

GENERALIZING SIERPIŃSKI NUMBERS TO BASE b

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ABSTRACT. Sierpiński proved that there are infinitely many odd integers k such that $k \cdot 2^n + 1$ is composite for all $n \geq 0$. These k are now called Sierpiński numbers. We define a Sierpiński number base b to be an integer $k > 1$ for which $\gcd(k+1, b-1) = 1$, k is not a rational power of b , and $k \cdot b^n + 1$ is composite for all $n > 0$. We discuss ways that these can arise, offer conjectured least Sierpiński number in each of the bases $2 < b \leq 100$ (34 are proven), and show that all bases b admit Sierpiński numbers. We also show 4 is the least base b Sierpiński number for infinitely many bases b .

1. INTRODUCTION AND HISTORY

In 1958, R. M. Robinson [25] formed a table of primes of the form $k \cdot 2^n + 1$ for odd integers $1 \leq k < 100$ and $0 \leq n \leq 512$. He found primes for all k values except 47. Some then wondered “Is there an odd k value such that $k \cdot 2^n + 1$ is always composite?” In 1960, W. Sierpiński [28] proved that there were indeed infinitely many such odd integers k . He did this by finding a small set of primes S such that for a suitable choice of k , every term of the sequence $k \cdot 2^n + 1$ ($n > 0$) is divisible by a prime in his “cover” S . The values k which make every term in the sequence composite are now called **Sierpiński numbers**. Sierpiński however neither gave a value of k nor sought the least such k .

In 1962, Selfridge [unpublished] showed that $k = 78557$ is also a Sierpiński number, and this is now believed to be the least Sierpiński number. For three decades mathematicians have been testing all of the values of k less than 78557 to prove this conjecture [2, 9, 20, 22]. All values except 10223, 21181, 22699, 24737, 55459, and 67607 have now been eliminated by finding a prime in the corresponding sequence [18].

There are two standard methods of generalizing Sierpiński numbers. Several have generalized this idea by altering the restrictions on k [10, 11, 14, 21]. For example, one may seek Sierpiński numbers k for which all of k, k^2, k^3, \dots, k^r are also Sierpiński numbers for arbitrarily large integers r [14]. The second author is pursuing this direction in a related paper. In this paper we will pursue the other approach to generalization—by changing the base.

On the Internet several groups have generalized Sierpiński’s result to other bases b [4, 29, 30]. See, for example, the results in Table 1. There was also a short note by Bowen in 1964 [5] which we will mention several times below. But at the time we began our investigation, none of these presented a systematic study of the generalization or even a careful study of the definition. In this paper we will fill this gap by providing a definition and then extending the studied bases systematically to include all of the bases up through 100.

We will prove that Sierpiński numbers exist for all bases $b > 1$, and offer conjectured least Sierpiński numbers for the bases $2 < b \leq 100$. For 34 of these bases we are able to prove that the conjectured values are indeed the least.

2. GENERALIZING SIERPIŃSKI NUMBERS TO BASE b

A Sierpiński number is an odd integer k such that $k \cdot 2^n + 1$ is composite for all $n > 0$. Before generalizing this definition of a Sierpiński number to other bases b , there are a couple of things we must consider.

First, when generalizing a definition it is traditional to exclude any cases that are too trivial. So we begin by requiring that the sequence $k \cdot b^n + 1$ ($n = 0, 1, 2, \dots$) does not have a “one-cover.” That is, there is no single prime p which divides every value of the sequence. For example, if k and b are odd, then 2 divides every term (and in fact if 2 divides any term with $n > 0$ of any such sequence, then it divides every term).

Key words and phrases. Sierpiński number, covering set, generalized Fermat number.

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TABLE 1. Conjectured least Sierpiński numbers k base b

b	N	k {cover}	k 's not yet eliminated	ref
2	36	78557 {3, 5, 7, 13, 19, 37, 73}	{10223, 21181, 22699, 24737, 55459, 67607}	[18]
3	144	125050976086 {5, 7, 13, 17, 19, 37, 41, 193, 767}	{2949008, 4273396, 4660218, 6363484, 8058998, 8182316, ...}	[4, 6]
4	12	66741 {5, 7, 13, 17, 241}	{18534, 20446, 21181, 22699, 23451, 49474, 55459, 60849, 64494}	[18, 30]
5	12	159986 {3, 7, 13, 31, 601}	{6436, 7528, 8644, 10918, 24032, 26798, 29914, 31712, 36412, ...}	[4]
6	12	174308 {7, 13, 31, 37, 97}	{10107, 13215, 14505, 26375, 31340, 33706, 36772, 50252, 51255, ...}	[4]
7	24	1112646039348 {5, 13, 19, 43, 73, 181, 193, 1201}	{66936, 95626, 242334, 270636, 303366, 357132, 468552, ...}	[4]

Theorem 2.1 (1-covers). *The prime p divides $k \cdot b^n + 1$ for all non-negative integers n if and only if p divides $\gcd(k+1, b-1)$.*

Proof. First, suppose p divides $k \cdot b^n + 1$ for all n , then it does so for $n = 0$ and 1 , that is p divides $k + 1$ and $k \cdot b + 1$. Subtracting these we see p divides $k(b - 1)$ so p divides $\gcd(k+1, b-1)$. If instead p divides $\gcd(k+1, b-1)$, then $k \cdot b^n + 1 \equiv k+1 \equiv 0 \pmod{p}$. \square

In 1964, Bowen [5] showed there were choices of k for which $k \cdot b^n + 1$ is composite for all $n \geq 0$, but he did so by using 1-covers for all bases except those relatively few which are a power of 2 plus one.

Second, some have suggested that the restriction “ k odd” appears in the above definition because any factor of 2 in k can be absorbed into the exponent n , but consider the number $2^m 2^n + 1$ for some fixed positive integers m and n . If this number is to be prime, then it must be a Fermat number $F_n = 2^{2^n} + 1$ and so $n+m$ must be a power of two. It is widely suspected that there are only finitely many Fermat primes, which would mean there would be infinitely many even Sierpiński numbers that are a power of 2. If the only prime Fermat numbers are the five known, then $2^{16} 2^n + 1$ would be composite for $n > 0$, and therefore $2^{16} = 65536$ would be the least Sierpiński number, not Selfridge’s 78557.

Since the existence of infinitely many Fermat primes is undecidable at this point in time, it seems best to define generalized Sierpiński numbers in such a way as to exclude the Fermat numbers and, for bases other than powers of 2, to exclude the generalized Fermat numbers $F_n(b) = b^{2^n} + 1$ [12]. At the end of this section (Theorem 2.4) we will show that this is equivalent to adding the requirement that k is not a rational power of b ($k \neq b^{\frac{e}{q}}$ for integers $p \geq 0$ and $q > 0$), and hence that $k > 1$. Those values so omitted (for our range $2 \leq b \leq 100$) are listed in Table 2.

Combining these considerations we generalize Sierpiński numbers as follows.

Definition 2.2. Let $b > 1$ be an integer. A **Sierpiński number base b** (or **b -Sierpiński**) is an integer $k > 1$ for which $\gcd(k+1, b-1) = 1$, k is not a rational power of b , and $k \cdot b^n + 1$ is composite for all $n > 0$.

Notice that this definition extends the definition of Sierpiński numbers in base 2 as well as the larger integer bases—yet still 78557 likely remains the least possible 2-Sierpiński number.

We end this section by showing we have properly characterized those pairs k and b which may generate infinitely many generalized Fermat numbers. Recall that the order of b modulo a relatively prime integer p , denoted $\text{ord}_p(b)$, is the least positive integer m for which p divides $b^m - 1$. So in particular $\text{ord}_p(b)$ divides $\phi(p)$ (Euler’s ϕ function of p).

Lemma 2.3. *Let $e > 1$, $f > 0$ and $c \neq 0$ be integers. Write $e = 2^n e'$ where e' is odd. Then $\gcd(c^e - 1, c^e + 1) > 1$ if and only if c is odd or 2^{n+1} divides f .*

Proof. Let $d = \gcd(c^f - 1, c^e + 1)$. First note that 2 divides d if and only if c is odd, so assume c is even. Note that since e' is odd, $c^{2^n} + 1$ divides $c^e + 1$. If 2^{n+1} divides f , then $c^{2^n} + 1$ divides d . Conversely, if any odd prime p divides d , then $\text{ord}_p(c)$ divides both $2e$ and f , but not e . This means 2^{n+1} divides $\text{ord}_p(c)$ and therefore divides f . \square

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TABLE 2. $k \cdot b^n + 1$ is infinitely often a generalized Fermat ‡

b	k	b	k
6	6, 36, 216, 1296, 7776, 46656	46	46, 2116
8	2, 4, 8, 16, 32	48	48
10	10, 100, 1000	52	52, 2704
12	12, 144	58	58, 3364
16	16, 256, 4096, 65536	60	60, 3600
18	18, 324	64	4, 16
22	22, 484	66	66, 4356, 287496, 18974736
24	24, 576, 13824	70	70, 4900
26	26	72	72
28	28, 784	78	78, 6084
30	30	80	80
32	2, 4, 8	82	82, 6724
36	36, 1296	88	88
40	40, 1600, 64000	96	96, 9216
42	42, 1764	100	100
<hr/>			
2	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536		
<hr/>			
4	4, 16, 64, 256, 1024, 4096, 16384, 65536		

‡ Just those smaller than conjectured least base b Sierpiński and with $\gcd(k + 1, b - 1) = 1$.

Theorem 2.4. *Let $b > 1$ and $k > 0$ be integers for which $\gcd(k + 1, b - 1) = 1$. There is an integer $c > 1$ for which $k \cdot b^n + 1 = F_r(c)$ for infinitely many integer values of r and n , if and only if k is a rational power of b .*

Proof. Let $b > 1$ and $k > 0$ be fixed integers for which $\gcd(k + 1, b - 1) = 1$.

Suppose there is an integer c for which $k \cdot b^n + 1$ ($n > 1$) is the generalized Fermat number $F_r(c)$ for infinitely many pairs of integers r and n . Choose two such pairs (r, n) and (s, m) with $n < m$. Then

$$k \cdot b^n + 1 = c^{2^r} + 1 \quad \text{and} \quad k \cdot b^m + 1 = c^{2^s} + 1.$$

Thus $b^{m-n} = c^{2^s - 2^r}$, and it follows $b = c^{\frac{2^s - 2^r}{m-n}}$, $k = c^{\frac{m2^r - n2^s}{m-n}}$, and therefore k is a rational power of b (and both are rational powers of c).

Conversely, suppose k is a rational power of b , say $k = b^{e/f}$ for relatively prime integers e and f with $e \geq 0$ and $f > 0$. Then because b is an integer, $b = c^f$ and $k = c^e$ for some integer c . Write $f = 2^t f'$ where f' is an odd integer. Now $\gcd(c^f - 1, c^e + 1) = 1$, so by Lemma 2.3 c is even and the power of 2 which divides e is at least as great as the power of 2 which divides f . So we may write $e = 2^t e'$ for some integer (not necessarily odd) e' . Note that if r is any positive multiple of $\text{ord}_{f'}(2)$, then $e' \equiv e'2^r \pmod{f'}$, so we may solve the following for a positive integer $n = n(r)$:

$$e' + f'n = e'2^r.$$

So it follows

$$e + fn = 2^t(e' + f'n) = e'2^{r+t},$$

and there are infinitely many choices of r and n for which

$$k \cdot b^n + 1 = c^{e+fn} + 1 = c^{e'2^{r+t}} + 1 = F_{r+t}(c^{e'}).$$

□

3. N -COVERS: COVERS AND THEOREMS

The use of covers was introduced by Paul Erdős in 1950 [13] for the related form $2^n + p$ (to disprove the de Polignac conjecture). Notice that I use the term cover to describe the set of primes rather than a set of congruences (modulo those primes).

Definition 3.1. A cover for the sequence $k \cdot b^n + 1$ ($n > 0$) is a finite set of primes $S = \{p_1, p_2, \dots, p_m\}$ for which each element of the sequence is divisible by a prime in S . We ask that covers be minimal in the sense that no subset of S will also cover the sequence. S is called an N -cover if N is the least positive integer for which each prime p in S divides $k \cdot b^n + 1$ if and only if p divides $k \cdot b^{n+N} + 1$. We will call this integer N the **period of the cover** S . Finally, we will say that the **base b has an N -cover** if there is an integer k for which $k \cdot b^n + 1$ has a non-trivial N -cover ($N > 1$).

Erdős apparently believed that all Sierpiński numbers arise from covers [16, Section F13]. That is probably not the case [14, 19]. In section 5 we will show that not all b -Sierpiński numbers appear to arise from covers. In practice, though, most small examples do come from covers.

There are two basic ways of constructing covers: Sierpiński's approach of using the Fermat numbers (generalized Fermat numbers in our case) and Selfridge's use of factors of $b^n - 1$. We begin with the latter.

Theorem 3.2. *Every element of an N -cover S of $k \cdot b^n + 1$ divides $b^N - 1$.*

Proof. Choose $p \in S$. This p must divide $k \cdot b^n + 1$ for some $n \leq N$. It then also divides $k \cdot b^{n+N} + 1$, so divides their difference $k \cdot b^n(b^N - 1)$. Since p does not divide $k \cdot b^n$, this completes the proof. \square

For example, Selfridge's Sierpiński 78557 arises from the cover $\{3, 5, 7, 13, 19, 37, 73\}$. Each prime of this cover divides $2^{36} - 1$, so to prove 78557 is a Sierpiński number base 2, it is sufficient to show that each of the first 36 terms in the sequence $78557 \cdot 2^n + 1$ ($n > 0$) are divisible by one of these seven primes.

The previous theorem also tells us that for every element p of an N -cover of $k \cdot b^n + 1$, $\text{ord}_p(b)$ divides N . It is easy to show $N = \text{lcm}_{p \in S}(\text{ord}_p(b))$.

Unless we say otherwise, in the rest of this article " N -cover" will mean non-trivial N cover, that is $N > 1$. Note that if S is an N -cover for one k, b pair, then it is also a cover for infinitely many other multipliers k and bases b .

Theorem 3.3. *An N -cover S of $k \cdot b^n + 1$ is also a cover of $K \cdot B^n + 1$ for all integers $K \equiv k, B \equiv b \pmod{P}$ where P is the product of the primes in the cover S .*

It follows by Dirichlet's theorem that there are infinitely many prime multipliers, and infinitely many prime bases covered by any given N -cover.

In what follows it will be helpful to recall the cyclotomic polynomials $\Phi_n(x)$. These are defined by

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)} \quad \text{and so} \quad x^n - 1 = \prod_{d|n} \Phi_d(x).$$

This makes $\Phi_n(x)$ the "primitive part" of $x^n - 1$ when factoring, and the $\phi(n)$ zeros of $\Phi_n(x)$ are the primitive n^{th} roots of unity. For integers n and b greater than one, if a prime p divides $\Phi_n(b)$ but not n , then $p \equiv 1 \pmod{n}$.

Theorem 3.2 can now be greatly sharpened for more specific values of N .

Theorem 3.4. *Let p be a prime number. The sequence base b has a p -cover S if and only if $\Phi_p(b)$ has at least p distinct prime divisors greater than p .*

Proof. Suppose first S is a p -cover of $k \cdot b^n + 1$. So there is an element of S which divides each element of $T = \{k \cdot b^1 + 1, k \cdot b^2 + 1, \dots, k \cdot b^p + 1\}$. If q is an element of S , the $\text{ord}_q(b)$ must divide the period of S , which is p . This order can not be one, or $\{q\}$ would be a trivial cover, so $\text{ord}_q(b) = p$ and therefore $p | q - 1$. This means q can only divide one element of T , hence there are at least p primes in S . Finally, these primes do not divide $b - 1$, so they each divide $(b^p - 1)/(b - 1) = \Phi_p(b)$.

On the other hand, if $\Phi_p(b)$ has the p distinct prime divisors: $q_1, q_2, q_3, \dots, q_p$, each greater than p , then none divide $b - 1$ so we can use the Chinese Remainder Theorem to show they form a p -cover by solving the system of linear equations

$$\begin{aligned} k \cdot b^1 + 1 &\equiv 0 \pmod{q_1} \\ k \cdot b^2 + 1 &\equiv 0 \pmod{q_2} \\ &\vdots \\ k \cdot b^p + 1 &\equiv 0 \pmod{q_p} \end{aligned}$$

for the multiplier k . \square

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For example, the base b has a 2-cover if and only if $b + 1$ has at least two distinct odd prime divisors. Examples of this include $b = 14, 20, 29, 32, 34, 38, 41, 44, 50, 54, 56, 59, 62, 64, 65, 68, 69, 74, 76, 77, 83, 84, 86, 89, 90, 92, 94, 98 \dots$ For all of these listed bases except 68 and 86, we have proven the 2-cover generates the least generalized Sierpiński number base b (see Table 3). Bowen [5] also used 2-covers to address the bases $b = 2^s + 1$ where $s \neq 2^m + 1$ and $s > 5$.

The following theorem shows that by using 2-covers we may completely solve the Sierpiński problem for infinitely many bases.

Theorem 3.5. *There are infinitely many bases b for which 4 is the least b -Sierpiński number.*

Proof. There are infinitely many primes of the form $p = 30m + 29$ by Dirichlet's theorem on primes in arithmetic progressions. Also the number of such primes below x is asymptotic to $\frac{x}{81 \log x}$, asymptotically much higher than the number of powers of 2 below x . So there are infinitely many choices of m where $p = 30m + 29$ is prime, and $b = 15m + 14$ is not a power of 2.

Since $\{3, 5\}$ covers $4 \cdot 14^n + 1$, it also covers $4 \cdot b^n + 1$ (for any such $b = b(m)$). Note $\gcd(b - 1, k + 1) = \gcd(15m + 13, 5) = 1$, and by our choice of b , 4 is not a rational power of b . So 4 is a b -Sierpiński number for all such choices of m .

To see it is the least b -Sierpiński number, note that first our definition of b -Sierpiński rules out $k = 1$. We can not have $k = 2$, as the first term in the sequence $2b^n + 1$ is $2b + 1 = 30m + 29$, the prime p . Finally, $k = 3$ is not possible as then $3b^n + 1$ has the trivial cover $\{2\}$. \square

The base b has a 3-cover if and only if $\Phi_3(b) = b^2 + b + 1$ has at least three distinct divisors greater than 3. The first such bases are $b = 74, 81, 87, 100, 102, 107, 121, \text{ and } 135$. Of those bases $b \leq 100$, only for 100 does the 3-cover yield the least generalized Sierpiński number. Bases 74, 81, and 87 have 3-covers, but these produce larger multipliers k than can be generated by other methods.

The minimal base for longer prime period covers grows quickly: $(p, \text{minimal base } b) = (2, 14), (3, 74), (5, 339), (7, 2601), (11, 32400), \text{ and } (13, 212574)$.

The structure of composite period covers are more interesting. For example, 4-covers usually arise from an odd prime factor p for which the base b has order 2 (a divisor of $\Phi_2(b) = b + 1$), and two primes q_1, q_2 for which b has order 4 (divisors of $\Phi_4(b) = b^2 + 1$). Then the terms of the sequence $k \cdot b^n + 1$ ($n = 1, 2, \dots$) are divisible by the primes of the 4-cover in a pattern like

$$\underline{p, q_1, p, q_2}, \underline{p, q_1, p, q_2}, \dots$$

So one choice of k could be found by solving the following system.

$$\begin{aligned} k \cdot b^1 + 1 &\equiv 0 \pmod{p} \\ k \cdot b^2 + 1 &\equiv 0 \pmod{q_1} \\ k \cdot b^4 + 1 &\equiv 0 \pmod{q_2} \end{aligned}$$

For 29 of the bases in Table 3, 4-covers provide the least known b -Sierpiński numbers.

Most 6-covers involve four primes. Often one prime in the cover, say p , has period 2 ($\text{ord}_p(b) = 2$), and there are three more of orders 3 or 6, say q_1, q_2, q_3 , dividing the terms of $k \cdot b^n + 1$ in a pattern similar to

$$\underline{p, q_1, p, q_2, p, q_3}, \underline{p, q_1, p, q_2, p, q_3}, \dots$$

Most bases have a 12-cover. One way one of these can arise is if each of $\Phi_2(b), \Phi_3(b), \Phi_4(b), \Phi_6(b)$ and $\Phi_{12}(b)$ have a primitive divisor, call them p_2, p_3, p_4, p_6 and p_{12} respectively. Then by solving the following system for k

$$\begin{aligned} k \cdot b^1 + 1 &\equiv 0 \pmod{p_2} \\ k \cdot b^2 + 1 &\equiv 0 \pmod{p_3} \\ k \cdot b^4 + 1 &\equiv 0 \pmod{p_4} \\ k \cdot b^6 + 1 &\equiv 0 \pmod{p_6} \\ k \cdot b^{10} + 1 &\equiv 0 \pmod{p_{12}} \end{aligned}$$

we have the divisibility pattern

$$\underline{p_2, p_3, p_2, p_4, p_2, p_6, p_2, p_3, p_2, p_{12}, p_2, p_4}, \dots$$

Many other such patterns are possible with these five primes, but this one is sufficient to prove the following.

Theorem 3.6. *Every base $b > 2$ which is not a Mersenne number has a 12-cover.*

The proof follows immediately from the congruences above and Bang's result [3] that $b^N - 1$ has a primitive divisor except when $N = 2$ and b is a Mersenne number ($2^n - 1$, n a positive integer); or $N = 6$ and $b = 2$.

We can also apply Bang's theorem to the Mersenne numbers by using a 144-cover as follows. Choose a primitive divisor p of $\Phi_n(b)$ and then solve the system of congruences $k \cdot b^e + 1 \equiv 0 \pmod{p}$ for each of the pairs $(n, e) = (3, 1), (4, 2), (6, 3), (8, 5), (9, 8), (12, 12), (16, 20), (18, 11), (24, 32), (36, 23), (48, 92)$, and $(72, 41)$. Similar systems are easily found for bases such as 120 and 180, but these require more primes. With Dirichlet's Theorem we now have the following.

Theorem 3.7. *There are infinitely many prime generalized Sierpiński numbers for every base b .*

Finally, when searching for possible covers the following results can be very useful.

Theorem 3.8. *If S is an N -cover of $k \cdot b^n + 1$, then $\sum_{p \in S} \frac{1}{\text{ord}_p(b)} \geq 1$*

Theorem 3.9. *If there is a non-trivial cover for $k \cdot b^n + 1$, then $k + 1$ has an odd prime divisor.*

The first of these was used by Stanton [31] in his analysis of possible covers for the $b = 2$ case.

4. A SIMPLE PROGRAM AND KNOWN RESULTS

Theorem 3.2 can be turned into a surprisingly effective program to find N -covers by looping on k until one is found for which $\gcd(k \cdot b^n + 1, b^N - 1) > 1$ for each of $k = 1, 2, \dots, N$. If this is done with a fairly large round value of N , such as 5040, then most small covers with relatively small k (say less than 10^8) will be easily spotted.

Daniel Adler, at the time a student at University of Tennessee at Martin, was enlisted to write a program *Sierpiński* in C++ using the multiprecision package GMP¹. When the program finds an N -cover, it outputs k and a vector of length N where the i^{th} component is $\gcd(k \cdot b^i + 1, b^N - 1)$ ($1 \leq i \leq N$). From this it was a simple hand calculation to find the actual covering set of primes.

This program was run on the 16 nodes of our Beowulf cluster for about 80-CPU days to find the constants k and the associated covers in the first columns of Table 3 except for $b = 3, 7$ and 15. Some individual values (e.g., 71), required substantially longer search times.

The program *Sierpiński* has several limitations. First, one must know something of N in advance because the program is set up to seek all covers with period N dividing a specified constant. For Table 3 we usually sought periods dividing $7! = 5040$. It is possible that we missed some covers for smaller k values.

Second, the program *Sierpiński* only seeks values of k belonging to covers. Such k values are Sierpiński numbers base b , but there may be smaller b -Sierpiński numbers that do not arise from covers. We will discuss this in the next section.

Finally, the program *Sierpiński* is too slow to find the least covers for bases like 3. For those bases we may begin by factoring $b^N - 1$ for various small values of N and construct covers as described in the previous section. For example, Brennen [7] used this method to find 3574321403229074 (48-cover) for $b = 3$. (This improved earlier results of Bowen [5] and Saouter [27].) With an improved algorithm Bosma [6] reduced this to $k = 125050976086$ (144-cover).

5. POLYNOMIAL FACTORIZATION AND PARTIAL FACTORIZATION

Another way that generalized Sierpiński numbers can arise is through factorization as polynomials. For example, when $b = 27$ and $k = 8$, each term factors as a difference of cubes:

$$8 \cdot 27^n + 1 = (2 \cdot 3^n + 1)(4 \cdot 3^{2n} - 2 \cdot 3^n + 1).$$

Similarly 8 is b^3 -Sierpiński number for all positive multiples of 3. Such b -Sierpiński numbers arise whenever b is a perfect cube.

Consider also the factorization $4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1)$. Anytime b is fourth power and the multiplier k is 4 times a fourth power, every term of the sequence $k \cdot b^n + 1$ will all factor in this

¹<http://gmp11b.org/>

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manner. Small examples for which this factorization generates the least known generalized Sierpiński numbers base b include $(k, b) = (2500, 16)$, and $(2500, 81)$.

The cases where the least Sierpiński numbers arise by polynomial factorization are marked by $\#$ in Table 3.

A final possibility is a “partial factorization,” where part of the sequence is covered by a set of primes, and the remainder of the terms factor as above. For example, the least known b -Sierpiński number for base $b = 63$, $k = 3511808$, comes from the partial 3-cover $\{37, 109\}$ (which divide $3511808 \cdot 63^n + 1$ when $n \equiv 1, 2 \pmod{3}$) and the factorable $x^3 + 1$ (for $n \equiv 0 \pmod{3}$). This was discussed for the usual base 2 Sierpiński numbers by Izotov [19] (see also [14]).

Another example is $k \cdot 2070^n + 1$ whose least base b Sierpiński appears to be 324. Here $324 \cdot 2070^n + 1$ factors as $4x^4 + 1$ when $n \equiv 0 \pmod{4}$, and then values of $n \not\equiv 0 \pmod{4}$ are covered by $\{17, 19\}$. To prove 324 is the least Sierpiński base 2070, we must find a prime for each of the following values of k : 77, 96, 132, 153, and 305. All others are known to generate primes.

The generalized Fermat numbers base b allow neither factorizations nor finite covers, yet it seems very likely that there are bases b such that all $F_n(b)$ ($n \geq 0$) are composite. These have been excluded by our definition, but we see no obvious reason that there could not be examples of b -Sierpiński numbers that have neither covers nor factorizations.

6. CHECKING THE RESULTS

The conjectured minimal values in Table 3 were compared against the published results [4, 30] and against the results of Robert Gerbicz’s program which finds covers very quickly².

To prove the multipliers k conjectured in Table 3 are the least b -Sierpiński numbers is “simple:” just find a prime of the form $K \cdot b^n + 1$ ($n > 0$) for each potential $K < k$. Though conceptually trivial, the amount of effort this can take may be truly massive! This is shown by the original case $b = 2$, still unsettled after 45 years, and still one of the larger distributive computing projects: Seventeen or Bust [18]. The largest prime that they have had to find so far to eliminate a k value was $19249 \cdot 2^{13018586} + 1$ with 3,918,990 digits. They estimate they may need to search to an exponent of $n = 3,400,000,000,000$ just to get a 50% chance of finishing of the remaining cases [17].

To eliminate these K values, we began with a Maple program. We then used `OpenPFGW` [24] for anything larger than a dozen digits. This was done in two passes: the first to trial factor by small primes and perform a probable primality test (this took about five CPU years). Second we reran `OpenPFGW` on our list of probable primes to provide classical $n \pm 1$ primality proofs [8].

We compared our results against all published sources that we could find, especially [4, 30]. For many of the smaller bases, Barnes [4] has results from more extensive searches than ours—so we include those results in Table 3 also. When comparing tables it is necessary to be sensitive of the variety of different definitions of generalized Sierpiński numbers being used.

7. CONCLUSIONS

Of the many possible generalizations of the Sierpiński numbers, we have discussed what seemed the most natural to us. It would be interesting, but difficult, to study the generalized Fermat cases that we excluded in our definition. By Dickson’s conjecture, it seems likely that bases b can be found so that the least Sierpiński number is arbitrarily large. One can also ask the reverse question: given a value k , can we find a base b for which k is a base b Sierpiński? A partial answer has been provided by one of the authors [23].

Note that every cover of a sequence of the form $k \cdot b^n + 1$ ($n > 0$) is also a cover of a sequence $k' \cdot b^n - 1$ ($n > 0$), and vice versa. Positive odd integers k for which $k \cdot b^n - 1$ are composite for all $n > 0$ are called Riesel numbers after an article by Riesel [26] in 1956 (so the Riesel numbers predate the Sierpiński numbers). Thus another generalization to study would be the generalized Riesel numbers (k which make $k \cdot b^n - 1$ composite for all $n > 0$ with suitable restrictions on k and b); as well as the numbers that are both b -Riesels and b -Sierpińskis. Part of this work is being done informally by Barnes and others [4], as are restrictive cases like seeking the smallest b -Sierpiński numbers which are prime.

A. de Polignac conjectured (and quickly retracted) the guess that every positive odd number can be written in the form $2^n + p$ for a prime p and integer $n > 0$. He did this even though Euler had previously

²<http://robert.gerbicz.googlepages.com/coveringsets>

shown that this was not the case for 127 or 929 [1]. Again every cover of $k \cdot b^n + 1$ ($n > 0$) is also a cover of a sequence $b^n + k$ ($n > 0$), and vice versa.

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GENERALIZING SIERPIŃSKI NUMBERS TO BASE b Table 3: Conjectured Least Sierpiński Numbers k base b

b	N	k {cover}	k 's not yet eliminated	ref
8	4	47 {3, 5, 13}	proven	
9	6	2344 {5, 7, 13, 73}	{2036}	
10	6	9175 {7, 11, 13, 73}	{7666}	[4]
11	6	1490 {3, 7, 19, 37}	proven	[4]
12	4	521 {5, 13, 29}	{404}	
13	4	132 {5, 7, 17}	proven	
14	2	4 {3, 5}	proven	
15	24	91218919470156 {13, 17, 113, 211, 241, 1489, 3877}	{114258, 148458, 215432, 405556, 424074, ...}	
16	#	2500	proven	
17	4	278 {3, 5, 29}	{244}	[4]
18	4	398 {5, 13, 19}	{122}	
19	12	765174 {5, 7, 13, 127, 769}	{634, 1446, 2526, 2716, 3714, 4506, ...}	
20	2	8 {3, 7}	proven	
21	4	1002 {11, 13, 17}	proven	
22	4	6694 {5, 23, 97}	{1611, 1908, 4233, 5128}	[4]
23	4	182 {3, 5, 53}	{8, 68}	
24	12	30651 {5, 7, 13, 73, 79}	{319, 621, 656, 821, 1099, 1851, 1864, 2164, 2351, 2586, 3031, 3051, 3404, 3526, ...}	
25	6	262638 {7, 13, 31, 601}	{222, 5550, 6082, 6436, 7528, 8644, 10218, 10918, 12864, 12988, 13026, 13548, ...}	
26	6	221 {3, 7, 19, 37}	{32, 65, 155}	
27	#	8	proven	
28	4	4554 {5, 29, 157}	{871, 2377, 3394, 4233, 4552}	
29	2	4 {3, 5}	proven	
30	6	867 {7, 13, 19, 31}	{278, 588}	
31	12	6360528 {7, 13, 19, 37, 331}	{10366, 13240, 69120, 70612, 76848, 99450, 101980, 122806, 124812, ...}	
32	2	10 {3, 11}	proven	
33	4	1854 {5, 17, 109}	{766, 1678, 1818}	
34	2	6 {5, 7}	proven	
35	6	214018 {3, 13, 97, 397}	{46, 1610, 2006, 2272, 2588, 3046, 3700, 3812, 5518, 8632, 8800, 9542, 10222, ...}	
36	6	1886 {13, 31, 37, 43}	proven	
37	4	2604 {5, 19, 137}	{94, 1272, 1866, 2224}	
38	2	14 {3, 13}	proven	
39	6	166134 {5, 7, 223, 1483}	{2264, 2414, 2434, 3254, 3986, 4226, ...}	
40	6	826477 {7, 41, 223, 547}	{4468, 7092, 9964, 11112, 18285, ...}	
41	2	8 {3, 7}	proven	
42	4	13372 {5, 43, 353}	{116, 988, 1117, 1421, 2794, 2903, 3046, 3226, 3897, 4127, 4297, 4643, ...}	
43	4	2256 {5, 11, 37}	{166, 648}	
44	2	4 {3, 5}	proven	
45	6	53474 {7, 19, 23, 109}	{474, 1908, 2444, 3106, 4530, 4990, 6510, 6586, 6624, 7108, 8026, 9774, ...}	
46	6	14992 {7, 19, 47, 103}	{892, 976, 1132, 1798, 3261, 3477, 3961, 4842, 5395, 6015, 6391, 6816, ...}	
47	4	8 {3, 5, 13}	proven	
48	6	1219 {7, 13, 61, 181}	{29, 36, 62, 153, 422, 1174}	
49	12	2944 {5, 19, 73, 181, 193}	{1134, 1414, 1456, 2694, 2746}	

† partial factorization, # factorization

Table 3: Conjectured Least Sierpiński Numbers k base b – Continued

b	N	k {cover}	k 's not yet eliminated	ref
50	2	16 {3, 17}	proven	
51	6	5183582 {7, 13, 379, 2551}	{5498, 6280, 6696, 7682, 8126, 8412, ...}	
52	4	28674 {5, 53, 541}	{1483, 1591, 2386, 3181, 3232, 3418, 5619, 5776, 5988, 6147, 6891, 7147, 8638, ...}	
53	4	1966 {3, 5, 281}	{1816, 1838, 1862, 1892}	
54	2	21 {5, 11}	proven	
55	4‡	2500 {7, 17}	{1980, 2274}	
56	2	20 {3, 19}	proven	
57	4	1188 {5, 13, 29}	{378}	
58	4	43071 {5, 59, 673}	{222, 787, 886, 1102, 1923, 2182, 2656, 2713, 3246, 3511, 3541, 4021, 5274, 6046, ...}	
59	2	4 {3, 5}	proven	
60	4	16957 {13, 61, 277}	{853, 1646, 2075, 2497, 4025, 4406, 4441, 5064, 5767, 6772, 7262, 7931, 8923, ...}	
61	6	15168 {7, 13, 31, 97}	{1570, 1642, 3390, 3442, 3936, 6852, 7348, 8710, 8772, 8902, 9208, 9268, 9952, ...}	
62	2	8 {3, 7}	proven	
63	3‡	3511808 {37, 109}	{3092, 3230, 4106, 7622, ...}	
64	2	51 {5, 13}	proven	
65	2	10 {3, 11}	proven	
66	24	21314443 {7, 17, 37, 67, 73, 4357}	{470, 2076, 4153, 5442, 6835, 13201, 17035, ...}	
67	4	18342 {5, 17, 449}	{154, 460, 1494, 2196, 2362, 2806, 2872, 2874, 3384, 4062, 4618, 4996, 5668, ...}	
68	2	22 {3, 23}	{12, 17}	
69	2	6 {5, 7}	proven	
70	4	11077 {13, 29, 71}	{3762, 4119, 5608, 9231, 10438}	
71	18	5917678826 {3, 19, 37, 73, 1657, 5113}	{172, 502, 508, 1942, 2782, 3776, 4490, 5002, 5078, 5266, 5330, 5632, 5950, 6338, ...}	
72	4	731 {5, 61, 73}	{493, 647}	
73	4	1444 {5, 13, 37}	{778, 1344}	
74	2	4 {3, 5}	proven	
75	6	4086 {7, 13, 19, 61}	{2336, 2564, 3782}	
76	2	43 {7, 11}	proven	
77	2	14 {3, 13}	proven	
78	4	186123 {5, 79, 1217}	{2371, 4820, 4897, 5294, 5531, 6353, ...}	
79	6	2212516 {5, 7, 43, 6163}	{24, 594, 724, 1086, 1654, 1774, 1896, ...}	
80	12	1039 {3, 7, 13, 43, 173}	{86, 92, 166, 188, 295, 326, 370, 433, 472, 556, 623, 628, 692, 770, 778, 787, 818, 857, 968}	
81	#	2500	{558, 1650, 2036, 2182, 2350, 2378}	
82	12	19587 {5, 7, 13, 37, 83}	{1251, 1327, 1570, 1716, 1798, 1908, 2251, 2352, 2461, 2491, 2731, 2989, 3342, ...}	
83	2	8 {3, 7}	proven	
84	2	16 {5, 17}	proven	
85	6	346334170 {37, 43, 193, 2437}	{7612, 11740, 27168, 31776, 32550, 34014, 35088, 36508, 43474, 48204, 50352, ...}	
86	2	28 {3, 29}	{8}	
87	6	274 {7, 11, 19, 31}	{32}	
88	12	4093 {5, 7, 31, 37, 89}	{192, 244, 958, 978, 1452, 1585, 1678, 1779, 2007, 2617, 2838, 3396, ...}	
89	2	4 {3, 5}	proven	
90	2	27 {7, 13}	proven	

‡ partial factorization,

factorization

GENERALIZING SIERPIŃSKI NUMBERS TO BASE b Table 3: Conjectured Least Sierpiński Numbers k base b – Continued

b	N	k {cover}	k 's not yet eliminated	ref
91	4	89586 {23, 41, 101}	{252, 1678, 2008, 6970, 8902, 11706, 12306, 14236, 22932, 23520, 26472, 29488, ...}	
92	2	32 {3, 31}	proven	
93	4	24394 {5, 47, 173}	{62, 306, 706, 866, 894, 902, 1652, 2208, 2678, 3218, 3244, 3384, 3750, 3996, ...}	
94	2	39 {5, 19}	proven	
95	6	41354 {3, 7, 13, 229}	{244, 376, 692, 790, 848, 908, 926, 1004, 1012, 1024, 1096, 1312, 1396, 1662, ...}	
96	4	353081 {13, 97, 709}	{1262, 2952, 3028, 4461, ...}	
97	4	15996 {5, 7, 941}	{120, 202, 538, 666, 736, 762, 1042, 1044, 1098, 1114, 1156, 1252, 1308, 1518, ...}	
98	2	10 {3, 11}	proven	
99	4	684 {5, 13, 29}	{284}	
100	3	2469 {7, 13, 37}	{64, 433, 684, 922, 2145}	

‡ partial factorization, † factorization

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