_n(x)}\left(\frac{F_{m+1}(x)}{G_{m+n+1}(x)} - \frac{F_m(x)}{G_{m+n}(x)}\right) = \frac{(-1)^m}{G_{m+n}(x)G_{m+n+1}(x)}. \tag{2.9}$$

Without loss of generality, assume that $m \geq t$. We establish (2.8) by induction on m, the case $m = t$ being valid trivially. Assuming (2.8) holds up to m, to prove it holds for $m + 1$ is equivalent to proving that

$$\frac{(-1)^{m+1}F_t(x)}{G_{m+1}(x)G_{m+1+t}(x)} = \sum_{n=1}^{t} \frac{(-1)^n}{G_n(x)} \left(\frac{F_{m+1}(x)}{G_{n+m+1}(x)} - \frac{F_m(x)}{G_{n+m}(x)} \right).$$

Using (2.9), this is equivalent to proving that

$$\frac{F_t(x)}{G_{m+1}(x)G_{m+1+t}(x)} = \sum_{n=1}^{t} \frac{(-1)^{n+1}}{G_{m+n}(x)G_{n+m+1}(x)}. \tag{2.10}$$

Since $F_1(x) = 1$, (2.10) clearly holds when $t = 1$. Assuming it holds up to t, to show it holds at $t + 1$, we must verify that

$$\frac{F_{t+1}(x)}{G_{m+1}(x)G_{m+t+2}(x)} = \frac{F_t(x)}{G_{m+1}(x)G_{m+t+1}(x)} + \frac{(-1)^t}{G_{m+t+1}(x)G_{m+t+2}(x)}.$$

This last equation follows easily from (2.9). □

Besides beautiful identities about reciprocal sums of Fibonacci and Lucas numbers, there are equally beautiful identities relating such numbers with trigonometric and hyperbolic functions such as those in [29]. To end this section, we derive their generalizations to Fibonacci and Lucas polynomials.

Proposition 2. *Let $n \in \mathbb{N}$. Then*

1. $\tan^{-1} F_{n+4}(x) - \tan^{-1} F_n(x) = \tan^{-1} \left(\dfrac{xL_{n+2}(x)}{F_{n+2}(x)^2 + 1 - (-1)^n x^2} \right).$

2. $\tan^{-1} \left(\dfrac{1}{F_n(x)} \right) + \tan^{-1} \left(\dfrac{1}{F_{n+4}(x)} \right) = \tan^{-1} \left(\dfrac{(x^2 + 2)F_{n+2}(x)}{F_{n+2}(x)^2 - 1 - (-1)^n x^2} \right)$
provided $F_{n+4}(x)F_n(x) > 1$.

3. $\tanh^{-1} \left(\dfrac{1}{F_n(x)} \right) + \tanh^{-1} \left(\dfrac{1}{F_{n+4}(x)} \right) = \tanh^{-1} \left(\dfrac{(x^2 + 2)F_{n+2}(x)}{F_{n+2}(x)^2 + 1 - (-1)^n x^2} \right).$

4. $\tanh^{-1} \left(\dfrac{1}{F_n(x)} \right) - \tanh^{-1} \left(\dfrac{1}{F_{n+4}(x)} \right) = \tanh^{-1} \left(\dfrac{xL_{n+2}(x)}{F_{n+2}(x)^2 - 1 - (-1)^n x^2} \right).$

Proof. The four identities follow from the following facts, with $n \in \mathbb{N}$ and $u, v \in \mathbb{R}$:

$$\tan^{-1} u - \tan^{-1} v = \tan^{-1} \left(\frac{u - v}{1 + uv} \right) \text{ provided } uv > -1,$$

$$\tan^{-1} u + \tan^{-1} v = \tan^{-1} \left(\frac{u + v}{1 - uv} \right) \text{ provided } uv < 1,$$

$$\tanh^{-1} u + \tanh^{-1} v = \tanh^{-1} \left(\frac{u + v}{1 + uv} \right),$$

$$\tanh^{-1} u - \tanh^{-1} v = \tanh^{-1} \left(\frac{u - v}{1 - uv} \right),$$

$$F_n(x)F_{n+4}(x) + (-1)^n x^2 = F_{n+2}(x)^2,$$
$$F_{n+4}(x) - F_n(x) = xL_{n+2}(x),$$
$$F_{n+4}(x) + F_n(x) = (x^2 + 2)F_{n+2}(x)$$

and the observation that, for $x \neq 0$, $F_n(x)F_{n+4}(x) > 0$. □

Corollary 3. *Let $t \in \mathbb{N}$ and $x > 0$. Then*

1. $\displaystyle\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{xL_{t+4i-2}(x)}{F_{t+4i-2}(x)^2 + 1 - (-1)^t x^2}\right) = \frac{\pi}{2} - \tan^{-1} F_t(x).$

2. $\displaystyle\sum_{i=1}^{\infty} (-1)^{i-1} \tan^{-1}\left(\frac{(x^2+2)F_{t+4i-2}(x)}{F_{t+4i-2}(x)^2 - 1 - (-1)^t x^2}\right) = \tan^{-1}\left(\frac{1}{F_t(x)}\right)$

    provided $F_{t+4}(x)F_t(x) > 1$.

3. $\displaystyle\sum_{i=1}^{\infty} (-1)^{i-1} \tanh^{-1}\left(\frac{(x^2+2)F_{t+4i-2}(x)}{F_{t+4i-2}(x)^2 + 1 - (-1)^t x^2}\right) = \tanh^{-1}\left(\frac{1}{F_t(x)}\right).$

4. $\displaystyle\sum_{i=1}^{\infty} \tanh^{-1}\left(\frac{xL_{t+4i-2}(x)}{F_{t+4i-2}(x)^2 - 1 - (-1)^t x^2}\right) = \tanh^{-1}\left(\frac{1}{F_t(x)}\right).$

Proof. Putting $n = t,\ t+4,\ t+8, \ldots, t+4N-4$ in the first identity of Proposition 2 and summing, we get

$$\sum_{i=1}^{N} \tan^{-1}\left(\frac{xL_{t+4i-2}(x)}{F_{t+4i-2}(x)^2 + 1 - (-1)^t x^2}\right) = \tan^{-1} F_{t+4N}(x) - \tan^{-1} F_t(x).$$

Noting that for $x > 0$, we have $F_{t+4N}(x) \to \infty$ as $N \to \infty$. The first assertion thus follows by letting $N \to \infty$. The remaining three assertions follow in the same manner. $\qquad\square$

3   Continued fractions

As elements satisfying the recurrence (2.1) are closely related to non-regular continued fraction expansions, it is meaningful to transfer the above results into the language of continued fractions. Define the sequence $\{S_n\}_{n \geq 1}$ by

$$S_n = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{\ddots}{\quad + \cfrac{a_n}{b_n}}}}} \qquad (n \geq 1).$$

If the sequence $\{S_n\}$ converges with respect to some appropriate topology, we write

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} := \lim_{n \to \infty} S_n, \tag{3.1}$$

and call it a (non-regular) continued fraction of the element it represents. The elements a_n, b_n, are referred to as its nth **partial numerator**, respectively, n^{th} **partial denominator**, and S_n is called the nth **approximant** (or **convergent**) of the continued fraction (3.1). If

$$a_1 = a_2 = \cdots = 1 \quad \text{and} \quad b_i \in \mathbb{N}\ (i \geq 1),$$

then (3.1) is usually called a simple continued fraction, customarily denoted by $[b_1, b_2, b_3, \ldots]$. For a fixed b_0, define

$$A_{-1} = 1,\ A_0 = b_0,\ B_{-1} = 0,\ B_0 = 1, \tag{3.2}$$

and let

$$\frac{A_n}{B_n} := b_0 + S_n \quad (n \geq 1).$$

It is well-known, see e.g. Chapter 2 of [22] or Chapter 1 of [26], that the two sequences $\{A_n\}$ and $\{B_n\}$ satisfy the same system of second order linear recurrence relations but with different initial conditions, viz.,

$$A_n = b_n A_{n-1} + a_n A_{n-2} \quad (n \geq 1), \tag{3.3}$$

$$B_n = b_n B_{n-1} + a_n B_{n-2} \quad (n \geq 1). \tag{3.4}$$

The element A_n/B_n is called the nth convergent of the continued fraction (3.1) with A_n being the **numerator** and B_n the **denominator** of the nth convergent, respectively. Since the two recurrence relations (3.3) and (3.4) are of the same form as (2.1), Theorem 1 at once yields:

Theorem 2. *Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 0}$ be two sequences in a field all of whose elements except b_0 are nonzero. Let $\{A_n\}$, $\{B_n\}$ be two sequences satisfying (3.2), (3.3) and (3.4) and the sequence $\{C_{m,n}\}$ be as defined in (2.2). Let $f : \mathbb{N} \cup \{0\} \to \mathbb{Z}$. Assume that $B_{f(k)} \neq 0$ for all $k \in \mathbb{N} \cup \{0\}$.*
I. For $t \in \mathbb{N}$ we have

$$\sum_{k=0}^{t-1} \frac{(-1)^{f(k)} C_{f(k),\,f(k+1)-f(k)} \prod_{i=1}^{f(k)+1} a_i}{B_{f(k)} B_{f(k+1)}} = \frac{(-1)^{f(0)} C_{f(0),\,f(t)-f(0)} \prod_{i=1}^{f(0)+1} a_i}{B_{f(0)} B_{f(t)}} \tag{3.5}$$

$$= \frac{A_{f(t)}}{B_{f(t)}} - \frac{A_{f(0)}}{B_{f(0)}}. \tag{3.6}$$

II. If $f(k)$ is such that $\lim_{k\to\infty} f(k) = +\infty$ and if the sequence $\{A_n/B_n\}$ converges to ξ, then it is a sequence of approximants of the continued fraction representing ξ and

$$\sum_{k=0}^{\infty} \frac{(-1)^{f(k)} C_{f(k),\,f(k+1)-f(k)} \prod_{i=1}^{f(k)+1} a_i}{B_{f(k)} B_{f(k+1)}} = \xi - \frac{A_{f(0)}}{B_{f(0)}}.$$

III. If $f(k)$ is such that $\lim_{k\to\infty} f(k) = -\infty$ and if the sequence $\{A_{-n}/B_{-n}\}_{n\in\mathbb{N}}$ converges to ψ, then

$$\sum_{k=0}^{\infty} \frac{(-1)^{f(k)} C_{f(k),\,f(k+1)-f(k)} \prod_{i=1}^{f(k)+1} a_i}{B_{f(k)} B_{f(k+1)}} = \psi - \frac{A_{f(0)}}{B_{f(0)}}.$$

The results of Theorem 2 are applicable to the theory of continued fractions in the field of Laurent series over a field equipped with the degree valuation which we briefly discuss now.

Let K be a field, x an indeterminate, $K((1/x))$ the field of all formal Laurent series equipped with the degree valuation, $|\cdot|_\infty$, so normalized that $|x|_\infty = e^1$. In $K((1/x))$, there is a continued fraction expansion ([12] or [40]), whose basic properties are similar to the one in the real case, which is constructed as follows: since each element $\xi \in K((1/x)) \setminus \{0\}$ has a unique representation of the form

$$\xi = c_m x^m + c_{m-1} x^{m-1} + c_{m-2} x^{m-2} + \ldots \quad (m \in \mathbb{Z}),$$

with coefficients $c_m (\neq 0)$, c_{m-1}, $c_{m-2}, \ldots \in K$. Define

$$\xi = [\xi] + (\xi),$$

where

$$[\xi] := c_m x^m + c_{m-1} x^{m-1} + \ldots + c_1 x + c_0,$$

$$(\xi) := c_{-1} x^{-1} + c_{-2} x^{-2} + \ldots,$$

with the customary convention that empty sums are interpreted as 0. Clearly, $[\xi]$ and (ξ) are uniquely determined. Let

$$\beta_0 = [\xi] \in K[x],$$

so that $|\beta_0|_\infty = |\xi|_\infty \geq 1$, provided $[\xi] \neq 0$. If $(\xi) = 0$, then the process stops. If $(\xi) \neq 0$, then write

$$\xi = \beta_0 + \frac{1}{\xi_1},$$

where $\xi_1^{-1} = (\xi)$ with $|\xi_1|_\infty > 1$. Next write

$$\xi_1 = [\xi_1] + (\xi_1)$$

and let

$$\beta_1 = [\xi_1] \in K[x] \setminus K,$$

so that $|\beta_1|_\infty = |\xi_1|_\infty > 1$. If $(\xi_1) = 0$, then the process stops. If $(\xi_1) \neq 0$, then write

$$\xi_1 = \beta_1 + \frac{1}{\xi_2},$$

where $\xi_2^{-1} = (\xi_1)$ with $|\xi_2|_\infty > 1$ and let

$$\beta_2 = [\xi_2] \in K[x] \setminus K,$$

so that $|\beta_2|_\infty = |\xi_2|_\infty > 1$. Again, if $(\xi_2) = 0$, then the process stops; otherwise, continue in the same manner. By so doing, we obtain the unique representation

$$\xi = [\beta_0;\ \beta_1, \ldots,\ \beta_{n-1},\ \xi_n] := \beta_0 + \cfrac{1}{\beta_1 + \cfrac{1}{\beta_2 + \cfrac{\ddots}{\ + \cfrac{1}{\beta_{n-1} + \cfrac{1}{\xi_n}}}}},$$

where

$$\beta_i \in K[x] \setminus K \quad (i \geq 1), \quad \xi_n \in K((1/x)), \quad |\xi_n|_\infty > 1$$

if the process does not stop before, and ξ_n is referred to as the nth *complete quotient*. The sequence $\{\beta_n\}$ is uniquely determined, called the sequence of *partial quotients* of ξ.

The two sequences of partial numerators, $\{C_n\}$, and partial denominators, $\{D_n\}$, are defined by

$$C_{-1} = 1, \ C_0 = \beta_0, \ C_{n+1} = \beta_{n+1} C_n + C_{n-1} \quad (n \geq 0)$$
$$D_{-1} = 0, \ D_0 = 1, \quad D_{n+1} = \beta_{n+1} D_n + D_{n-1} \quad (n \geq 0).$$

As an example, note that the partial numerators and partial denominators of the continued fraction

$$[0;\ x,\ x,\ x, \ldots] := 0 + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{\ddots}}}}$$

are merely two shifted sequences of Fibonacci polynomials, namely,

$$C_n = F_n(x), \ D_n = F_{n+1}(x).$$

This continued fraction converges, with respect to the degree valuation, to $(\sqrt{x^2 + 4} - x)/2$.

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