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On the existence of shock curves in 2×2 hyperbolic systems of conservation laws

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Abstract. We consider shock curves in 2×2 hyperbolic systems of conservation laws in one space variable: $u_t+f(u,v)_x=0, v_t+g(u,v)_x=0$. Our assumptions are that $f_v g_u<0$ and the system is strictly hyperbolic. We also assume that the system is genuinely nonlinear, satisfies the Smoller-Johnson condition and the half-plane condition. Under those assumptions, by applying an argument as in [10], we prove the existence of shock curves and the stability condition for shock speeds.

1 Introduction

In this paper we consider shock curves in 2×2 hyperbolic systems of conservation laws in one space variable,

$$u_t+f(u,v)_x=0,\quad v_t+g(u,v)_x=0,\qquad t>0,\ -\infty<x<\infty. \tag{1.1}$$

Here u and v are functions of t and x, and f and g are C^2 functions of two real variables u and v. Our assumptions are that $f_v g_u<0$ and system (1.1) is strictly hyperbolic. We also require that system (1.1) is genuinely nonlinear, satisfies the Smoller-Johnson condition and the half-plane condition. The Smoller-Johnson condition is given in [14] and implies a certain convexity of rarefaction curves. Moreover, it is shown in [14] that for sufficiently weak shocks the condition implies the Glimm-Lax shock interaction condition [6], which states that the interaction of two shocks of the same characteristic family produces a shock of that characteristic family plus a rarefaction wave of the opposite characteristic family. The half-plane condition is given in [8] and implies that the right eigenvectors of dF, where dF denotes the Fréchet derivative (Jacobian) of the mapping F from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $F=(f,g)$, point into opposite fixed half-planes. Systems (1.1) satisfying $f_v g_u>0$ satisfy the half-plane condition.

The Riemann problem for system (1.1) consists in finding a solution of (1.1) with piecewise constant initial data of the form

$$(u(x,0),v(x,0))=\begin{cases}(u_l,v_l) & x<0\ ,\\ (u_r,v_r) & x>0\ .\end{cases} \tag{1.2}$$

In general, the significance of the Riemann problem is that it is served to solve the Cauchy problem (1.1) with general initial data. In fact, the Riemann problem is the building block in the proof of existence theorems in [3], [5] and [15].

Since both (1.1) and (1.2) are invariant under uniform stretching of the spatial and temporal coordinates, the Riemann problem possesses self-similar solutions. The classical method of solution of the Riemann problem is based on the construction of shock and rarefaction curves of system (1.1). Thus, the shock curves play an essential role in the study of the existence and uniqueness of self-similar solutions to the Riemann problem ([2], [8], [11] and [12]). In general, however, it is not easy to construct shock curves without additional assumptions. In [14], for system (1.1) satisfying $f_v g_u > 0$, the genuine nonlinearity and the Smoller-Johnson conditions, Smoller and Johnson give a proof of the existence of the two shock curves. It is noticed that system (1.1) satisfying $f_v g_u > 0$ is strictly hyperbolic. But, by their method, the remaining two curves cannot be shown without assuming the stability condition for shock speeds (see [4] for the stability condition). The existence of shock curves and the proof of the stability condition appear in [2] and [8] on strictly hyperbolic system (1.1) satisfying the genuine nonlinearity, the Smoller-Johnson and the half-plane conditions. Their methods of proof are based on the following property of the shock curves: For a given point U_0 in $\mathbb{R}^2$, $U - U_0$ is never parallel to right eigenvectors of dF at any point U on shock curves. Note that there are many systems (1.1) of interest, in which the property is violated and for such systems their methods cannot be applied. In [10], for system (1.1) satisfying $f_v g_u > 0$, the genuine nonlinearity and the Smoller-Johnson conditions, Ohwa gives a new method for proving the existence of shock curves and the stability condition without using that property. The method of proof relies on the continuous dependence on initial points and the geometric properties of shock curves. Accordingly, the method will be expected to be relaxed the genuine nonlinearity and the Smoller-Johnson condition.

The purpose of this paper is to apply the method to strictly hyperbolic system (1.1) satisfying $f_v g_u < 0$, the genuine nonlinearity, the Smoller-Johnson and the half-plane conditions .

2 Statement of the theorem

In this section we shall state the main result of this paper.

Let F be the mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $F : (u, v) \to (f(u, v), g(u, v))$, and denote by $dF(u, v)$ the Fréchet derivative (Jacobian) of F. We assume that

$$f_v g_u < 0 \quad \text{in} \quad \mathbb{R}^2, \tag{2.1}$$

and system (1.1) is strictly hyperbolic, that is, $dF(u, v)$ has real and distinct eigenvalues $\lambda_1(u, v) < \lambda_2(u, v)$ for all $(u, v) \in \mathbb{R}^2$. Without loss of generality, we assume that

$$f_v < 0, \quad g_u > 0, \quad g_v < \lambda_1 < \lambda_2 < f_u \quad \text{in} \quad \mathbb{R}^2. \tag{2.2}$$

We denote by $r_i(u, v)$, $i = 1, 2$, the corresponding right eigenvectors and assume that

$$d\lambda_i(u, v) \cdot r_i(u, v) > 0, \quad (u, v) \in \mathbb{R}^2, \quad i = 1, 2. \tag{2.3}$$

Condition (2.3) implies that system (1.1) is genuinely nonlinear in the sense of Lax [9].

Let l_i, $i=1,2$, be the left eigenvectors of $dF(u,v)$, normalized by $l_i r_i > 0$, $i=1,2$. We then impose that system (1.1) satisfies the Smoller-Johnson condition

$$l_j(u,v)d^2F(r_i(u,v),r_i(u,v)) > 0, \quad (u,v)\in\mathbb{R}^2, \quad i,j=1,2, \quad i\neq j, \tag{2.4}$$

where d^2F is the second Fréchet derivative of F. In [14], it is shown that the genuine nonlinearity condition (2.3) is equivalent to

$$l_i(u,v)d^2F(r_i(u,v),r_i(u,v)) > 0, \quad (u,v)\in\mathbb{R}^2, \quad i=1,2. \tag{2.5}$$

Therefore, we can write (2.3) and (2.4) in the form

$$l_j(u,v)d^2F(r_i(u,v),r_i(u,v)) > 0, \quad (u,v)\in\mathbb{R}^2, \quad i,j=1,2. \tag{2.6}$$

Moreover, we impose that system (1.1) satisfies

$$r_1(u_1,v_1)\neq r_2(u_2,v_2) \tag{2.7}$$

for every pair of points (u_1,v_1) and (u_2,v_2). Condition (2.7) is called the half-plane condition. In [8], it is shown that the half-plane condition (2.7) is satisfied if and only if there exists a fixed vector w, independent of (u,v), such that $r_1\cdot w>0$ and $r_2\cdot w<0$ for all (u,v). Systems (1.1) satisfying $f_v g_u > 0$ clearly satisfy condition (2.7).

Under these assumptions, denoting

$$a_i = \frac{\lambda_i - f_u}{f_v} = \frac{g_u}{\lambda_i - g_v} > 0, \quad i=1,2, \tag{2.8}$$

we represent respectively the right and left eigenvectors in the form

$$r_i = \operatorname{sgn}(d\lambda_i\cdot\hat{r}_i)\,\hat{r}_i, \quad i=1,2, \tag{2.9}$$

$$l_1 = \operatorname{sgn}(d\lambda_1\cdot\hat{r}_1)\,\hat{l}_1, \quad l_2 = \operatorname{sgn}(d\lambda_2\cdot\hat{r}_2)\,\hat{l}_2, \tag{2.10}$$

where $\hat{r}_i=(1,a_i)^t$, $\hat{l}_1=(-a_2,1)$ and $\hat{l}_2=(a_1,-1)$. It is easy to check that

$$l_i(u,v)\cdot r_j(u,v) = 0, \quad i,j=1,2, \quad i\neq j.$$

Now, let $U_0=(u_0,v_0)$ be a point in $\mathbb{R}^2$. By i-rarefaction curves through U_0 we mean curves in $U=(u,v)$ that satisfy the following differential equation

$$\frac{dv}{du} = a_i, \quad v(u_0)=v_0, \quad i=1,2. \tag{2.11}$$

We denote i-rarefaction curves by $R_i(U_0)$. On i-rarefaction curves, we have

$$\frac{d^2v}{du^2} = b_i, \quad i=1,2, \tag{2.12}$$

where

$$b_1 = \operatorname{sgn}(d\lambda_2\cdot\hat{r}_2)\frac{l_2 d^2F(r_1,r_1)}{\lambda_2-\lambda_1}, \quad b_2 = \operatorname{sgn}(d\lambda_1\cdot\hat{r}_1)\frac{l_1 d^2F(r_2,r_2)}{\lambda_2-\lambda_1}.$$

From (2.6) it follows that all rarefaction curves of both families are convex.

Next, by i-shock curves originating at U_0 we mean curves in $U = (u, v)$ that satisfy the Rankine-Hugoniot condition

$$\sigma_i(U - U_0) = F(U) - F(U_0), \quad i = 1, 2, \tag{2.13}$$

where $\sigma_i \equiv \sigma_i(U; U_0) \equiv \sigma_i(u, v; u_0, v_0)$ is the i-shock speed. We eliminate σ_i in (2.13) to get

$$(u - u_0)\big[g(u, v) - g(u_0, v_0)\big] = (v - v_0)\big[f(u, v) - f(u_0, v_0)\big]. \tag{2.14}$$

From (2.14), we see that differential equation of i-shock curve is

$$\frac{dv}{du} = \begin{cases} h_i & u \neq u_0 \ , \\ a_i & u = u_0 \ , \end{cases} \tag{2.15}$$

where

$$\begin{aligned} h_i = h_i(u, v; U_0) &= \frac{g(u, v) - g(u_0, v_0) + (u - u_0)g_u - (v - v_0)f_u}{f(u, v) - f(u_0, v_0) + (v - v_0)f_v - (u - u_0)g_v} \\ &= \frac{(u - u_0)g_u + (\sigma_i - f_u)(v - v_0)}{(v - v_0)f_v + (\sigma_i - g_v)(u - u_0)}. \end{aligned}$$

By applying an argument as in [14], if $u - u_0$ is small, then the solution v of (2.15) exists and is described by

$$v = v_0 + a_i(u - u_0) + \frac{1}{2}b_i(u - u_0)^2 + O((u - u_0)^3). \tag{2.16}$$

From (2.9), there are four shock curves which leave U_0 in $\pm r_i$ direction. We denote the shock curves by $S_i(U_0)$, $S_i^*(U_0)$, $i = 1, 2$, where $S_i(U_0)$ are those which leave U_0 in $-r_i$ direction, and $S_i^*(U_0)$ are those which leave U_0 in r_i direction. In general, $S_i(U_0)$ are called i-shock curves and $S_i^*(U_0)$ are called i-rarefaction shock curves (see [13]). Since

$$\left.\frac{d\sigma_i}{d\mu_i}\right|_{U_0} = \left.\frac{1}{2}\frac{d\lambda_i}{d\mu_i}\right|_{U_0} \quad i = 1, 2, \tag{2.17}$$

where $\dfrac{d}{d\mu_i} = \dfrac{\partial}{\partial u} + h_i\dfrac{\partial}{\partial v}$, we see that the shock speeds of $S_i(U_0)$ decrease as the corresponding eigenvalues decrease along $S_i(U_0)$ and the shock speeds of $S_i^*(U_0)$ increase as the corresponding eigenvalues increase along $S_i^*(U_0)$.

Since the shock curves are not always monotonic with respect to u (see Example 4.2), it is convenient to choose arc length s in the U-plane as a parameter for the shock curves. On a smooth arc of the shock curves, we may differentiate equation (2.14) with respect to s so that

$$\frac{du}{ds}\Big\{(\sigma_i - f_u)(v - v_0) + g_u(u - u_0)\Big\} = \frac{dv}{du}\Big\{(\sigma_i - g_v)(u - u_0) + f_v(v - v_0)\Big\}.$$

Since

$$\left|\frac{dU}{ds}\right| = \sqrt{\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2} = 1,$$

we have

$$\frac{dU}{ds} = \begin{pmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{pmatrix} = \pm K_i(U) \begin{pmatrix} (\sigma_i - g_v)(u-u_0) + f_v(v-v_0) \\ (\sigma_i - f_u)(v-v_0) + g_u(u-u_0) \end{pmatrix}, \tag{2.18}$$

where

$$K_i(U) = \frac{1}{\sqrt{\{(\sigma_i - f_u)(v-v_0) + g_u(u-u_0)\}^2 + \{(\sigma_i - g_v)(u-u_0) + f_v(v-v_0)\}^2}}.$$

Moreover, we have

$$\frac{d\sigma_i}{ds} = \mp K_i(U)(\sigma_i - \lambda_1)(\sigma_i - \lambda_2). \tag{2.19}$$

Note that (2.18) and (2.19) are different signs.

From (2.9), (2.15), (2.17), (2.18) and (2.19), we see that the shock curves are described by the following differential equations and the shock speeds satisfy the following equations:

(i) For $U = (u, v) \in S_1(U_0)$,

$$\frac{dU}{ds} = \begin{cases} -K_1(U) \begin{pmatrix} (\sigma_1 - g_v)(u-u_0) + f_v(v-v_0) \\ (\sigma_1 - f_u)(v-v_0) + g_u(u-u_0) \end{pmatrix} & U \neq U_0, \\ \dfrac{-\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1)}{\sqrt{1+a_1^2}} \begin{pmatrix} 1 \\ a_1 \end{pmatrix} & U = U_0, \end{cases} \tag{2.20}$$

$$\frac{d\sigma_1}{ds} = \begin{cases} K_1(U)(\sigma_1 - \lambda_1)(\sigma_1 - \lambda_2) & U \neq U_0, \\ \dfrac{-dr_1 \cdot r_1}{2\sqrt{1+a_1^2}} & U = U_0. \end{cases} \tag{2.21}$$

(ii) For $U = (u, v) \in S_2(U_0)$,

$$\frac{dU}{ds} = \begin{cases} K_2(U) \begin{pmatrix} (\sigma_2 - g_v)(u-u_0) + f_v(v-v_0) \\ (\sigma_2 - f_u)(v-v_0) + g_u(u-u_0) \end{pmatrix} & U \neq U_0, \\ \dfrac{-\operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)}{\sqrt{1+a_2^2}} \begin{pmatrix} 1 \\ a_2 \end{pmatrix} & U = U_0, \end{cases} \tag{2.22}$$

$$\frac{d\sigma_2}{ds} = \begin{cases} -K_2(U)(\sigma_2 - \lambda_1)(\sigma_2 - \lambda_2) & U \neq U_0, \\ \dfrac{-dr_2 \cdot r_2}{2\sqrt{1+a_2^2}} & U = U_0. \end{cases} \tag{2.23}$$

(iii) For $U = (u, v) \in S_1^*(U_0)$,

$$\frac{dU}{ds} = \begin{cases} -K_1(U)\begin{pmatrix}(\sigma_1 - g_v)(u-u_0) + f_v(v-v_0)\\(\sigma_1 - f_u)(v-v_0) + g_u(u-u_0)\end{pmatrix} & U \neq U_0,\\ \dfrac{\operatorname{sgn}(d\lambda_1\cdot\hat{r}_1)}{\sqrt{1+a_1^2}}\begin{pmatrix}1\\a_1\end{pmatrix} & U = U_0,\end{cases} \tag{2.24}$$

$$\frac{d\sigma_1}{ds} = \begin{cases} K_1(U)(\sigma_1-\lambda_1)(\sigma_1-\lambda_2) & U \neq U_0,\\ \dfrac{dr_1\cdot r_1}{2\sqrt{1+a_1^2}} & U = U_0.\end{cases} \tag{2.25}$$

(iv) For $U = (u, v) \in S_2^*(U_0)$,

$$\frac{dU}{ds} = \begin{cases} K_2(U)\begin{pmatrix}(\sigma_2 - g_v)(u-u_0) + f_v(v-v_0)\\(\sigma_2 - f_u)(v-v_0) + g_u(u-u_0)\end{pmatrix} & U \neq U_0,\\ \dfrac{\operatorname{sgn}(d\lambda_2\cdot\hat{r}_2)}{\sqrt{1+a_2^2}}\begin{pmatrix}1\\a_2\end{pmatrix} & U = U_0,\end{cases} \tag{2.26}$$

$$\frac{d\sigma_2}{ds} = \begin{cases} -K_2(U)(\sigma_2-\lambda_1)(\sigma_2-\lambda_2) & U \neq U_0,\\ \dfrac{dr_2\cdot r_2}{2\sqrt{1+a_2^2}} & U = U_0.\end{cases} \tag{2.27}$$

We are now in a position to state the main result on the existence of i-shock curves and i-rarefaction shock curves and the stability condition for shock speeds.

Theorem 2.1. *Let the system* (1.1) *satisfies* (2.2), (2.6) *and* (2.7) *in* $\mathbb{R}^2$, *and let w be a fixed vector of independent of U such that $r_1 \cdot w > 0$ and $r_2 \cdot w < 0$ for all U. Then, for any point $U_0 = (u_0, v_0)$ in $\mathbb{R}^2$, there exist four globally defined curves $S_i(U_0)$ and $S_i^*(U_0)$, $i = 1, 2$, satisfying the following properties:*

(i) *For* $U = (u, v) \in S_1(U_0)\backslash U_0$,

$$\frac{dU}{ds} = \alpha_1 r_1 + \beta_1 r_2 \quad \text{with } \alpha_1 < 0,\ \beta_1 < 0, \tag{2.28}$$

$$w\cdot\big(U - U_0\big) < 0, \tag{2.29}$$

$$\begin{cases} a_2 < a_1 < \dfrac{v-v_0}{u-u_0},\quad h_1 > 0 & \textit{if } \operatorname{sgn}(d\lambda_1\cdot\hat{r}_1) = \operatorname{sgn}(d\lambda_2\cdot\hat{r}_2),\\ a_2 < \dfrac{v-v_0}{u-u_0} < a_1 & \textit{if } \operatorname{sgn}(d\lambda_1\cdot\hat{r}_1) \neq \operatorname{sgn}(d\lambda_2\cdot\hat{r}_2),\end{cases} \tag{2.30}$$

$$\begin{cases} \dfrac{d}{ds}\left(\dfrac{v-v_0}{u-u_0}\right) < 0 & \textit{if } \operatorname{sgn}(d\lambda_1\cdot\hat{r}_1) = \operatorname{sgn}(d\lambda_2\cdot\hat{r}_2),\\ \dfrac{d}{ds}\left(\dfrac{v-v_0}{u-u_0}\right) > 0 & \textit{if } \operatorname{sgn}(d\lambda_1\cdot\hat{r}_1) \neq \operatorname{sgn}(d\lambda_2\cdot\hat{r}_2),\end{cases} \tag{2.31}$$

$$\frac{d\sigma_1}{ds} < 0, \tag{2.32}$$

$$\lambda_1(U) < \sigma_1(U;U_0) < \lambda_1(U_0), \tag{2.33}$$

$$\sigma_1(U;U_0) < \lambda_2(U). \tag{2.34}$$

(ii) *For* $U = (u,v) \in S_2(U_0)\backslash U_0$,

$$\frac{dU}{ds} = \alpha_2 r_1 + \beta_2 r_2 \quad \text{with } \alpha_2 < 0,\ \beta_2 < 0, \tag{2.35}$$

$$w \cdot (U - U_0) > 0, \tag{2.36}$$

$$\begin{cases} a_2 < \dfrac{v - v_0}{u - u_0} < a_1, \quad h_2 > 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \\ 0 < \dfrac{v - v_0}{u - u_0} < a_2 < a_1, \quad h_2 > 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \end{cases} \tag{2.37}$$

$$\begin{cases} \dfrac{d}{ds}\left(\dfrac{v - v_0}{u - u_0}\right) < 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \\ \dfrac{d}{ds}\left(\dfrac{v - v_0}{u - u_0}\right) > 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \end{cases} \tag{2.38}$$

$$\frac{d\sigma_2}{ds} < 0, \tag{2.39}$$

$$\lambda_2(U) < \sigma_2(U;U_0) < \lambda_2(U_0), \tag{2.40}$$

$$\lambda_1(U_0) < \sigma_2(U;U_0). \tag{2.41}$$

(iii) *For* $U = (u,v) \in S_1^*(U_0)\backslash U_0$,

$$\frac{dU}{ds} = \alpha_1^* r_1 + \beta_1^* r_2 \quad \text{with } \alpha_1^* > 0,\ \beta_1^* > 0, \tag{2.42}$$

$$w \cdot (U - U_0) > 0, \tag{2.43}$$

$$\begin{cases} a_2 < \dfrac{v - v_0}{u - u_0} < a_1, \quad h_1 > 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \\ a_2 < a_1 < \dfrac{v - v_0}{u - u_0}, \quad h_1 > 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \end{cases} \tag{2.44}$$

$$\begin{cases} \dfrac{d}{ds}\left(\dfrac{v - v_0}{u - u_0}\right) > 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \\ \dfrac{d}{ds}\left(\dfrac{v - v_0}{u - u_0}\right) < 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \end{cases} \tag{2.45}$$

$$\frac{d\sigma_1}{ds} > 0, \tag{2.46}$$

$$\lambda_1(U_0) < \sigma_1(U;U_0) < \lambda_1(U), \tag{2.47}$$

$$\sigma_1(U;U_0) < \lambda_2(U_0). \tag{2.48}$$

(iv) *For* $U = (u, v) \in S_2^*(U_0) \backslash U_0$,

$$\frac{dU}{ds} = \alpha_2^* r_1 + \beta_2^* r_2 \text{ with } \alpha_2^* > 0,\ \beta_2^* > 0, \tag{2.49}$$

$$w \cdot (U - U_0) < 0, \tag{2.50}$$

$$\begin{cases} 0 < \dfrac{v - v_0}{u - u_0} < a_2 < a_1, \quad h_2 > 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \\ a_2 < \dfrac{v - v_0}{u - u_0} < a_1 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \end{cases} \tag{2.51}$$

$$\begin{cases} \dfrac{d}{ds}\left(\dfrac{v - v_0}{u - u_0}\right) > 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \\ \dfrac{d}{ds}\left(\dfrac{v - v_0}{u - u_0}\right) < 0 & \textit{if } \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2), \end{cases} \tag{2.52}$$

$$\frac{d\sigma_2}{ds} > 0, \tag{2.53}$$

$$\lambda_2(U_0) < \sigma_2(U; U_0) < \lambda_2(U), \tag{2.54}$$

$$\lambda_1(U) < \sigma_2(U; U_0). \tag{2.55}$$

We note that inequalities (2.33) and (2.40) are *the shock condition*, and inequalities (2.34) and (2.41) are *the stability condition for shock speeds.*

Remark 2.2. It should be noted that if $\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)$, then the half-plane condition (2.7) is satisfied, but if $\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)$, then the half-plane condition (2.7) is not always satisfied (see Example 4.3).

3 Proof of the theorem

In this section we shall give a proof of Theorem 2.1. Before proving the theorem, we shall state a couple of preliminary results.

Using (2.2) we can easily prove the following result on the Hugoniot locus which is defined by

$$H(U_0) = \{U \mid \exists\, \sigma : \sigma(U - U_0) = F(U) - F(U_0)\}.$$

Lemma 3.1. *Let* $U_0 = (u_0, v_0)$ *and let the system* (2.1) *satisfies* (2.2) *in* $\mathbb{R}^2$. *We have the following:*

(i) *For any* $v \neq v_0$, $(u_0, v) \notin H(U_0)$.

(ii) *For any* $u \neq u_0$, $(u, v_0) \notin H(U_0)$.

The following result gives necessary conditions for the singular points of $S_i(U_0)$ and $S_i^*(U_0)$, $i = 1, 2$:

Lemma 3.2. *Let* $U_0 = (u_0, v_0)$ *and let the system* (2.1) *satisfies* (2.2) *in* $\mathbb{R}^2$. *If for* $U \neq U_0$, *the denominator of* $K_i(U)$, $i = 1, 2$, *is zero, then* $\sigma_i(U; U_0) = \lambda_1(U)$ *or* $\lambda_2(U)$.

Proof. Let $U = (u, v) \neq U_0$. Since the denominator of $K_i(U)$ is zero, we have

$$(\sigma_i - g_v)(u - u_0) + f_v(v - v_0) = (\sigma_i - f_u)(v - v_0) + g_u(u - u_0) = 0.$$

This means that $u = u_0$ if and only if $v = v_0$. Since $u \neq u_0$ by assumption, we have

$$\frac{v - v_0}{u - u_0} = -\frac{\sigma_i - g_v}{f_v} = -\frac{g_u}{\sigma_i - f_u}.$$

From this it follows that

$$(\sigma_i - \lambda_1)(\sigma_i - \lambda_2) = 0.$$

Thus, Lemma 3.2 is proved. □

By Lemma 3.2, we see that the shock curves are defined and nonsingular except at points where the shock speeds are equal to an eigenvalue of dF.

We now prove Theorem 2.1. The proof of Theorem 2.1 shall be given in four steps. In the rest of this paper, we assume that conditions (2.2), (2.6) and (2.7) are satisfied.

Step 1. Let $U_0 = (u_0, v_0)$ be a point in $\mathbb{R}^2$. We provisionally assume the following conditions:

$$\lambda_1(U) < \sigma_1(U; U_0) < \lambda_2(U) \quad \text{for } U \in S_1(U_0)\backslash U_0, \tag{3.1}$$
$$\lambda_2(U) < \sigma_2(U; U_0) \quad \text{for } U \in S_2(U_0)\backslash U_0, \tag{3.2}$$
$$\sigma_1(U; U_0) < \lambda_1(U) \quad \text{for } U \in S_1^*(U_0)\backslash U_0, \tag{3.3}$$
$$\lambda_1(U) < \sigma_2(U; U_0) < \lambda_2(U) \quad \text{for } U \in S_2^*(U_0)\backslash U_0. \tag{3.4}$$

By Lemma 3.2, $S_i(U_0)$ and $S_i^*(U_0)$, $i = 1, 2$ must either extend as a simple arc without singularities to infinity or return eventually to U_0. In this step, we show that conditions (3.1)–(3.4) guarantee the global existence of $S_i(U_0)$ and $S_i^*(U_0)$, $i = 1, 2$.

Proposition 3.3. *Let $U_0 = (u_0, v_0)$ in $\mathbb{R}^2$. We have the following:*

(i) *If* (3.1) *holds, then there exists a globally defined curve $S_1(U_0)$ satisfying* (2.28)-(2.34) *for $U = (u, v) \in S_1(U_0)\backslash U_0$.*

(ii) *If* (3.2) *holds, then there exists a globally defined curve $S_2(U_0)$ satisfying* (2.35)-(2.40) *for $U = (u, v) \in S_2(U_0)\backslash U_0$.*

(iii) *If* (3.3) *holds, then there exists a globally defined curve $S_1^*(U_0)$ satisfying* (2.42)-(2.47) *for $U = (u, v) \in S_1^*(U_0)\backslash U_0$.*

(iv) *If* (3.4) *holds, then there exists a globally defined curve $S_2^*(U_0)$ satisfying* (2.49)-(2.55) *for $U = (u, v) \in S_2^*(U_0)\backslash U_0$.*

Proof. We only prove (i) and (ii), because (iii) and (iv) are proved by arguments similar to those of (i) and (ii).

We begin with the proof of (i). By Lemma 3.2, it is obvious that (2.32), (2.33) and (2.34) hold for $U \in S_1(U_0)\backslash U_0$. Moreover, by Lemma 3.1, it follows that

$$\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1)(u - u_0) < 0, \quad \operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1)(v - v_0) < 0 \tag{3.5}$$

for $U = (u, v) \in S_1(U_0)\backslash U_0$.

The proofs of (2.30) and (2.31) are given in the following cases:

Case 1. $\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)$,

Case 2. $\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)$.

Inequalities (2.30) and (2.31) in Case 1: Since

$$\left.\frac{da_1}{ds}\right|_{U_0} = \frac{-\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1)b_1}{\sqrt{1+a_1^2}} = \frac{-l_2 d^2F(r_1, r_1)}{\sqrt{1+a_1^2}(\lambda_2 - \lambda_1)} < 0,$$

$$\left.\frac{d}{ds}\left(\frac{v - v_0}{u - u_0}\right)\right|_{U_0} = \frac{-\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1)b_1}{2\sqrt{1+a_1^2}} = \frac{-l_2 d^2F(r_1, r_1)}{2\sqrt{1+a_1^2}(\lambda_2 - \lambda_1)} < 0,$$

$$\lim_{U \to U_0} \frac{v - v_0}{u - u_0} = a_1(U_0),$$

we have

$$a_2 < a_1 < \frac{v - v_0}{u - u_0}, \tag{3.6}$$

$$\frac{d}{ds}\left(\frac{v - v_0}{u - u_0}\right) < 0 \tag{3.7}$$

for U close to U_0.

If inequality (3.6) is not true all along $S_1(U_0)$, then there is the first point $U_1 = (u_1, v_1) \neq U_0$ such that

$$a_1(U_1) = \frac{v_1 - v_0}{u_1 - u_0}.$$

We then have

$$\left.\frac{dU}{ds}\right|_{U_1} = -K_1(U_1)\begin{pmatrix} (\sigma_1 - \lambda_2)(u_1 - u_0) \\ a_1(\sigma_1 - \lambda_2)(u_1 - u_0) \end{pmatrix}$$

so that

$$\left.\frac{d}{ds}\left(\frac{v - v_0}{u - u_0}\right)\right|_{U_1} = \frac{-K_1(U_1)}{(u_1 - u_0)^2}\left\{a_1(\sigma_1 - \lambda_2)(u_1 - u_0)^2 - a_1(\sigma_1 - \lambda_2)(u_1 - u_0)^2\right\} = 0.$$

However, since

$$\left.\frac{da_1}{ds}\right|_{U_1} < 0,$$

we have

$$\left.\frac{d}{ds}\left[a_1 - \frac{v - v_0}{u - u_0}\right]\right|_{U_1} < 0.$$

This implies a contradiction. Thus, inequality (3.6) is proved.

Next, if inequality (3.7) is not true all along $S_1(U_0)$, then there is the first point $U_1 = (u_1, v_1) \neq U_0$ such that

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\bigg|_{U_1} = 0.$$

We then have $\dfrac{v_1 - v_0}{u_1 - u_0} = a_1$ or a_2. This contradicts to inequality (3.6) and the proof of (3.7) is complete.

Inequality $h_1 > 0$ follows from (3.5) and (3.6). Thus, (2.30) and (2.31) in Case 1 are proved.

Inequalities (2.30) and (2.31) in Case 2: Since

$$\frac{da_1}{ds}\bigg|_{U_0} = \frac{-\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) b_1}{\sqrt{1+a_1^2}} = \frac{l_2 d^2 F(r_1, r_1)}{\sqrt{1+a_1^2}(\lambda_2 - \lambda_1)} > 0,$$

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\bigg|_{U_0} = \frac{-\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) b_1}{2\sqrt{1+a_1^2}} = \frac{l_2 d^2 F(r_1, r_1)}{2\sqrt{1+a_1^2}(\lambda_2 - \lambda_1)} > 0,$$

$$\lim_{U \to U_0} \frac{v-v_0}{u-u_0} = a_1(U_0),$$

we have

$$a_2 < \frac{v-v_0}{u-u_0} < a_1, \tag{3.8}$$

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right) > 0 \tag{3.9}$$

for U close to U_0.

If the right side of inequality (3.8) is not true all along $S_1(U_0)$, then there is the first point $U_1 = (u_1, v_1) \neq U_0$ such that

$$a_1(U_1) = \frac{v_1 - v_0}{u_1 - u_0}.$$

We then have

$$\frac{dU}{ds}\bigg|_{U_1} = -K_1(U_1)\begin{pmatrix}(\sigma_1 - \lambda_2)(u_1 - u_0)\\ a_1(\sigma_1 - \lambda_2)(u_1 - u_0)\end{pmatrix}$$

so that

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\bigg|_{U_1} = \frac{-K_1(U_1)}{(u_1-u_0)^2}\left\{a_1(\sigma_1 - \lambda_2)(u_1 - u_0)^2 - a_1(\sigma_1 - \lambda_2)(u_1 - u_0)^2\right\} = 0.$$

However, since

$$\frac{da_1}{ds}\bigg|_{U_1} > 0,$$

we have

$$\frac{d}{ds}\left[a_1 - \frac{v-v_0}{u-u_0}\right]\bigg|_{U_1} > 0.$$

This implies a contradiction. Thus, the right side of inequality (3.8) is proved.

If the left side of inequality (3.8) is not true all along $S_1(U_0)$, then there is the first point $U_1 = (u_1, v_1) \neq U_0$ such that

$$a_2(U_1) = \frac{v_1 - v_0}{u_1 - u_0}.$$

We then have

$$\left.\frac{dU}{ds}\right|_{U_1} = -K_1(U_1)\begin{pmatrix}(\sigma_1 - \lambda_1)(u_1 - u_0)\\ a_2(\sigma_1 - \lambda_1)(u_1 - u_0)\end{pmatrix}$$

so that

$$\left.\frac{d}{ds}\left(\frac{v - v_0}{u - u_0}\right)\right|_{U_1} = 0.$$

Noting that $\frac{v - v_0}{u - u_0} = a_1$ or a_2 for $U = (u, v) \neq U_0$ such that

$$\frac{d}{ds}\left(\frac{v - v_0}{u - u_0}\right) = 0,$$

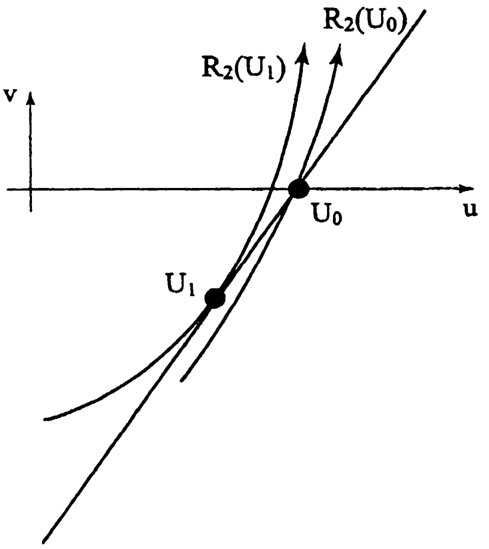
we see that

$$a_2(U_0) < a_1(U_0) < \frac{v_1 - v_0}{u_1 - u_0} = a_2(U_1). \tag{3.10}$$

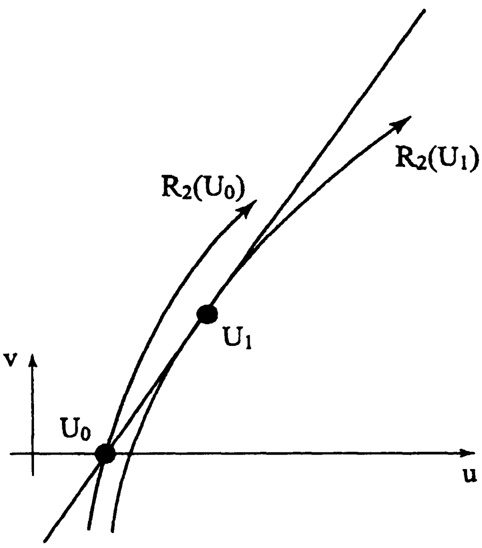
However, since 2-rarefaction curves are convex, U_0 is outside $R_2(U_1)$. Hence, $R_2(U_0)$ intersects the line jointing U_0 and U_1 at U_0, as in Figure 1, so that

$$\frac{v_1 - v_0}{u_1 - u_0} < a_2(U_0).$$

This contradicts to inequality (3.10) and the proof of (3.8) is complete.



The situation for $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1)=1$.



The situation for $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1)=-1$.

Figure 1: The figure where $R_2(U_0)$ intersects the line jointing U_0 and U_1 at U_0.

Next, if inequality (3.9) is not true all along $S_1(U_0)$, then there is the first point $U_1 = (u_1, v_1) \neq U_0$ such that

$$\frac{d}{ds}\left(\frac{v - v_0}{u - u_0}\right)\bigg|_{U_1} = 0.$$

We then have $\dfrac{v_1 - v_0}{u_1 - u_0} = a_1$ or a_2. This contradicts to inequality (3.8) and the proof of (3.9) is complete. Thus, (2.30) and (2.31) in Case 2 are proved.

Now, we prove (2.28) and (2.29). Since

$$\frac{dU}{ds} \cdot l_1 = -\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) K_1(U)(\lambda_2 - \sigma_1)\{a_2(u - u_0) - (v - v_0)\} < 0,$$
$$\frac{dU}{ds} \cdot l_2 = -\mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2) K_1(U)(\sigma_1 - \lambda_1)\{a_1(u - u_0) - (v - v_0)\} < 0,$$

we see that

$$\frac{dU}{ds} = \alpha_1 r_1 + \beta_1 r_2 \quad \text{with } \alpha_1 < 0,\ \beta_1 < 0.$$

Hence, (2.28) is proved. Moreover, by differentiating (2.13) with respect to s, we have

$$\frac{d\sigma_1}{ds}(U - U_0) = (dF - \sigma_1 I)\frac{dU}{ds}$$

so that

$$\frac{d\sigma_1}{ds}(U - U_0) = (dF - \sigma_1 I)(\alpha_1 r_1 + \beta_1 r_2) = \alpha_1(\lambda_1 - \sigma_1) r_1 + \beta_1(\lambda_2 - \sigma_1) r_2.$$

Therefore, for w satisfying $w \cdot r_1 > 0$ and $w \cdot r_2 < 0$, it follows from (2.28), (2.32), (2.33) and (2.34) that

$$\frac{d\sigma_1}{ds}\, w \cdot (U - U_0) > 0.$$

Hence, (2.29) is proved.

Finally, it follows from (2.28) that $S_1(U_0)$ cannot return to U_0. Thus, $S_1(U_0)$ is a simple arc extending from U_0 to infinity.

We next proceed to prove (ii). By Lemma 3.2, it is obvious that (2.39) and (2.40) hold for $U \in S_2(U_0)\backslash U_0$. Moreover, by Lemma 3.1, it follows that

$$\mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2)(u - u_0) < 0, \quad \mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2)(v - v_0) < 0 \tag{3.11}$$

for $U = (u, v) \in S_2(U_0)\backslash U_0$.

The proofs of (2.37) and (2.38) are given in the cases of $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) = \mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2)$ and $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2)$. The proofs are only given in the case of $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) = \mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2)$, because the proof in the case of $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2)$ is similar to those of (2.30) and (2.31) in the case of $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) = \mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2)$.

If $\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)$, then we have

$$\left.\frac{da_2}{ds}\right|_{U_0} = \frac{-\operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)b_2}{\sqrt{1+a_2^2}} = \frac{-l_1 d^2F(r_2, r_2)}{\sqrt{1+a_2^2}(\lambda_2 - \lambda_1)} < 0,$$

$$\left.\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\right|_{U_0} = \frac{-\operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)b_2}{2\sqrt{1+a_2^2}} = \frac{-l_1 d^2F(r_2, r_2)}{2\sqrt{1+a_2^2}(\lambda_2 - \lambda_1)} < 0,$$

$$\lim_{U \to U_0} \frac{v-v_0}{u-u_0} = a_2(U_0)$$

so that

$$a_2 < \frac{v-v_0}{u-u_0} < a_1, \tag{3.12}$$

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right) < 0 \tag{3.13}$$

for U close to U_0.

If the left side of inequality (3.12) is not true all along $S_2(U_0)$, then there is the first point $U_1 = (u_1, v_1) \neq U_0$ such that

$$a_2(U_1) = \frac{v_1 - v_0}{u_1 - u_0}.$$

We then have

$$\left.\frac{dU}{ds}\right|_{U_1} = -K_2(U_1)\begin{pmatrix}(\sigma_2 - \lambda_1)(u_1 - u_0)\\ a_2(\sigma_2 - \lambda_1)(u_1 - u_0)\end{pmatrix}$$

so that

$$\left.\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\right|_{U_1} = \frac{-K_2(U_1)}{(u_1-u_0)^2}\left\{a_2(\sigma_2 - \lambda_1)(u_1-u_0)^2 - a_2(\sigma_2 - \lambda_1)(u_1-u_0)^2\right\} = 0.$$

However, since

$$\left.\frac{da_2}{ds}\right|_{U_1} < 0,$$

we have

$$\left.\frac{d}{ds}\left[a_2 - \frac{v-v_0}{u-u_0}\right]\right|_{U_1} < 0.$$

This implies a contradiction. Thus, the left side of inequality (3.12) is proved.

If the right side of inequality (3.12) is not true all along $S_2(U_0)$, then there is the first point $U_1 = (u_1, v_1) \neq U_0$ such that

$$a_1(U_1) = \frac{v_1 - v_0}{u_1 - u_0}.$$

We then have

$$\left.\frac{dU}{ds}\right|_{U_1} = -K_2(U_1)\begin{pmatrix}(\sigma_2-\lambda_2)(u_1-u_0)\\ a_1(\sigma_2-\lambda_2)(u_1-u_0)\end{pmatrix}$$

so that

$$\left.\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\right|_{U_1} = 0.$$

Noting that $\dfrac{v-v_0}{u-u_0} = a_1$ or a_2 for $U = (u,v) \neq U_0$ such that

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right) = 0,$$

we see that

$$a_1(U_1) = \frac{v_1-v_0}{u_1-u_0} < a_2(U_0) < a_1(U_0).$$

By the continuous dependence of $S_2(U_0)$, there exists a point $\hat{U} \in S_2(U_0)$ such that $a_1(\hat{U}) = a_2(U_0)$. This contradicts to the half-plane condition (2.7) and the proof of (3.12) is complete.

Next, if inequality (3.13) is not true all along $S_2(U_0)$, then there is the first point $U_1 = (u_1, v_1) \neq U_0$ such that

$$\left.\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\right|_{U_1} = 0.$$

We then have $\dfrac{v_1-v_0}{u_1-u_0} = a_1$ or a_2. This contradicts to inequality (3.12) and the proof of (3.13) is complete.

By (3.11) and (3.12) it is easy to check that $h_2 > 0$. Thus, (2.37) and (2.38) are proved.

Now, we prove (2.35) and (2.36). Since

$$\frac{dU}{ds}\cdot l_1 = \operatorname{sgn}(d\lambda_1\cdot\hat{r}_1)K_2(U)(\lambda_2-\sigma_2)\{a_2(u-u_0)-(v-v_0)\} < 0,$$
$$\frac{dU}{ds}\cdot l_2 = \operatorname{sgn}(d\lambda_2\cdot\hat{r}_2)K_2(U)(\sigma_2-\lambda_1)\{a_1(u-u_0)-(v-v_0)\} < 0,$$

we see that

$$\frac{dU}{ds} = \alpha_2 r_1 + \beta_2 r_2 \quad\text{with } \alpha_2 < 0,\ \beta_2 < 0.$$

Hence, (2.35) is proved. Moreover, from (2.35) and (2.38), we see that $S_2(U_0)$ lies on the outside of $R_2(U_0)$ and is starlike with respect to U_0 so that

$$w\cdot(U-U_0) > 0.$$

Hence, (2.36) is proved.

Finally, it follows from (2.35) that $S_2(U_0)$ cannot return to U_0. Thus, $S_2(U_0)$ is a simple arc extending from U_0 to infinity. □

Step 2. In this step we shall prove (3.2) and (3.3). Because of Proposition 3.3, it is sufficient to prove the existence of $S_2(U_0)$ and $S_1^*(U_0)$.

We only prove (3.2), because (3.3) is proved by arguments similar to the proof of (3.2).

Let $U = (u, v) \in S_2(U_0) \backslash U_0$. Since

$$\left.\frac{d\sigma_2}{ds}\right|_{U_0} = \frac{1}{2}\left.\frac{d\lambda_2}{ds}\right|_{U_0},$$

it is obvious that (3.2) holds for U close to U_0. If this inequality is not true all along $S_2(U_0)$, then there is the first point $U_1 = (u_1, v_1) \neq U_0$ such that $\sigma_2(U_1; U_0) = \lambda_2(U_1)$. Then, by the dependence of $S_2(U_0)$, we see that $\frac{v_1 - v_0}{u_1 - u_0} \leq a_1(U_1)$.

If $\frac{v_1 - v_0}{u_1 - u_0} < a_1(U_1)$, then we can define $K_2(U_1)$ so that

$$\left.\frac{d\sigma_2}{ds}\right|_{U_1} = 0.$$

Since

$$\left.\frac{dU}{ds}\right|_{U_1} = \{f_v(v_1 - v_0) + (\lambda_2 - g_v)(u_1 - u_0)\}K_2(U_1)\begin{pmatrix}1\\ a_2\end{pmatrix},$$

we have

$$\left.\frac{d\lambda_2}{ds}\right|_{U_1} = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)\{f_v(v_1 - v_0) + (\lambda_2 - g_v)(u_1 - u_0)\}K_2(U_1)D\lambda_2 \cdot r_2|_{U_1} < 0.$$

From this it follows that

$$\frac{d}{ds}\{\sigma_2 - \lambda_2\}|_{U_1} < 0.$$

This implies a contradiction.

If $\frac{v_1 - v_0}{u_1 - u_0} = a_1(U_1)$, then we have $\operatorname{sgn}(d\lambda_1 \cdot \hat{r}_1) = \operatorname{sgn}(d\lambda_2 \cdot \hat{r}_2)$. Therefore, we see that

$$a_1(U_1) = \frac{v_1 - v_0}{u_1 - u_0} < a_2(U_0) < a_1(U_0).$$

By the continuous dependence of $S_2(U_0)$, there exists a point $\hat{U} \in S_2(U_0)$ such that $a_1(\hat{U}) = a_2(U_0)$. This contradicts to the half-plane condition (2.7) and the proof of (3.2) is complete.

Step 3. In Step 1 and Step 2, it is shown that there exist globally defined curves $S_2(U_0)$ and $S_1^*(U_0)$ satisfying (2.35)–(2.40) and (2.42)–(2.47), respectively. In this step, we shall prove the stability conditions (2.41) and (2.48). The key is the continuous dependence on initial points and the geometric properties of the shock curves.

To prove (2.41) and (2.48), we need the following result:

Lemma 3.4. *Let $U_0 = (u_0, v_0)$ in $\mathbb{R}^2$. We have the following:*

(i) *For $U_1 \in S_1^*(U_0)\backslash U_0$, $S_2(U_1)$ does not intersect $S_2(U_0)$.*

(ii) *For $U_1 \in S_2(U_0)\backslash U_0$, $S_1^*(U_1)$ does not intersect $S_1^*(U_0)$.*

Proof. Without loss of generality, we may suppose that $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) = 1$. We only prove (i), because (ii) is proved by arguments similar to the proof of (i).

Let $U_1 \in S_1^*(U_0)\backslash U_0$ and suppose that $S_2(U_1)$ intersects $S_2(U_0)$ at $U_2 = (u_2, v_2)$. By the Rankine-Hugoniot condition, we have

$$\begin{aligned} \sigma_{11}(U_1 - U_0) &= F(U_1) - F(U_0), \\ \sigma_{21}(U_2 - U_0) &= F(U_2) - F(U_0), \\ \sigma_{22}(U_2 - U_1) &= F(U_2) - F(U_1), \end{aligned}$$

where $\sigma_{11} = \sigma_1(U_1; U_0)$, $\sigma_{21} = \sigma_2(U_2; U_0)$ and $\sigma_{22} = \sigma_2(U_2; U_1)$. Therefore, we have

$$(\sigma_{21} - \sigma_{11})(U_2 - U_0) + (\sigma_{11} - \sigma_{22})(U_2 - U_1) = 0. \tag{3.14}$$

Since the vectors $U_2 - U_0$ and $U_2 - U_1$ are linearly independent, this means that

$$\sigma_{11} = \sigma_{12} = \sigma_{22}.$$

Next, if for $\bar{U}_1 = (\bar{u}_1, \bar{v}_1) \in S_1^*(U_0)$ with $\bar{u}_1 \geq u_1$, $S_2(\bar{U}_1)$ intersects $S_2(U_0)$ at $\bar{U}_2 = (\bar{u}_2, \bar{v}_2)$, then $\sigma_1(\bar{U}_1; U_0) = \sigma_2(\bar{U}_2; U_0)$. From (2.36), (2.39), (2.43) and (2.46), it follows that $S_2(\bar{U}_1)$ does not intersect $S_2(U_0)$ for $\bar{U}_1 = (\bar{u}_1, \bar{v}_1) \in S_1^*(U_0)$ with sufficiently large $\bar{u}_1 > u_1$ (see Figure 2).

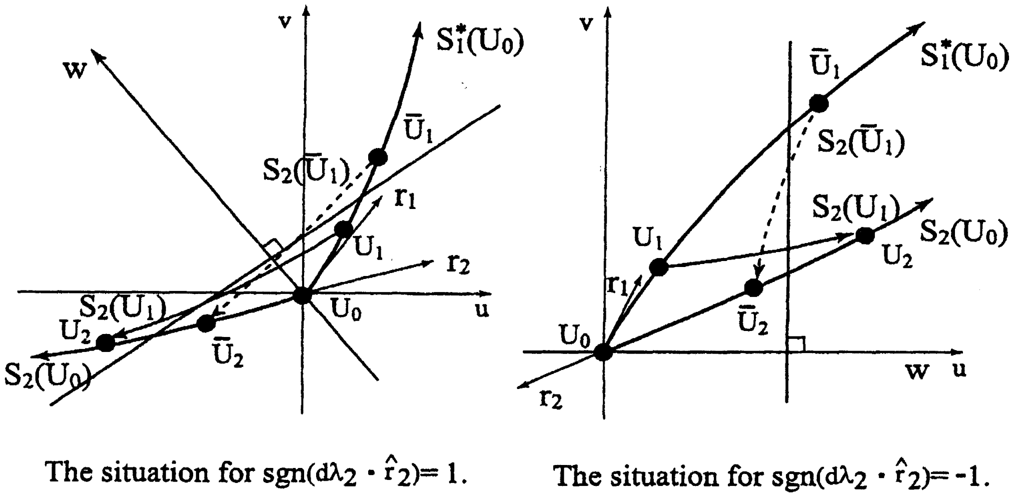
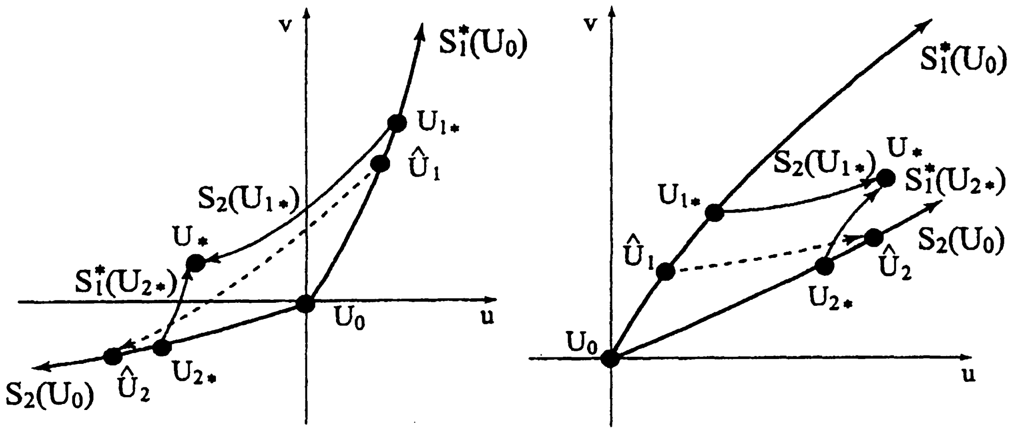


Figure 2: The situation for sufficiently large $\bar{u}_1 > u_1$.

Therefore, we see that there exist $\hat{U}_1 = (\hat{u}_1, \hat{v}_1) \in S_1^*(U_0)$ and $U_{1*} = (u_{1*}, v_{1*}) \in S_1^*(U_0)$ close to $\hat{U}_1$ with $\hat{u}_1 < u_{1*}$ satisfying the following: $S_2(\hat{U}_1)$ intersects $S_2(U_0)$ at $\hat{U}_2$ and $S_2(U_{1*})$ does not intersect $S_2(U_0)$. By the continuous dependence of shock curves on initial points, we then see that there exists $U_{2*} \in S_2(U_0)$ such that $\sigma_{11} = \sigma_{21}$ where $\sigma_{11} = \sigma_1(U_{1*}; U_0)$ and $\sigma_{21} = \sigma_2(U_{2*}; U_0)$, and $S_2(U_{1*})$ intersects $S_1^*(U_{2*})$ at some U_*, as in Figure 3. Here we used the fact that $\sigma_1(\hat{U}_1; U_0) = \sigma_2(\hat{U}_2; U_0)$.



The situation for sgn($d\lambda_2 \cdot \hat{r}_2$)= 1. The situation for sgn($d\lambda_2 \cdot \hat{r}_2$)= -1.

Figure 3: The figure where $S_2(U_{1*})$ intersects $S_1^*(U_{2*})$ at U_*.

By the Rankine-Hugoniot condition, we have

$$\begin{aligned}
\sigma_{12}(U_* - U_{2*}) &= F(U_*) - F(U_{2*}), \\
\sigma_{22}(U_* - U_{1*}) &= F(U_*) - F(U_{1*}), \\
\sigma_{11}(U_{1*} - U_{2*}) &= F(U_{1*}) - F(U_{2*}),
\end{aligned}$$

where $\sigma_{12} = \sigma_1(U_*; U_{2*})$ and $\sigma_{22} = \sigma_2(U_*; U_{1*})$. Therefore, it follows that

$$(\sigma_{22} - \sigma_{11})(U_* - U_{1*}) + (\sigma_{11} - \sigma_{12})(U_* - U_{2*}) = 0. \tag{3.15}$$

Since the vectors $U_* - U_{1*}$ and $U_* - U_{2*}$ are linearly independent, this means that

$$\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22}.$$

However, because of (2.40) and (2.47), we have

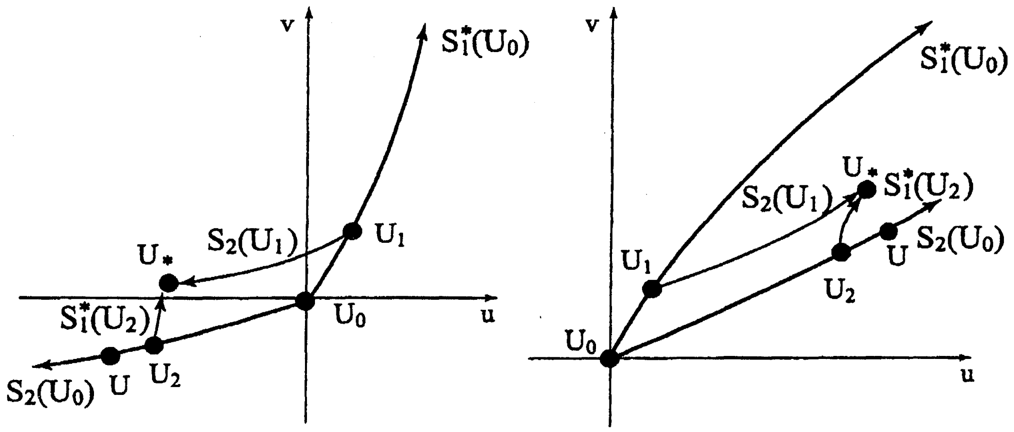
$$\lambda_1(U_{2*}) < \sigma_{12} < \lambda_1(U_*) < \lambda_2(U_*) < \sigma_{22} < \lambda_2(U_{1*}).$$

This is a contradiction. Therefore, it is proved that $S_1^*(U_2)$ does not intersect $S_1^*(U_0)$. $\square$

By using Lemma 3.4, we can prove (2.41) and (2.48). Without loss of generality, we may suppose that $\text{sgn}(d\lambda_1 \cdot \hat{r}_1) = 1$. We only prove (2.41), because (2.48) is proved by arguments similar to the proof of (2.41).

Because of $\lambda_1(U_0) < \lambda_2(U_0) = \sigma_2(U_0; U_0)$, we have $\lambda_1(U_0) < \sigma_2(U; U_0)$ for U close to U_0. Suppose that (2.41) is not true all along $S_2(U_0)$ and let U be the first point such that $\sigma_2(U; U_0) = \lambda_1(U_0)$. It follows from (2.39) and (2.46) that there exist $U_1 = (u_1, v_1) \in$

$S_1^*(U_0)$ and $U_2 = (u_2, v_2) \in S_2(U_0)$ close to U such that $\sigma_{11} = \sigma_{21}$ where $\sigma_{11} = \sigma_1(U_1; U_0)$ and $\sigma_{21} = \sigma_2(U_2; U_0)$, with $u_0 < u_1$. By Lemma 3.4 and the continuous dependence of shock curves on initial points, we see that $S_2(U_1)$ intersects $S_1^*(U_2)$ at some U_*, as in Figure 4.



The situation for sgn($d\lambda_2 \cdot \hat{r}_2$)= 1. The situation for sgn($d\lambda_2 \cdot \hat{r}_2$)= -1.

Figure 4: The figure where $S_2(U_1)$ intersects $S_1^*(U_2)$ at U_*.

By the Rankine-Hugoniot condition, we have

$$\begin{aligned}
\sigma_{12}(U_* - U_2) &= F(U_*) - F(U_2),\\
\sigma_{22}(U_* - U_1) &= F(U_*) - F(U_1),\\
\sigma_{11}(U_1 - U_2) &= F(U_1) - F(U_2),
\end{aligned}$$

where $\sigma_{12} = \sigma_1(U_*; U_2)$ and $\sigma_{22} = \sigma_2(U_*; U_1)$. Therefore, it follows that

$$(\sigma_{22} - \sigma_{11})(U_* - U_1) + (\sigma_{11} - \sigma_{12})(U_* - U_2) = 0. \tag{3.16}$$

Since the vectors $U_* - U_1$ and $U_* - U_2$ are linearly independent, this means that

$$\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22}.$$

This implies a contradiction and the proof of (2.41) is complete.

Step 4. In this step, we shall prove (3.1) and (3.4). Because of Proposition 3.3, it is sufficient to prove the statements for $S_1(U_0)$ and $S_2^*(U_0)$ in Theorem 2.1.

The key to prove (3.1) and (3.4) is the following lemma which represents "the reciprocity relationship" between shock curves and rarefaction shock curves (cf. [8]).

Lemma 3.5. *Let $U_0 = (u_0, v_0)$ in $\mathbb{R}^2$. We have the following:*

(i) *If (3.1) holds, then $U \in S_1(U_0)\backslash U_0$ implies $U_0 \in S_1^*(U)$.*

(ii) *If (3.4) holds, then $U \in S_2^*(U_0)\backslash U_0$ implies $U_0 \in S_2(U)$.*

Proof. Without loss of generality, we may suppose that $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) = 1$. We only prove (i), because (ii) is proved by arguments similar to the proof of (i).

Let $U \in S_1(U_0)\backslash U_0$. We first show that $S_1^*(U)$ does not intersect $S_1(U_0)$. To this end, suppose that $S_1^*(U)$ intersects $S_1(U_0)$ at $U_1 \neq U_0$. By the Rankine-Hugoniot condition, we have

$$\begin{aligned}\sigma_{11}(U - U_0) &= F(U) - F(U_0),\\ \sigma_{12}(U_1 - U_0) &= F(U_1) - F(U_0),\\ \sigma_{13}(U_1 - U) &= F(U_1) - F(U),\end{aligned}$$

where $\sigma_{11} = \sigma_1(U; U_0)$, $\sigma_{12} = \sigma_1(U_1; U_0)$ and $\sigma_{13} = \sigma_1(U_1; U)$. Therefore, it follows that

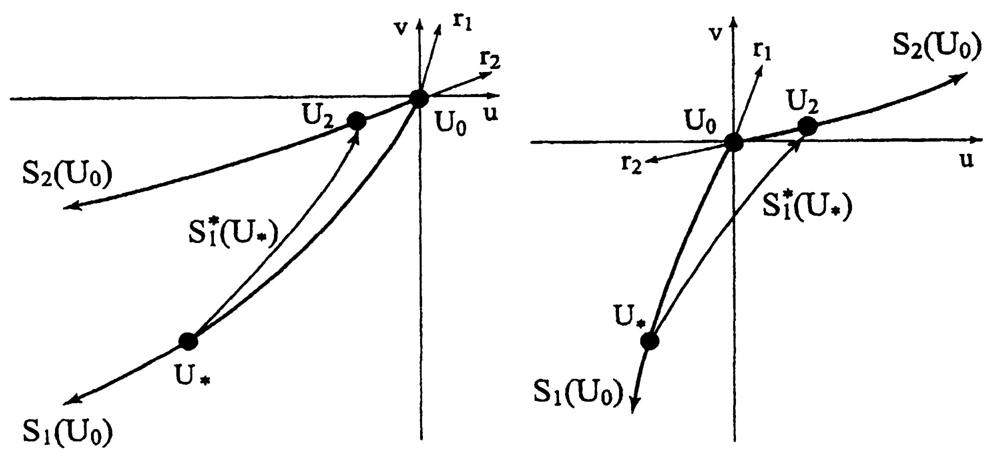
$$(\sigma_{11} - \sigma_{13})(U - U_0) + (\sigma_{13} - \sigma_{12})(U_1 - U_0) = 0. \tag{3.17}$$

Because of (2.31), the vectors $U - U_0$ and $U_1 - U_0$ are linearly independent. This means that

$$\sigma_{11} = \sigma_{12} = \sigma_{13}.$$

But this contradicts to the fact that $\sigma_{11} < \sigma_{12}$.

Now, suppose that (i) is not true all along $S_1(U_0)$. Since it is known (cf. [8] and [9]) that $S_1^*(U)$ passes through U_0 for U close to U_0, we then see that there exists a point $U_* \in S_1(U_0)$ such that $S_1^*(U_*)$ does not pass through U_0. From property (2.28) and the fact that $S_1^*(U_*)$ does not intersect $S_1(U_0)$, it follows that $S_1^*(U_*)$ intersects $S_2(U_0)$ at $U_2 \neq U_0$ (see Figure 5).



The situation for $\mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2) = 1$. The situation for $\mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2) = -1$.

Figure 5: The figure where $S_1^*(U_*)$ intersects $S_2(U_0)$ at U_2.

By the Rankine-Hugoniot condition, we have

$$\begin{aligned}\sigma_{11}(U_* - U_0) &= F(U_*) - F(U_0),\\ \sigma_{12}(U_2 - U_*) &= F(U_2) - F(U_*),\\ \sigma_{21}(U_2 - U_0) &= F(U_2) - F(U_0),\end{aligned}$$

where $\sigma_{11} = \sigma_1(U_*; U_0)$, $\sigma_{12} = \sigma_1(U_2; U_*)$ and $\sigma_{21} = \sigma_2(U_2; U_0)$. Therefore, we obtain

$$(\sigma_{12} - \sigma_{11})(U_* - U_2) + (\sigma_{11} - \sigma_{21})(U_0 - U_2) = 0. \tag{3.18}$$

Since the vectors $U_* - U_2$ and $U_0 - U_2$ are linearly independent, this means that

$$\sigma_{11} = \sigma_{12} = \sigma_{21}.$$

But this contradicts to the fact that $\sigma_{12} < \lambda_1(U_2) < \lambda_2(U_2) < \sigma_{21}$. Thus, (i) is proved. □

By using Lemma 3.5, we can prove (3.1) and (3.4). We only prove (3.1), because (3.4) is proved by arguments similar to the proof of (3.1).

Suppose that the left side of inequality (3.1) does not hold. Since $\lambda_1(U) < \sigma_1(U; U_0) < \lambda_2(U)$ for U close to U_0, we then see that there exists the first point $U_1 \in S_1(U_0)$ such that $\sigma_1(U_1; U_0) = \lambda_1(U_1)$. By Lemma 3.5 and the continuous dependence of shock curves on initial points, we find that $U_0 \in S_1^*(U_1)$ and hence

$$\sigma_1(U_1; U_0)(U_1 - U_0) = F(U_1) - F(U_0),$$
$$\sigma_1(U_0; U_1)(U_0 - U_1) = F(U_0) - F(U_1),$$

where $\sigma_1(U_1; U_0)$ and $\sigma_1(U_0; U_1)$ denote the shock speeds for $U_1 \in S_1(U_0)$ and $U_0 \in S_1^*(U_1)$, respectively. This means that

$$\sigma_1(U_1; U_0) = \sigma_1(U_0; U_1) = \lambda_1(U_1).$$

This contradicts to the left side of inequality $\lambda_1(U_1) < \sigma_1(U_0; U_1)$ in (2.47).

Next, suppose that the right side of inequality (3.1) does not hold. Since $\lambda_1(U) < \sigma_1(U; U_0) < \lambda_2(U)$ for U close to U_0, we then see that there exists the first point $U_1 \in S_1(U_0)$ such that $\sigma_1(U_1; U_0) = \lambda_2(U_1)$. By Lemma 3.5 and the continuous dependence of shock curves on initial points, we find that $U_0 \in S_1^*(U_1)$ such that $\sigma_1(U_0; U_1) = \lambda_2(U_1)$. By (2.48) we have $\sigma_1(U_0; U_1) < \lambda_2(U_1)$. This is a contradiction and the proof of (3.1) is complete. The main result of this paper, Theorem 2.1, is now fully proved.

4 Examples

In this section, we give a few examples of 2×2 hyperbolic systems of conservation laws in which condition (2.6) is satisfied.

Example 4.1. The following example corresponds to the case $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) = \mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2)$. Consider the hyperbolic system

$$u_t + \left(u + e^{-v}\right)_x = 0, \quad v_t + \left(u + 2e^{-v}\right)_x = 0. \tag{4.1}$$

The Fréchet derivative dF has real and distinct eigenvalues such that $-\infty < \lambda_1 < 0$, $\frac{1}{2} < \lambda_2 < 1$. The corresponding right and left eigenvectors are

$$r_1 = (1, -e^v(\lambda_1 - 1))^t, \quad r_2 = (1, -e^v(\lambda_2 - 1))^t,$$
$$l_1 = (-e^v(\lambda_1 + 2e^{-v}), 1), \quad l_2 = (e^v(\lambda_2 + 2e^{-v}), 1).$$

It is easy to check that

$$d\lambda_i \cdot r_i = -\frac{(\lambda_i - 1)(2\lambda_i - 1)}{\lambda_i - \lambda_j} > 0, \quad i \neq j,$$

so that the genuine nonlinearity condition (2.3) is satisfied. Also we easily check that

$$l_j d^2 F(r_i, r_i) = \pm e^{2v} \lambda_j (\lambda_i - 1)^2 > 0, \quad i \neq j,$$

with the upper sign for $i = 1$ and the lower sign for $i = 2$, so that the Smoller-Johnson condition (2.4) is satisfied. Moreover, since

$$-e^v(\lambda_2 - 1) < 2 < -e^v(\lambda_1 - 1),$$

the half-plane condition (2.7) is satisfied. The geometry of shock curves is illustrated in Figure 6.

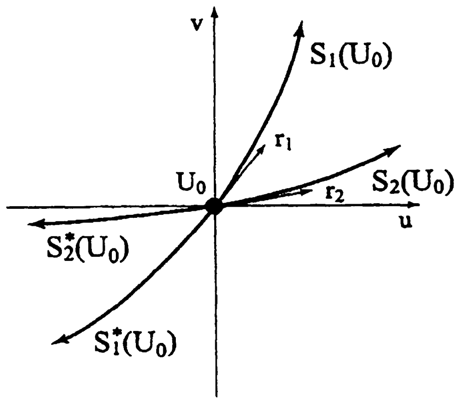


Figure 6: The geometry of shock curves in system (4.1).

Example 4.2. The following example corresponds to the case $\text{sgn}(d\lambda_1 \cdot \hat{r}_1) \neq \text{sgn}(d\lambda_2 \cdot \hat{r}_2)$. Consider the hyperbolic system

$$u_t + \big(-v\big)_x = 0, \quad v_t + \big(u + 2e^{-v} - 2v\big)_x = 0. \tag{4.2}$$

The Fréchet derivative dF has real and distinct eigenvalues such that $-\infty < \lambda_1 < \lambda_2 < 0$. The corresponding right and left eigenvectors are

$$r_1 = \big(1, -\lambda_1\big)^t, \quad r_2 = \big(-1, \lambda_2\big)^t, \quad l_1 = \big(\lambda_2, 1\big), \quad l_2 = \big(\lambda_1, 1\big).$$

It is easy to check that the genuine nonlinearity condition (2.3) is satisfied. Also we easily check that

$$l_j d^2 F(r_i, r_i) = 2e^{-v} \lambda_i^2 > 0, \quad i \neq j.$$

Thus, the Smoller-Johnson condition (2.4) is satisfied. It is obvious that the half-plane condition (2.7) is satisfied. Shock curves $S_1(U_0)$ are not monotonic with respect to u and the geometry of shock curves is illustrated in Figure 7.

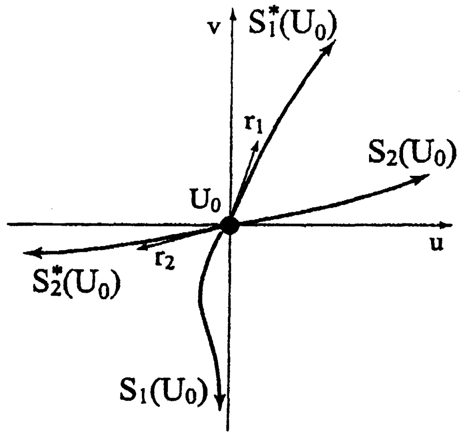


Figure 7: The geometry of shock curves in system (4.2).

Example 4.3. The following example corresponds to the case in which $\mathrm{sgn}(d\lambda_1 \cdot \hat{r}_1) = \mathrm{sgn}(d\lambda_2 \cdot \hat{r}_2)$ and the half-plane condition (2.7) is not satisfied. Consider the hyperbolic system

$$u_t + \left(u + 4e^{\frac{u}{2}} - v\right)_x = 0, \quad v_t + \left(e^u\right)_x = 0. \tag{4.3}$$

The Fréchet derivative dF has real and distinct eigenvalues such that $0 < \lambda_1 < \lambda_2 < \infty$. The corresponding right and left eigenvectors are

$$r_1 = \left(1, \lambda_2\right)^t, \quad r_2 = \left(1, \lambda_1\right)^t, \quad l_1 = \left(-\lambda_1, 1\right), \quad l_2 = \left(\lambda_2, -1\right).$$

It is easy to check that the genuine nonlinearity condition (2.3) is satisfied. Also we easily check that

$$l_j d^2F(r_i, r_i) = \lambda_j(e^{\frac{u}{2}} - \lambda_i) > 0, \quad i \neq j.$$

Thus, the Smoller-Johnson condition (2.4) is satisfied. It is easy to check that the half-plane condition (2.7) is not satisfied. Shock curves originating at $O = (0,0)$ form a loop and the geometry is illustrated in Figure 8.

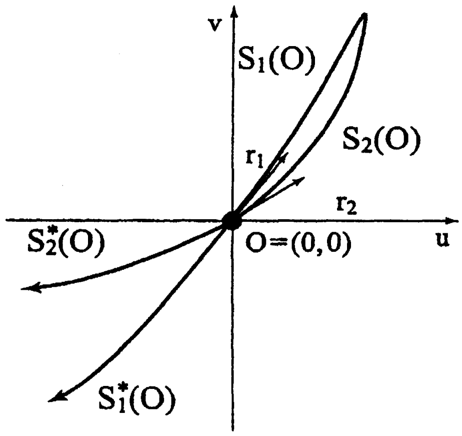


Figure 8: The geometry of shock curves in system (4.3).

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