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Certain double series of Euler type and of Eisenstein type and Hurwitz numbers

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1. Introduction

Special values of the Lerch zeta function for $j \geq 2$ are given by

$$\sum_{n\in\mathbf{Z}\setminus\{0\}} \frac{e^{2\pi i n z}}{(n\tau)^j} = -\frac{\mathcal{B}_j(z;\tau)}{j!}, \tag{1.1}$$

where $B_j(z)$ is the ordinary jth Bernoulli polynomial and

$$\mathfrak{B}(\xi) = \frac{2\pi i}{\tau}\frac{e^{2\pi i\xi z/\tau}}{e^{2\pi i\xi/\tau}-1} = \sum_{j=0}^{\infty}\frac{\mathcal{B}_j(z;\tau)}{j!}\xi^{j-1}, \tag{1.2}$$

$$\mathcal{B}_j(z;\tau) = \left(\frac{2\pi i}{\tau}\right)^j B_j(z), \tag{1.3}$$

(see [2] for details). An analogous formula for Eisenstein series is also considered by various authors such as Hurwitz [10], Kronecker (Weil [28, Chapter 8]), Herglotz [9], Katayama [12], which reads for $j \geq 3$

$$\sum_{(m,n)\in\mathbf{Z}^2\setminus\{(0,0)\}} \frac{e^{2\pi i(mx+ny)}}{(m+n\tau)^j} = -\frac{\mathcal{H}_j(x,y;\tau)}{j!}, \tag{1.4}$$

where $\theta(x;\tau)$ is the Jacobi theta function and $\mathcal{H}_j(x,y;\tau)$ is given by

$$\mathfrak{F}(\xi) = e^{2\pi i x\xi}\frac{\theta'(0;\tau)\theta(\xi+x\tau-y;\tau)}{\theta(\xi;\tau)\theta(x\tau-y;\tau)} = \sum_{j=0}^{\infty}\frac{\mathcal{H}_j(x,y;\tau)}{j!}\xi^{j-1}. \tag{1.5}$$

In particular when $\tau = i$, we have

$$\mathcal{H}_{4j}(0,0;i) = -(2\varpi)^{4j}H_{4j},$$

where $\{H_{4k} \in \mathbb{Q} \mid k \in \mathbb{N}\}$ are the Hurwitz numbers defined as coefficients of the Laurent series expansion of the Weierstrass $\wp$-function, namely,

$$
\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\substack{\lambda \in \mathbb{Z}\varpi + \mathbb{Z}\varpi i \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \\
&= \frac{1}{z^2} + \sum_{m=1}^{\infty} \frac{2^{4m} H_{4m}}{4m} \frac{z^{4m-2}}{(4m-2)!},
\end{aligned} \tag{1.6}
$$

and

$$
\varpi = 2 \int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.622057\cdots \tag{1.7}
$$

is the lemniscate constant (see, for example, [18]). From the property of $\wp(z)$, we can see that

$$
H_4 = \frac{1}{10},\ H_8 = \frac{3}{10},\ H_{12} = \frac{567}{130},\ H_{16} = \frac{43659}{170},\ H_{20} = \frac{392931}{10}, \ldots \tag{1.8}
$$

As other analogous formulas of (1.1), Cauchy [7], Mellin [22, 23], Ramanujan, Berndt [4, 5, 6] gave

$$
\sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^m}{\sinh(m\pi) m^{4k+3}} = (2\pi)^{4k+3} \sum_{j=0}^{2k+2} (-1)^{j+1} \frac{B_{2j}(1/2)}{(2j)!} \frac{B_{4k+4-2j}(1/2)}{(4k+4-2j)!}, \tag{1.9}
$$

$$
\sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{\coth(m\pi)}{m^{4k+3}} = (2\pi)^{4k+3} \sum_{j=0}^{2k+2} (-1)^{j+1} \frac{B_{2j}(0)}{(2j)!} \frac{B_{4k+4-2j}(0)}{(4k+4-2j)!} \tag{1.10}
$$

for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\coth x = \cosh x / \sinh x$.

As certain double analogues of these results, the third-named author proved

$$
\sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ m \neq 0}} \frac{(-1)^n}{\sinh(m\pi)(m+ni)^{2k+1}} \in \frac{1}{\pi} \mathbb{Q}[\pi, \varpi^4] \quad (k \in \mathbb{N}_0) \tag{1.11}
$$

(see [26]) and

$$
\sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ m \neq 0}} \frac{\coth(m\pi)}{(m+ni)^{2k+1}} \in \frac{1}{\pi} \mathbb{Q}[\pi, \varpi^4] \quad (k \in \mathbb{N}) \tag{1.12}
$$

(see [27]). Actually he gave some explicit formulas for these values (for example, see Section 4).

Our aim in the former part is to study their principles and consider generalized double series, for example,

$$
\sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ m \neq 0}} \frac{(-1)^{\ell n}}{(\sinh(m\pi))^{\ell} (m+ni)^{j}}, \tag{1.13}
$$

(see Section 2). Furthermore we evaluate special values of the q-zeta function defined by

$$\zeta_q(s,t)=\sum_{m=1}^{\infty}\frac{q^{mt}}{[m]^s}, \tag{1.14}$$

where

$$[x]=\frac{1-q^x}{1-q}$$

(see Section 3). As explicit examples, we give evaluation formulas for (1.13) and for (1.14) (see Section 4). Note that our special values of the q-zeta function are at positive integers while those at negative integers are studied by Kaneko, Kawagoe, Kurokawa, Wakayama, Yamasaki [11, 14] and the third-named author [25].

In the latter part of this paper, we consider the double series of Eisenstein type defined by

$$\zeta_2(s_1,s_2;\omega_1,\omega_2)=\sum_{m=1}^{\infty}\frac{1}{(m\omega_1)^{s_1}}\sum_{n=1}^{\infty}\frac{1}{(m\omega_1+n\omega_2)^{s_2}} \tag{1.15}$$

for $(s_1,s_2)\in\mathbb{C}^2$ with $\Re(s_1+s_2)>2$ and $\Re s_2>1$. In particular when $\omega_1=\omega_2=1$, this coincides with the double zeta-function of Euler type. In [21], the second-named author proved a certain functional equation for the double zeta-function of Euler type. As a generalization of this fact, we prove a functional equation for (1.15) of reflection-type (see Theorem 6.2), which can be regarded as a double analogue of the functional equation of the Riemann zeta-function $\zeta(s)$. Furthermore, we give some functional relations for certain double series involving hyperbolic functions (see Theorem 8.1). In particular, we can see that these functional relations include value-relations for double series in (1.11).

Finally we give some functional equations for double L-functions of Euler type (see Theorem 9.2). These can be regarded as χ-analogues of those for double zeta functions of Euler type.

2. General principle and double series

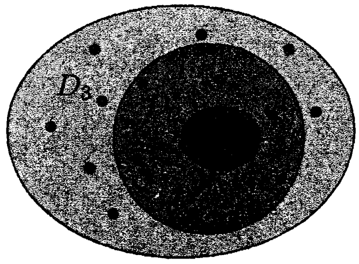
In this section, we give a very simple general principle to calculate special values of a certain family of series which includes those appearing in the introduction. The following can be easily shown by use of the Cauchy integral formula.

Lemma 2.1. *Let $f(\xi)$ be a meromorphic function. Assume that there exists a family of compact sets $\{D_N\}_{N=1}^{\infty}$ such that*

(1) *∂D_N is piecewise smooth and $\displaystyle\lim_{N\to\infty}\int_{\partial D_N} f(\xi)d\xi=0$,*

(2) *D_N contains D_{N-1}.*

Then
$$\lim_{N\to\infty}\sum_{c\in \mathring{D}_N}\operatorname*{Res}_{\xi=c} f(\xi)=0.$$
In particular, if the sum is absolutely convergent,
$$\sum_{c\in\bigcup_{N=1}^{\infty}\mathring{D}_N}\operatorname*{Res}_{\xi=c} f(\xi)=0.$$



As applications of this lemma we give special values appearing in the introduction.

Example 2.2. We show (1.1). The function

$$\mathfrak{G}(\xi)=\frac{2\pi i}{\tau}\frac{e^{2\pi i\xi z/\tau}}{e^{2\pi i\xi/\tau}-1} \tag{2.1}$$

has simple poles on $\tau\mathbb{Z}$ with $\operatorname{Res}_{\xi=n\tau}\mathfrak{G}(\xi)=e^{2\pi i n z}$ and the periodicity

$$\mathfrak{G}(\xi+\tau)=\mathfrak{G}(\xi)e^{2\pi i z}. \tag{2.2}$$

Put $f(\xi)=\mathfrak{G}(\xi)\xi^{-j}$ for $j\geq 2$. Then Lemma 2.1 implies

$$\operatorname*{Res}_{\xi=0} f(\xi)+\sum_{n\in\mathbb{Z}\setminus\{0\}}\operatorname*{Res}_{\xi=n\tau} f(\xi)=0 \qquad\Rightarrow\qquad \frac{\mathcal{B}_j(z;\tau)}{j!}+\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{e^{2\pi i n z}}{(n\tau)^j}=0. \tag{2.3}$$

Example 2.3. We show (1.4). The function

$$\mathfrak{F}(\xi)=e^{2\pi i x\xi}\frac{\theta'(0;\tau)\theta(\xi+x\tau-y;\tau)}{\theta(\xi;\tau)\theta(x\tau-y;\tau)} \tag{2.4}$$

has simple poles on $\mathbb{Z}+\tau\mathbb{Z}$ with $\operatorname{Res}_{\xi=m+n\tau}\mathfrak{F}(\xi)=e^{2\pi i(mx+ny)}$ and the periodicities

$$\mathfrak{F}(\xi+1)=\mathfrak{F}(\xi)e^{2\pi i x},\qquad \mathfrak{F}(\xi+\tau)=\mathfrak{F}(\xi)e^{2\pi i y}. \tag{2.5}$$

Putting $f(\xi)=\mathfrak{F}(\xi)\xi^{-j}$ for $j\geq 3$ and applying Lemma 2.1 to this function, we obtain

$$\operatorname*{Res}_{\xi=0} f(\xi)+\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\operatorname*{Res}_{\xi=m+n\tau} f(\xi)=0 \quad\Rightarrow$$
$$\frac{\mathcal{H}_j(x,y;\tau)}{j!}+\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{e^{2\pi i(mx+ny)}}{(m+n\tau)^j}=0. \tag{2.6}$$

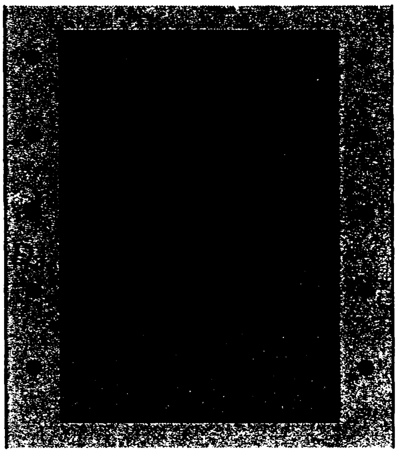
Example 2.4 (Ramanujan's formula [5, p. 275]). Let $f(\xi)=\dfrac{1}{e^{\alpha^{1/2}\xi}-1}\dfrac{1}{e^{i\beta^{1/2}\xi}-1}\xi^{-2j-1}$ with $\alpha\beta=\pi^2$. Then for $j\geq 1$, Lemma 2.1 implies the following famous Ramanujan's formula

$$\alpha^{-j}\left\{\frac{1}{2}\zeta(2j+1)+\sum_{k=1}^{\infty}\frac{1}{\left(e^{2k\alpha}-1\right)k^{2j+1}}\right\}$$
$$=(-\beta)^{-j}\left\{\frac{1}{2}\zeta(2j+1)+\sum_{k=1}^{\infty}\frac{1}{\left(e^{2k\beta}-1\right)k^{2j+1}}\right\}$$
$$-2^{2j}\sum_{k=0}^{j+1}(-1)^{k}\frac{B_{2k}(0)}{(2k)!}\frac{B_{2j+2-2k}(0)}{(2j+2-2k)!}\alpha^{j+1-k}\beta^{k}. \quad (2.7)$$

Example 2.5 (Cauchy, Mellin, Ramanujan, Berndt). For $j \geq 0$,

$$\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{(-1)^m}{\sinh(m\pi)m^{4j+3}}=(2\pi)^{4j+3}\sum_{k=0}^{2j+2}(-1)^{k+1}\frac{B_{2k}(1/2)}{(2k)!}\frac{B_{4j+4-2k}(1/2)}{(4j+4-2k)!}. \quad (2.8)$$

To give explicit evaluations of the series (1.13), we need to construct an appropriate generating function. By apply Lemma 2.1 to the function $f(\xi)=\mathfrak{F}(\xi)\mathfrak{G}(\xi)^{\ell}\xi^{-j}$, we find for $j \geq 3$



$$\begin{aligned}0&=\lim_{N\to\infty}\frac{1}{2\pi i}\int_{\partial D_N}f(\xi)d\xi\\&=\underbrace{\operatorname*{Res}_{\xi=0}f(\xi)}_{\substack{\text{coefficient of }\xi^{j-1}\\ \text{of }\mathfrak{F}(\xi)\mathfrak{G}(\xi)^{\ell}}}+\underbrace{\lim_{N\to\infty}\sum_{\substack{|m|,|n|<N\\(m,n)\neq(0,0)}}\operatorname*{Res}_{\xi=m+n\tau}f(\xi)}_{\text{wanted series}}\\&\quad+\underbrace{\sum_{n\in\mathbb{Z}\setminus\{0\}}\operatorname*{Res}_{\xi=n\tau}f(\xi)}_{\substack{\text{unwanted series}\\ \text{(Riemann zeta functions)}}}.\end{aligned} \quad (2.9)$$

By subtracting the generating function of the unwanted series we obtain the following theorem.

Theorem 2.6. *Assume* $(x,y)\in\mathbb{R}^2\setminus\mathbb{Z}^2$ *and* $0<z<1$. *Then*

$$\mathfrak{F}(\xi)\mathfrak{G}(\xi)^{\ell}-\operatorname*{Res}_{\eta=0}\left(\mathfrak{F}(\eta)\mathfrak{G}(\eta)^{\ell}\mathfrak{G}(\xi-\eta;\{y+\ell z\};\tau)\right)=\sum_{j=1}^{\infty}\frac{\mathcal{K}_{j,\ell}(x,y,z;\tau)}{j!}\xi^{j-1}, \quad (2.10)$$

where the residue is taken for $\xi\notin\tau\mathbb{Z}$. *For* $j\geq 3$,

$$-j!\lim_{N\to\infty}\sum_{\substack{|m|,|n|<N\\(m,n)\neq(0,0)}}\frac{\mathfrak{G}(m)^{\ell}e^{2\pi i(mx+ny+n\ell z)}}{(m+n\tau)^j}=\mathcal{K}_{j,\ell}(x,y,z;\tau), \quad (2.11)$$

which is also valid for $(x,y)\in\mathbb{R}^2$ *and* $0\leq z\leq 1$ *by taking the limit of* $\mathcal{K}_{j,\ell}(x,y,z;\tau)$ *appropriately.*

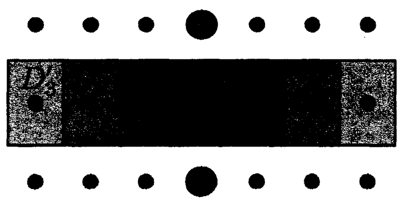
Remark 2.7. In the cases $j = 1, 2$, the above theorem also holds in some conditions. However the conditions are a little complicated and omitted here for simplicity. See [17] for details.

If $j = 0$, then the limit of the integrals (2.9) does not exist. However, we show that it is possible to give a series interpretation even in the case $j = 0$. We explain this fact in the following. Assume $y + \ell z \in \mathbb{Z}$. Then we have

$$\mathfrak{F}(\xi+\tau)\mathfrak{G}(\xi+\tau)^\ell = \mathfrak{F}(\xi)\mathfrak{G}(\xi)^\ell e^{2\pi i(y+\ell z)} = \mathfrak{F}(\xi)\mathfrak{G}(\xi)^\ell. \tag{2.12}$$

By this periodicity, we obtain

$$\begin{aligned} 0 &= \lim_{N\to\infty} \frac{1}{2\pi i}\int_{\partial D'_N} \mathfrak{F}(\xi)\mathfrak{G}(\xi)^\ell d\xi \\ &= \operatorname*{Res}_{\xi=0} \mathfrak{F}(\xi)\mathfrak{G}(\xi)^\ell + \sum_{m\in\mathbb{Z}\setminus\{0\}} \mathfrak{G}(m)^\ell e^{2\pi i m x}. \end{aligned}$$



Therefore, under this assumption, if we put

$$\mathfrak{F}(\xi)\mathfrak{G}(\xi)^\ell = \sum_{j=0}^{\infty} c_{j,\ell}\xi^{j-1}, \tag{2.13}$$

the coefficient $c_{j,\ell}$ is interpreted as a special value of a certain single series in the case $j = 0$ while it is interpreted as a special value of a certain double series in the case $j \geq 1$. Note that throughout this interpretation, ℓ is fixed.

In the next sections, we will change the interpretation and we will regard $\mathfrak{F}(\xi)$ itself as a generating function of all $c_{0,\ell}$, that is,

$$\mathfrak{F}(\xi) = \sum_{\ell=0}^{\infty} c_{0,\ell}(\mathfrak{G}(\xi)^{-1})'(\mathfrak{G}(\xi)^{-1})^{\ell-1}. \tag{2.14}$$

Furthermore we will show that the coefficients $c_{0,\ell}$ are special values of the q-zeta functions.

3. Generalized Laurent expansion and q-zeta function

First we give a general framework of expansions of the form (2.14). Let φ be a biholomorphic function between U and $\varphi(U)$ with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$ (i.e. a local coordinate).

Lemma 3.1. *Assume that f is a holomorphic function on $U \setminus \{0\}$. For $|z| < \varepsilon$, we have the uniformly absolutely convergent series*

$$f(z) = \sum_{\ell=-\infty}^{\infty} a_\ell \varphi'(z)\varphi(z)^\ell, \qquad a_\ell = \frac{1}{2\pi i}\int_{|\xi|=\varepsilon} \frac{f(\xi)}{\varphi(\xi)^{\ell+1}} d\xi. \tag{3.1}$$

We see that

$$\mathfrak{G}(\xi)^{-1} = \frac{\tau}{2\pi i}\frac{e^{2\pi i\xi/\tau}-1}{e^{2\pi i\xi z/\tau}} = 0 + \xi + O(\xi^2). \tag{3.2}$$

From this expression, we can apply Lemma 3.1 to the pair $f(\xi) = \mathfrak{F}(\xi)$, $\varphi(\xi) = \mathfrak{G}(\xi)^{-1}$ to get

$$\mathfrak{F}(\xi) = \sum_{\ell=-1}^{\infty} a_\ell (\mathfrak{G}(\xi)^{-1})' (\mathfrak{G}(\xi)^{-1})^{\ell}, \tag{3.3}$$

$$a_\ell = c_{0,\ell+1} = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \mathfrak{F}(\xi)\mathfrak{G}(\xi)^{\ell+1} d\xi. \tag{3.4}$$

As announced in the last part of the previous section, we regard the coefficient of ξ^{-1} in (2.13) as that of $(\mathfrak{G}(\xi)^{-1})'(\mathfrak{G}(\xi)^{-1})^{\ell}$ in (2.14).

Next we clarify a relation between these coefficients and special values of the q-zeta function. Let

$$[\ell]_q = \frac{1-q^\ell}{1-q}, \tag{3.5}$$

$$[\ell]_q! = [1]_q \cdots [\ell]_q, \tag{3.6}$$

$$(a;q)_k = \prod_{j=0}^{k-1} (1 - aq^j). \tag{3.7}$$

The following formula is known:

$$\lim_{q\uparrow 1} [\ell]_q! = \lim_{q\uparrow 1} \frac{(q;q)_\ell}{(1-q)^\ell} = \ell!. \tag{3.8}$$

Then the Lerch-type q-zeta function is defined as

$$\zeta_q(s,z;x) = \sum_{m=1}^{\infty} \frac{e^{2\pi i m x} q^{mz}}{[m]_q^s}. \tag{3.9}$$

For $\tau \in \mathbb{C}$ with $\Im\tau > 0$, put $q = e^{-2\pi i/\tau}$. Then we obtain the following.

Theorem 3.2. *Assume $(x, y) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ and $0 < z < 1$. We have*

$$\mathfrak{F}(\xi; x, y; \tau) = \sum_{\ell=0}^{\infty} \frac{\mathfrak{Q}_\ell(x,y,z;\tau)}{(q;q)_\ell} (2\pi i/\tau)^\ell (\mathfrak{G}(\xi;z;\tau)^{-1})' (\mathfrak{G}(\xi;z;\tau)^{-1})^{\ell-1}. \tag{3.10}$$

Assume $y + \ell z \in \mathbb{Z}$, $0 < z < 1$ and $x \in \mathbb{R}$. Then by taking the limit of $\mathfrak{Q}_\ell(x, y, z; \tau)$ appropriately, we have

$$-[\ell]_q! \big(\zeta_q(\ell, \ell(1-z); x) + (-1)^\ell \zeta_q(\ell, \ell z; -x)\big) = \mathfrak{Q}_\ell(x,y,z;\tau). \tag{3.11}$$

It should be noted that Theorem 3.2 is regarded as a q-analogue of the classical formulas between the Lerch zeta function and Bernoulli polynomials

$$f(\xi;x;\tau) = \mathfrak{G}(\xi;x;\tau) = \frac{2\pi i}{\tau} \frac{e^{2\pi i x\xi/\tau}}{e^{2\pi i \xi/\tau} - 1} = \sum_{\ell=0}^{\infty} \frac{(2\pi i/\tau)^\ell B_\ell(x)}{\ell!} \xi^{\ell-1}, \tag{3.12}$$

$$-\ell! (\phi(\ell;x) + (-1)^\ell \phi(\ell;-x)) = (2\pi i/\tau)^\ell B_\ell(x), \tag{3.13}$$

$$\phi(s;x) = \sum_{m=1}^{\infty} \frac{e^{2\pi i m x}}{(\tau m)^s}. \tag{3.14}$$

Similarly to the classical case ($\ell \in 2\mathbb{N}$)

$$\zeta(\ell) = -\frac{(2\pi i)^\ell B_\ell}{2 \cdot \ell!}, \tag{3.15}$$

we can derive a formula for special values of the q-zeta function, by setting $(x, y, z, \tau) = (0, 0, 1/2, i)$ in (3.11).

Corollary 3.3. *For* $\ell \in 2\mathbb{N}$,

$$\zeta_q(\ell, \ell/2; 0) = -\frac{Q_\ell(0, 0, 1/2; \tau)}{2[\ell]_q!}. \tag{3.16}$$

This value in the case $\tau = i$ can be described in terms of Hurwitz numbers and Bernoulli polynomials of higher order as follows.

Corollary 3.4. *For* $\ell \in 2\mathbb{N}$,

$$\zeta_{q^\ell}(\ell, 1/2; 0) = -\frac{(1-q)^\ell}{2}\left(\frac{B_\ell^{(\ell)}(1/2)}{\ell!} + \pi\frac{B_{\ell-2}^{(\ell)}(1/2)}{(2\pi)^2(\ell-2)!} - \sum_{\substack{j=4\\ j\in 4\mathbb{N}}}^{\ell} \frac{(2\varpi)^j H_j}{j!}\frac{B_{\ell-j}^{(\ell)}(1/2)}{(2\pi)^j(\ell-j)!}\right), \tag{3.17}$$

where H_j is the Hurwitz number defined by (1.6), *ϖ is the lemniscate constant defined by* (1.7), *and the Bernoulli polynomial of higher order $B_j^{(\ell)}(z)$ is defined by*

$$\left(\frac{\xi e^{z\xi}}{e^\xi - 1}\right)^\ell = \sum_{j=0}^{\infty} B_j^{(\ell)}(z)\frac{\xi^j}{j!}. \tag{3.18}$$

4. Examples

We consider the case when $(x, y, l, \tau) = (0, 0, 1, i)$ in Equation (2.11) of Theorem 2.6. When $j = 3$, we obtain

$$\begin{aligned}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ m\neq 0}} &\frac{(-1)^n}{\sinh(m\pi)}\frac{e^{2\pi(m+in)(z-1/2)}}{(m+ni)^3} \\ &= \left(\frac{2\pi^3}{3} - 2\pi^2\right)z^2 + \left(-\frac{2\pi^3}{3} + 2\pi^2\right)z + \frac{\varpi^4}{15\pi} + \frac{4}{45}\pi^3 - \frac{1}{3}\pi^2\end{aligned} \tag{4.1}$$

for $z \in \mathbb{R}$ with $0 \le z \le 1$. In particular, putting $z = 1/2$ in (4.1), we have

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ m\neq 0}} \frac{(-1)^n}{\sinh(m\pi)(m+ni)^3} = \frac{\varpi^4}{15\pi} - \frac{7}{90}\pi^3 + \frac{1}{6}\pi^2, \tag{4.2}$$

which was already given in [26]. Also, put $z = 0$ in (4.1) and replace (m, n) by $(-m, -n)$. Then, by noting $\coth(m\pi) = (e^{m\pi} + e^{-m\pi})/(e^{m\pi} - e^{-m\pi})$, we obtain

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\neq 0}} \frac{\coth(m\pi)}{(m+ni)^3} = \frac{\varpi^4}{15\pi} + \frac{4}{45}\pi^3 - \frac{1}{3}\pi^2, \tag{4.3}$$

which was also given in [27].

As another example, setting $(x, y, \tau) = (0, 0, i)$, $l = 2$ and $z = 1/2$ in Equation (2.11) of Theorem 2.6, we obtain

$$\sum_{\substack{(m,n)\in\mathbf{Z}^2 \\ m\neq 0}} \frac{(-1)^n}{\sinh(m\pi)^2(m+ni)^4} = -\frac{\varpi^4}{45} + \frac{37}{945}\pi^4 - \frac{4}{45}\pi^3. \tag{4.4}$$

We can also treat the case $\tau = \rho = e^{2\pi i/3}$, for example,

$$\sum_{\substack{(m,n)\in\mathbf{Z}^2 \\ m\neq 0}} \frac{(-1)^n}{\sinh(m\pi i/\rho)(m+n\rho)^5} = \rho i \left(\frac{31}{2520}\pi^5 - \frac{7\sqrt{3}}{540}\pi^4 - \frac{\tilde{\varpi}^6}{35\pi} \right), \tag{4.5}$$

$$\sum_{\substack{(m,n)\in\mathbf{Z}^2 \\ m\neq 0}} \frac{1}{(\sinh(m\pi i/\rho))^2(m+n\rho)^4} = \frac{1}{\rho} \left(\frac{37}{945}\pi^4 - \frac{8\sqrt{3}}{135}\pi^3 - \frac{\tilde{\varpi}^6}{35\pi} \right), \tag{4.6}$$

where

$$\tilde{\varpi} = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428651\cdots. \tag{4.7}$$

Next we give explicit evaluation formulas for the q-zeta function $\zeta_q(s,t)$ defined by (1.14) (see Kaneko-Kurokawa-Wakayama [11] and Kawagoe-Wakayama-Yamasaki [14]). Let $q = e^{-2\pi}$ in Corollary 3.3. Then we obtain

$$\zeta_q(2,1) = \frac{(1-e^{-2\pi})^2}{4} \sum_{m=1}^{\infty} \frac{1}{(\sinh(m\pi))^2} = (1-e^{-2\pi})^2 \Big(\frac{1}{24} - \frac{1}{8\pi} \Big), \tag{4.8}$$

$$\zeta_q(4,2) = (1-e^{-2\pi})^4 \Big(\frac{\varpi^4}{480\pi^4} - \frac{11}{1440} + \frac{1}{48\pi} \Big), \tag{4.9}$$

$$\zeta_q(6,3) = (1-e^{-2\pi})^6 \Big(-\frac{\varpi^4}{1920\pi^4} + \frac{191}{120960} - \frac{1}{240\pi} \Big), \tag{4.10}$$

$$\zeta_q(8,4) = (1-e^{-2\pi})^8 \Big(\frac{\varpi^8}{268800\pi^8} + \frac{7\varpi^4}{57600\pi^4} - \frac{2497}{7257600} + \frac{1}{1120\pi} \Big). \tag{4.11}$$

It is noted that these values can be regarded as character analogues of the Witten zeta function of type A_1.

Remark 4.1. Special q-zeta values at negative integers are obtained by many authors.

5. Summary of the former part

Consider the coefficient a_ℓ of the generalized Laurent expansion

$$\mathfrak{A}(\xi) = \sum_{\ell=-1}^{\infty} a_\ell (\mathfrak{B}(\xi)^{-1})' (\mathfrak{B}(\xi)^{-1})^\ell, \qquad a_\ell = \frac{1}{2\pi i} \int_{|\xi|=\varepsilon} \mathfrak{A}(\xi)\mathfrak{B}(\xi)^{\ell+1} d\xi \tag{5.1}$$

associated with $(\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are

$$\mathfrak{F}(\xi) = e^{2\pi i x\xi} \frac{\theta'(0;\tau)\theta(\xi + x\tau - y;\tau)}{\theta(\xi;\tau)\theta(x\tau - y;\tau)} = \frac{1}{\xi} + O(1), \tag{5.2}$$

$$\mathfrak{G}(\xi) = \frac{2\pi i}{\tau} \frac{e^{2\pi i \xi z/\tau}}{e^{2\pi i \xi/\tau} - 1} = \frac{1}{\xi} + O(1), \tag{5.3}$$

$$\mathfrak{H}(\xi) = \frac{1}{\xi}. \tag{5.4}$$

We interpret the coefficient a_ℓ as a series by summing up all the residues of $\mathfrak{A}(\xi)\mathfrak{B}(\xi)^{\ell+1}$. Here are the pairs which are treated in this paper.

- $(\mathfrak{G}, \mathfrak{H}) \Longrightarrow$ Lerch zeta function.
- $(\mathfrak{G}\mathfrak{G}, \mathfrak{H}) \Longrightarrow$ Ramanujan's formula, etc.
- $(\mathfrak{F}, \mathfrak{H}) \Longrightarrow$ Eisenstein series.
- $(\mathfrak{F}, \mathfrak{G}) \Longrightarrow$ Lerch q-zeta function (a q-analogue of the Witten zeta function of type A_1).
- $(\mathfrak{F}\mathfrak{G}^\ell, \mathfrak{H}) \Longrightarrow$ Double series in (1.13).

6. Functional equations for double series of Eisenstein type

In this section, we consider certain functional equations for double series of Eisenstein type defined as follows (for details, see [15]).

Throughout this paper, we interpret z^s as $e^{s \log z}$, where $\log z = \log|z| + i \arg z$ with $-\pi < \arg z \leq \pi$ unless otherwise indicated. To ensure the convergence of (6.1) (defined below), we assume the following condition for $\omega_1, \omega_2 \in \mathbb{C}$. Let l_0 be any line on the complex plane crossing the origin. Then l_0 divides the plane into two half-planes. Let $H(l_0)$ be one of those half-planes, not including l_0 itself. Using this notation, we assume that

$$\omega_1, \omega_2 \in H(l_0)$$

for some l_0 (see [19, Theorem 8.1]).

Definition 6.1. (Double series of Eisenstein type)

$$\zeta_2(s_1, s_2; \omega_1, \omega_2) = \sum_{m=1}^{\infty} \frac{1}{(m\omega_1)^{s_1}} \sum_{n=1}^{\infty} \frac{1}{(m\omega_1 + n\omega_2)^{s_2}}, \tag{6.1}$$

for $(s_1, s_2) \in \mathbb{C}^2$ with $\Re(s_1 + s_2) > 2$ and $\Re s_2 > 1$. When $\omega_1 = \omega_2 = 1$,

$$\zeta_2(s_1, s_2; 1, 1) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1}(m+n)^{s_2}}$$

which is the ordinary double zeta function $\zeta_2(s_1, s_2)$ of Euler type.

Theorem 6.2. (Functional equations) *Suppose* $\Re\omega_1 > 0$, $\Re\omega_2 > 0$. *Then* $\zeta_2(s_1, s_2; \omega_1, \omega_2)$ *is continued meromorphically to* $\mathbb{C}^2$, *and*

$$\begin{aligned}
&\left(\frac{2\pi i}{\omega_1\omega_2}\right)^{\frac{1-s_1-s_2}{2}} \Gamma(s_2)\left\{\zeta_2(s_1, s_2; \omega_1, \omega_2) - \frac{\Gamma(1-s_1)\Gamma(s_1+s_2-1)}{\omega_1\omega_2^{s_1+s_2-1}\Gamma(s_2)}\zeta(s_1+s_2-1)\right\} \\
&= \left(\frac{2\pi i}{\omega_1\omega_2}\right)^{\frac{s_1+s_2-1}{2}} \Gamma(1-s_1)\left\{\zeta_2(1-s_2, 1-s_1; \omega_1, \omega_2)\right. \\
&\qquad\qquad \left. - \frac{\Gamma(s_2)\Gamma(1-s_1-s_2)}{\omega_1\omega_2^{1-s_1-s_2}\Gamma(1-s_1)}\zeta(1-s_1-s_2)\right\}
\end{aligned} \tag{6.2}$$

holds on the hyperplane

$$\Omega_{2k+1} = \{(s_1, s_2) \in \mathbb{C}^2 \mid s_1 + s_2 = 2k+1\} \quad (k \in \mathbb{Z} \setminus \{0\}).$$

Remark 6.3. Denote the left-hand side of (6.2) by $\xi(s_1, s_2; \omega_1, \omega_2)$. Then

$$\xi(s_1, s_2; \omega_1, \omega_2) = \xi(1-s_2, 1-s_1; \omega_1, \omega_2).$$

When $\omega_1 = \omega_2 = 1$, this implies the functional equation for the ordinary double zeta function $\zeta_2(s_1, s_2)$ of Euler type and the Riemann zeta function $\zeta(s)$.

We can easily see that

$$\zeta_2(s_1, s_2; \omega_1, \omega_2) = \omega_1^{-s_1-s_2}\zeta_2(s_1, s_2; 1, \omega_2/\omega_1).$$

On the other hand, if $\Im\tau > 0$ then we can write $\tau = \omega_2/\omega_1$ for some $\Re\omega_1 > 0$, $\Re\omega_2 > 0$. Therefore, by Theorem 6.2, we obtain the following.

Corollary 6.4. *For* $\tau \in \mathbb{C}$ *with* $\Im\tau > 0$ *or* $\tau \in (0, \infty)$, *the following functional equation*

$$\begin{aligned}
&\left(\frac{2\pi i}{\tau}\right)^{\frac{1-s_1-s_2}{2}} \Gamma(s_2)\left\{\zeta_2(s_1, s_2; 1, \tau) - \frac{\Gamma(1-s_1)\Gamma(s_1+s_2-1)}{\tau^{s_1+s_2-1}\Gamma(s_2)}\zeta(s_1+s_2-1)\right\} \\
&= \left(\frac{2\pi i}{\tau}\right)^{\frac{s_1+s_2-1}{2}} \Gamma(1-s_1)\left\{\zeta_2(1-s_2, 1-s_1; 1, \tau)\right. \\
&\qquad\qquad \left. - \frac{\Gamma(s_2)\Gamma(1-s_1-s_2)}{\tau^{1-s_1-s_2}\Gamma(1-s_1)}\zeta(1-s_1-s_2)\right\}
\end{aligned} \tag{6.3}$$

holds for $(s_1, s_2) \in \Omega_{2k+1}$ $(k \in \mathbb{Z} \setminus \{0\})$.

The above functional equations are deduced from a more general result as follows.

Definition 6.5. (Confluent hypergeometric functions)

$$\Psi(a,c;x)=\frac{1}{\Gamma(a)}\int_0^{e^{i\theta}\infty}e^{-xy}y^{a-1}(1+y)^{c-a-1}dy,$$

where $\Re a>0$, $-\pi<\theta<\pi$, $|\theta+\arg x|<\pi/2$ (see Erdélyi et al. [8, formula 6.5 (3)]), and

$$F_{\pm}(s_1,s_2;\tau)=\sum_{k=1}^{\infty}\sigma_{s_1+s_2-1}(k)\Psi(s_2,s_1+s_2;\pm 2\pi ik\tau), \tag{6.4}$$

where $\sigma_{s_1+s_2-1}(k)=\sum_{d|k}d^{s_1+s_2-1}$, which can be continued meromorphically to $\mathbb{C}^2$ (see [21]).

Using this notation, we obtain the following theorem about general forms of functional equations. In the next section, we will give a brief sketch of its proof. For details, see [15].

Theorem 6.6. (General forms of functional equations)
Suppose $\Re\omega_1>0$ and $\Re\omega_2>0$. For $(s_1,s_2)\in\mathbb{C}^2$, the functional equation holds:

$$\begin{aligned}&\zeta_2(s_1,s_2;\omega_1,\omega_2)-\frac{\Gamma(1-s_1)\Gamma(s_1+s_2-1)}{\omega_1\omega_2^{s_1+s_2-1}\Gamma(s_2)}\zeta(s_1+s_2-1)\\&=\frac{\Gamma(1-s_1)}{\omega_1\omega_2^{s_1+s_2-1}}\left\{F_+\left(1-s_2,1-s_1;\frac{\omega_2}{\omega_1}\right)+F_-\left(1-s_2,1-s_1;\frac{\omega_2}{\omega_1}\right)\right\}.\end{aligned} \tag{6.5}$$

Remark 6.7. The Hurwitz zeta function $\zeta(s,\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-s}$ $(\alpha>0)$ satisfies the functional equation

$$\zeta(s,\alpha)=\frac{\Gamma(1-s)}{i(2\pi)^{1-s}}\left\{e^{\pi is/2}\phi(1-s,\alpha)-e^{-\pi is/2}\phi(1-s,-\alpha)\right\}, \tag{6.6}$$

where $\phi(s,\alpha)=\sum_{n=1}^{\infty}e^{2\pi in\alpha}n^{-s}$ (see Titchmarsh [24, (2.17.3)]). Formula (6.5) may be regarded as the double analogue of (6.6). We know the asymptotic expansion

$$\Psi(a,c;x)=\sum_{k=0}^{N-1}\frac{(-1)^k(a-c+1)_k(a)_k}{k!}x^{-a-k}+\rho_N(a,c;x), \tag{6.7}$$

where $(a)_k=\Gamma(a+k)/\Gamma(a)$ and $\rho_N(a,c;x)$ is the remainder term. This implies that $\Psi(s_2,s_1+s_2;\pm 2\pi ik\tau)$ can be approximated by $(\pm 2\pi ik\tau)^{-s_2}$ (the term corresponding to $k=0$ in (6.7)), hence $F_{\pm}(s_1,s_2;\tau)$ can be approximated by the Dirichlet series

$$\sum_{k=1}^{\infty}\frac{\sigma_{s_1+s_2-1}(k)}{(\pm 2\pi ik\tau)^{s_2}}.$$

Therefore $F_{\pm}(s_1,s_2;\tau)$ may be considered as a "generalized Dirichlet series". From this viewpoint, formula (6.5) can be regarded as a duality formula among "generalized Dirichlet series".

In other words, since $n^{-s} = F(s,1,1;1-n)$ (where the right-hand side is the usual notation of the Gauss hypergeometric function), we see that a Dirichlet series is a special case of infinite series of hypergeometric functions. Therefore "functional equations" in general setting are perhaps to be understood as duality relations among infinite series of (confluent or non-confluent) hypergeometric functions.

Proof of Theorem 6.2. By the transformation formula for $\Psi(a,c;x)$, we have

$$F_{\pm}(1-s_2, 1-s_1; \omega_2/\omega_1) = (\pm 2\pi i\omega_2/\omega_1)^{s_1+s_2-1} F_{\pm}(s_1, s_2; \omega_2/\omega_1).$$

For $(s_1, s_2) \in \Omega_{2k+1}$, that is, $s_1 + s_2 = 2k+1$, we have

$$F_{\pm}(1-s_2, 1-s_1; \omega_2/\omega_1) = (2\pi i\omega_2/\omega_1)^{s_1+s_2-1} F_{\pm}(s_1, s_2; \omega_2/\omega_1). \tag{6.8}$$

From Theorem 6.6, we have

$$\frac{\omega_1\omega_2^{s_1+s_2-1}}{\Gamma(1-s_1)}\left\{\zeta_2(s_1,s_2;\omega_1,\omega_2) - \frac{\Gamma(1-s_1)\Gamma(s_1+s_2-1)}{\omega_1\omega_2^{s_1+s_2-1}\Gamma(s_2)}\zeta(s_1+s_2-1)\right\}$$
$$= F_+\left(1-s_2, 1-s_1; \frac{\omega_2}{\omega_1}\right) + F_-\left(1-s_2, 1-s_1; \frac{\omega_2}{\omega_1}\right).$$

Combining this relation and (6.8), we obtain Theorem 6.2. □

From Theorem 6.2, we obtain the following.

Corollary 6.8. *for $p, q \in \mathbb{N}_0$,*

$$\zeta_2(-2p, -2q-1; \omega_1, \omega_2) = \frac{B_{2p+2q+2}}{4(p+q+1)}\omega_1^{2p+2q+1}, \tag{6.9}$$

$$\zeta_2(-2p-1, -2q; \omega_1, \omega_2) = \frac{B_{2p+2q+2}}{4(p+q+1)}\omega_1^{2p+2q+1}, \tag{6.10}$$

where $\{B_n\}$ is the Bernoulli numbers.

This fact in the case $(\omega_1, \omega_2) = (1,1)$ coincides with the known result given by Akiyama, Egami and Tanigawa [1].

7. Brief sketch of the proof of Theorem 6.6

By the method introduced by the second-named author in [21], we have

$$\zeta_2(s_1, s_2; \omega_1, \omega_2)$$
$$= \frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty \frac{y^{s_2-1}}{e^{\omega_2 y}-1}\int_0^\infty \frac{x^{s_1-1}}{e^{\omega_1(x+y)}-1}dxdy,$$

where the right-hand side is convergent when $\Re s_1 > 0$, $\Re s_2 > 1$, $\Re(s_1+s_2) > 2$. Hence $\zeta_2(s_1, s_2; \omega_1, \omega_2)$ can be continued to this region. Let

$$h(z) = \frac{1}{e^{\omega_1 z}-1} - \frac{1}{\omega_1 z}.$$

Then

$$\zeta_2(s_1,s_2;\omega_1,\omega_2)=\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty\frac{y^{s_2-1}}{e^{\omega_2 y}-1}\int_0^\infty h(x+y)x^{s_1-1}dxdy \\ +\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty\frac{y^{s_2-1}}{e^{\omega_2 y}-1}\int_0^\infty\frac{x^{s_1-1}}{\omega_1(x+y)}dxdy. \tag{7.1}$$

We denote the first and the second integral on the right-hand side of (7.1) by $g(s_1,s_2;\omega_1,\omega_2)$ and $g_0(s_1,s_2;\omega_1,\omega_2)$, respectively.

When $0<\Re s_1<1$, $\Re(s_1+s_2)>2$, we have

$$\begin{aligned} g_0(s_1,s_2;\omega_1,\omega_2)&=\frac{\Gamma(1-s_1)}{\omega_1\Gamma(s_2)}\int_0^\infty\frac{y^{s_1+s_2-2}}{e^{\omega_2 y}-1}dy \\ &=\frac{\Gamma(1-s_1)\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1)}{\Gamma(s_2)\omega_1\omega_2^{s_1+s_2-1}}. \end{aligned}$$

Next we consider $g(s_1,s_2;\omega_1,\omega_2)$. Let C be the contour which starts from $+\infty$, goes along the positive real axis, rounds the origin counterclockwise, and then goes back along the positive real axis to $+\infty$. Then we have

$$g(s_1,s_2;\omega_1,\omega_2)=\frac{1}{\Gamma(s_1)\Gamma(s_2)\left(e^{2\pi i s_1}-1\right)\left(e^{2\pi i s_2}-1\right)} \\ \times\int_C\frac{y^{s_2-1}}{e^{\omega_2 y}-1}\int_C h(x+y)x^{s_1-1}dxdy$$

for $\Re s_1<1$ and $s_2\in\mathbb{C}$.

By residue calculus, we obtain

$$g(s_1,s_2;\omega_1,\omega_2)=-\frac{2\pi i\omega_1^{-1}}{\Gamma(s_1)\Gamma(s_2)\left(e^{2\pi i s_1}-1\right)\left(e^{2\pi i s_2}-1\right)}\sum_{n\neq 0}I_n,$$

where

$$I_n=\int_C\frac{y^{s_2-1}}{e^{\omega_2 y}-1}\left(-y+\frac{2\pi i n}{\omega_1}\right)^{s_1-1}dy.$$

By the method of Katsurada and Matsumoto [13], we have

$$\sum_{n=1}^{\infty}I_n=\left(e^{2\pi i s_2}-1\right)e^{\pi i(s_1-s_2-1)/2}\left(\frac{2\pi}{\omega_1}\right)^{s_1+s_2-1}\Gamma(s_2)F_-\left(s_1,s_2;\frac{\omega_2}{\omega_1}\right),$$

$$\sum_{n=-\infty}^{-1}I_n=\left(e^{2\pi i s_2}-1\right)e^{\pi i(3s_1+s_2-3)/2}\left(\frac{2\pi}{\omega_1}\right)^{s_1+s_2-1}\Gamma(s_2)F_+\left(s_1,s_2;\frac{\omega_2}{\omega_1}\right).$$

Hence we obtain

$$g(s_1,s_2;\omega_1,\omega_2)=(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)\omega_1^{-s_1-s_2} \\ \times\left\{e^{\pi i(1-s_1-s_2)/2}F_-\left(s_1,s_2;\frac{\omega_2}{\omega_1}\right)+e^{\pi i(s_1+s_2-1)/2}F_+\left(s_1,s_2;\frac{\omega_2}{\omega_1}\right)\right\}$$

for $\Re s_1 < 0$ and $\Re s_2 > 1$. Furthermore, we see that for any $N \in \mathbb{N}$,

$$F_{\pm}(s_1, s_2; \tau) = \sum_{j=0}^{N-1} \frac{(-1)^j}{j!}(1-s_1)_j(s_2)_j(\pm 2\pi i\tau)^{-s_2-j}$$
$$\times \zeta(1-s_1+j)\zeta(s_2+j) + R_N(s_1, s_2; \tau),$$

where $R_N(s_1, s_2; \tau)$ is a certain holomorphic function for $\Re s_1 < N$, $\Re s_2 > 1-N$. Since N is arbitrary, $F_{\pm}(s_1, s_2; \tau)$ can be continued meromorphically to the whole space $\mathbb{C}^2$, and so is $g(s_1, s_2; \omega_1, \omega_2)$.

Finally, based on the transformation formula for $\Psi(a, c; x)$, we can prove that

$$F_{\pm}\left(1-s_2, 1-s_1; \frac{\omega_2}{\omega_1}\right) = (\pm 2\pi i\tau)^{s_1+s_2-1} F_{\pm}\left(s_1, s_2; \frac{\omega_2}{\omega_1}\right).$$

Hence we have

$$g(s_1, s_2; \omega_1, \omega_2) = \Gamma(1-s_1)\omega_1^{-1}\omega_2^{1-s_1-s_2}$$
$$\times \left\{F_{+}\left(1-s_2, 1-s_1; \frac{\omega_2}{\omega_1}\right) + F_{-}\left(1-s_2, 1-s_1; \frac{\omega_2}{\omega_1}\right)\right\}.$$

Thus we obtain the proof of Theorem 6.6.

Remark 7.1. From the above observation, we see that the singular loci of $\zeta_2(s_1, s_2; \omega_1, \omega_2)$ are located on

$$s_1 + s_2 = 2 - 2k \ (k \in \mathbb{N}_0), \ s_1 + s_2 = 1, \ s_2 = 1.$$

In particular when $(\omega_1, \omega_2) = (1, 1)$, this fact for $\zeta_2(s_1, s_2)$ coincides with the known result given by Akiyama, Egami and Tanigawa [1] using the Euler-Maclaurin formula. We can also determine it by using the Mellin-Barnes formula.

8. Functional relations for certain double series

Now we consider double series of Eisenstein type defined by

$$\widetilde{\zeta}_2(s_1, s_2; \tau) = \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \sum_{n\in\mathbb{Z}} \frac{1}{m^{s_1}(m+n\tau)^{s_2}}$$

for $\tau = \eta e^{i\theta_1}$ ($\eta > 0$; $0 < \theta_1 < \pi$). Let $\xi = \sqrt{\eta}e^{i\theta_1/2}$. Then

$$\widetilde{\zeta}_2(s_1, s_2; \tau) = \left(1 + e^{-\pi i(s_1+s_2)}\right)\Big\{\xi^{-s_1-s_2}\zeta_2\left(s_1, s_2; \xi^{-1}, \xi\right)$$
$$+ (i\xi^{-1})^{s_1+s_2}\zeta_2\left(s_1, s_2; i\xi^{-1}, -i\xi\right) + \zeta(s_1+s_2)\Big\}$$

which shows the meromorphic continuation of $\widetilde{\zeta}_2(s_1, s_2; \tau)$.

For $\tau \in \mathbb{C}$ with $\Im\tau > 0$,

$$\mathcal{S}_2(s_1, s_2; \tau) = \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\sinh(m\pi i/\tau)m^{s_1}(m+n\tau)^{s_2}}.$$

Theorem 8.1. *For $k \in \mathbb{N}$ with $k \geq 2$, $\tau \in \mathbb{C}$ with $\Im\tau > 0$, and $s \in \mathbb{C}$,*

$$\mathcal{S}_2(s,k;\tau) = \frac{\tau}{\pi i}\sum_{j=0}^{k}\frac{(2\pi i/\tau)^{k-j}}{(k-j)!}B_{k-j}\left(\frac{1}{2}\right)\widetilde{\zeta}_2(s,j+1;\tau).$$

Example 8.2. For $k \in \mathbb{N}$,

$$\widetilde{\zeta}_2(0,2k;i) = G_{2k}(i) - 2(-1)^k\zeta(2k),$$

where

$$G_{2k}(\tau) = \begin{cases} \displaystyle\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{1}{(m+n\tau)^{2k}} & (k \geq 2),\\ \displaystyle\lim_{M\to\infty}\sum_{-M\leq m\leq M}\left(\lim_{N\to\infty}\sum_{\substack{-N\leq n\leq N\\(m,n)\neq(0,0)}}\frac{1}{(m+n\tau)^2}\right) & (k=1).\end{cases}$$

It is known that $G_2(i) = -\pi$ and $G_{4k}(i) = H_{4k}\frac{(2\varpi)^{4k}}{(4k)!}$, where $\{H_{4k} \in \mathbb{Q} \mid k \in \mathbb{N}\}$ are the Hurwitz numbers defined by (1.6) (see [18]). Note that $G_{4k+2}(i) = 0$ for $k \in \mathbb{N}$. By (1.8), we obtain

$$G_4(i) = \frac{1}{15}\varpi^4,\ G_8(i) = \frac{1}{525}\varpi^8,\ G_{12}(i) = \frac{2}{53625}\varpi^{12},\dots \tag{8.1}$$

From Theorem 8.1, we see that

$$\mathcal{S}_2(s,3;\tau) = \frac{\tau}{\pi i}\widetilde{\zeta}_2(s,4;\tau) + \frac{\pi}{6\tau i}\widetilde{\zeta}_2(s,2;\tau),$$

$$\mathcal{S}_2(s,5;\tau) = \frac{\tau}{\pi i}\widetilde{\zeta}_2(s,6;\tau) + \frac{\pi}{6\tau i}\widetilde{\zeta}_2(s,4;\tau) + \frac{\pi^3}{360\tau^3 i}\widetilde{\zeta}_2(s,2;\tau).$$

Letting $s \to 0$, we have

$$\mathcal{S}_2(0,3;i) = \frac{\varpi^4}{15\pi} - \frac{7}{90}\pi^3 + \frac{1}{6}\pi^2, \tag{8.2}$$

$$\mathcal{S}_2(0,5;i) = -\frac{1}{90}\varpi^4\pi + \frac{31}{2520}\pi^5 - \frac{7}{360}\pi^4, \tag{8.3}$$

where (8.2) coincides with (4.2). More generally we recover the known result (see [26]):

$$\begin{aligned}\mathcal{S}_2(0,2p-1;i) &= \sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{\sinh(m\pi)(m+ni)^{2p-1}}\\ &= \frac{2(-1)^{p+1}}{\pi}\sum_{j=1}^{p}\left(2^{1-2p+2j}-1\right)\zeta(2p-2j)\left\{(-1)^j G_{2j}(i) - 2\zeta(2j)\right\} \quad (p \in \mathbb{N}).\end{aligned}$$

This can be regarded as a double analogue of (1.9) which can be rewritten as

$$\begin{aligned}&\sum_{m\neq 0}\frac{(-1)^m}{\sinh(m\pi)m^{4k-1}}\\ &= \frac{2(-1)^{2k+1}}{\pi}\sum_{j=0}^{2k}\left(2^{1-4k+2j}-1\right)\zeta(4k-2j)(-1)^j\left(2^{1-2j}-1\right)\zeta(2j) \quad (k \in \mathbb{N}).\end{aligned}$$

9. Functional equations for double L-functions

Finally we introduce certain functional equations for double L-functions. For details, see [16].

Definition 9.1. (Double L-function) For Dirichlet characters χ_1, χ_2,

$$L_2^{\text{ш}}(s_1, s_2; \chi_1, \chi_2) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\chi_1(m)\chi_2(n)}{m^{s_1}(m+n)^{s_2}}. \tag{9.1}$$

More generally, multiple L-functions have already been studied (see, for example, [3]). The function (9.1) can be continued meromorphically to $\mathbb{C}^2$ (see, for example, [20]).

In order to prove functional equations for double L-functions, we recall the double Hurwitz-Lerch zeta functions introduced by the second-named author in [21].

Let

$$\zeta_2(s_1, s_2; \alpha, \beta, \omega_2) = \sum_{m=0}^{\infty} (\alpha + m)^{-s_1} \sum_{n=1}^{\infty} e^{2\pi i n \beta} (\alpha + m + n\omega_2)^{-s_2}, \tag{9.2}$$

where $0 < \alpha \le 1$, $0 \le \beta \le 1$ and $\omega_2 > 0$. The function $F_{\pm}(s_1, s_2; \alpha, \beta, \tau)$ is defined by replacing $\sigma_{s_1+s_2-1}(k)$ on the right-hand side of (6.4) by

$$\sigma_{s_1+s_2-1}(k; \alpha, \beta) = \sum_{d|k} e^{2\pi i d\alpha} e^{2\pi i (k/d)\beta} d^{s_1+s_2-1}.$$

Then the formula

$$\begin{aligned} &\zeta_2(s_1, s_2; \alpha, \beta, \omega_2) \\ &= \frac{\Gamma(1-s_1)}{\Gamma(s_2)} \Gamma(s_1+s_2-1) \phi(s_1+s_2-1, \beta) \omega_2^{1-s_1-s_2} \\ &\quad + \Gamma(1-s_1) \omega_2^{1-s_1-s_2} \\ &\quad \times \left\{ F_+ (1-s_2, 1-s_1; \beta, \alpha, \omega_2) + F_- (1-s_2, 1-s_1; \beta, -\alpha, \omega_2) \right\} \end{aligned} \tag{9.3}$$

can be easily deduced from Propositions 1 and 2 of [21]. On the other hand , we can easily obtain

$$L_2^{\text{ш}}(s_1, s_2; \chi_1, \chi_2) = \frac{1}{f^{s_1+s_2} \tau(\overline{\chi}_2)} \sum_{a=1}^{f} \sum_{b=1}^{f} \chi_1(a) \overline{\chi}_2(b) \zeta_2 \left(s_1, s_2; \frac{a}{f}, \frac{b}{f}, \frac{1}{f} \right). \tag{9.4}$$

Combining (9.3), (9.4) and (6.8), we can obtain the following.

Theorem 9.2. *For primitive Dirichlet characters χ_1, χ_2 of conductor $f > 1$, the functional equation*

$$\begin{aligned} &\left(\frac{2\pi i}{f}\right)^{\frac{1-s_1-s_2}{2}} \frac{\Gamma(s_2)}{\tau(\chi_1)} L_2^{\text{ш}}(s_1, s_2; \chi_1, \chi_2) \\ &\quad = \left(\frac{2\pi i}{f}\right)^{\frac{s_1+s_2-1}{2}} \frac{\Gamma(1-s_1)}{\tau(\overline{\chi}_2)} L_2^{\text{ш}}(1-s_2, 1-s_1; \overline{\chi}_2, \overline{\chi}_1), \end{aligned} \tag{9.5}$$

holds on the hyperplane

$$\begin{cases} \Omega_{2k+1} = \{(s_1, s_2) \mid s_1 + s_2 = 2k+1\} & \textit{if } \chi_1(-1)\chi_2(-1) = 1; \\ \Omega_{2k} = \{(s_1, s_2) \mid s_1 + s_2 = 2k\} & \textit{if } \chi_1(-1)\chi_2(-1) = -1. \end{cases}$$

Remark 9.3. It is also possible to give the functional equation in the case when χ_1 or χ_2 is the trivial character. In this case, an extra part including the Dirichlet L-function appears in the equation. Actually, when both χ_1 and χ_2 are trivial, extra parts including $\zeta(s)$ appear on both sides of (6.2). In [16], we further treat double L-functions of Eisenstein type.

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