数理解析研究所講究録
第 1640 巻 2009 年 116-128

Global Solutions with a Moving Singularity for a Semilinear Parabolic Equation

東北大学大学院・理学研究科　佐藤　翔大 (Shota Sato)
Mathematical Institute,
Tohoku University

1 Introduction

This article is based on a joint paper [12] with Eiji Yanagida (Tohoku University).

We consider singular solutions of the semilinear parabolic equation

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $p > 1$ is a parameter. It is known that for

$$N \geq 3, \quad p > p_{sg} := \frac{N}{N-2},$$

(1.1) has an explicit singular steady state $\varphi_\infty(x) \in C^\infty(\mathbb{R}^N \setminus \{0\})$ with a singular point $0 \in \mathbb{R}^N$ that is explicitly expressed as

$$\varphi_\infty(x) = L|x|^{-m}, \quad m = \frac{2}{p-1}, \quad L^{p-1} = m(N-m-2).$$

Since this singular steady state is radially symmetric with respect to 0, we may write φ_∞ as a function of $r = |x|$. Then $\varphi_\infty = \varphi_\infty(r)$ satisfies (1.1) in the distribution sense, and

$$(\varphi_\infty)_{rr} + \frac{N-1}{r}(\varphi_\infty)_r + (\varphi_\infty)^p = 0, \qquad r = |x| > 0. \tag{1.2}$$

Clearly, the spatial singularity of $u = \varphi_\infty$ persists for all $t > 0$, but the singular point does not move in time.

In [11], we studied the existence of a solution of (1.1) whose spatial singularity moves in time. More precisely, we define a solution with a moving singularity as follows.

Definition 1.
(a) The function $u(x,t)$ is said to be a solution of (1.1) with a moving singularity $\xi(t) \in \mathbb{R}^N$ for $t \in (0,T)$, where $0 < T \leq \infty$, if the following conditions hold:

(i) $u,\ u^p \in C([0,T); L^1_{loc}(\mathbb{R}^N))$ satisfy (1.1) in the distribution sense.

(ii) $u(x,t)$ is defined on $\{(x,t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\},\ t \in (0,T)\}$, and is twice continuously differentiable with respect to x and continuously differentiable with respect to t.

(iii) $u(x,t) \to \infty$ as $x \to \xi(t)$ for every $t \in [0,T)$.

(b) If the conditions (i)-(iii) hold for $T = \infty$, we call the function $u(x,t)$ a time-global solution of (1.1) with a moving singularity $\xi(t)$.

Concerning the existence of a solution with singularities, it is known that the exponent

$$p_* := \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}, \quad N > 2,$$

plays a important role. It was shown by Véron [14] that p_* is related to the linearized stability of the singular steady state, while it was shown by Chen-Lin [2] that p_* is crucial for the existence of positive solutions with a prescribed singular set of the Dirichlet problem

$$\begin{cases} \Delta u + u^p = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. In fact, in [2], they proved that if $N \geq 3$ and $p_{sg} < p < p_*$, then for any closed set $K \subset \Omega$, there exists a positive solution with K as a singular set. We note that p_* is larger than p_{sg} and is smaller than the Sobolev critical exponent $p_S := (N+2)/(N-2)$.

In [11], for $p_{sg} < p < p_*$, we established the time-local existence, uniqueness and comparison principle for a solution with a moving singularity of the Cauchy problem (1.1) with the initial condition

$$u(x,0) = u_0(x) \quad \text{in} \ \mathbb{R}^N, \tag{1.3}$$

where $u_0 \in L^1_{loc}(\mathbb{R}^N)$ is a nonnegative function. Given the motion $\xi(t)$ of a singularity and the initial data $u_0(x)$ satisfying some conditions, it can be shown that for some $T > 0$, there exists a solution of (1.1) and (1.3) with a moving singularity $\xi(t)$. However, in [11], the global existence of a solution with a moving singularity is not discussed.

The aim of this article is to find a time-global solution with a moving singularity. To this aim, we first consider a forward self-similar solution of the form

$$u = (t+1)^{-1/(p-1)}\varphi((t+1)^{-1/2}x - a), \tag{1.4}$$

where $a \in \mathbb{R}^N$ is a given point. If $\varphi(z)$ satisfies

$$\Delta_z \varphi + \frac{z+a}{2} \cdot \nabla_z \varphi + \frac{1}{p-1}\varphi + \varphi^p = 0, \qquad z \in \mathbb{R}^N, \tag{1.5}$$

in the distribution sense, then u defined by (1.4) may satisfy (1.1) in the distribution sense. Moreover, if

(A1) $\varphi(z)$ is defined on $\mathbb{R}^N \setminus \{0\}$ and is twice continuously differentiable, and

(A2) $\varphi(z) \to \infty$ as $z \to 0$,

then u defined by (1.4) may become a time-global solution with a singularity at $\xi(t) = (t+1)^{1/2}a$.

Equation (1.5) with $a = 0$ is called the Haraux-Weissler equation, which was introduced in [5], and has been extensively studied by many people. Among others, the Haraux-Wiessler equation is often used to study the large time behavior of global solutions to the Cauchy problem [7, 8], and to study solutions of (1.1) with singular initial data [9, 10, 13].

In order to state our result, we define Λ to be a set of $p > p_{sg}$ such that the equality

$$(-m+i)(N-m+i-2) + pm(N-m-2) = j(N+j-2) \tag{1.6}$$

holds for some

$$i \in \{1, 2, \ldots, [m]\} \quad \text{and} \quad j \in \{0, 1, 2, \ldots, i\},$$

where $[m]$ denotes the largest integer not greater than m. Clearly Λ is a finite set.

Concerning the existence of a forward self-similar solution with a moving singularity, we have the following result.

Theorem 1. *Let $N \geq 3$. Suppose that $p \notin \Lambda$ and*

$$p_{sg} < p < \begin{cases} p_* & \text{if} \quad N \leq 10, \\ \frac{N+2}{N-1} & \text{if} \quad N > 10. \end{cases} \tag{1.7}$$

Then there exists a constant $\delta > 0$ such that for any $|a| < \delta$, there exists a solution of (1.5) *satisfying* (A1), (A2). *Moreover, the function u defined by* (1.4) *satisfies* (1.1) *in the distribution sense.*

This theorem shows that we have a time-global solution of (1.1) with a singularity at $\xi(t) = (t+1)^{1/2}a$.

In this article, we study only a time-global solution with a moving singularity. When a solution with a moving singularity does not exist globally in time, it is interesting to ask what happens at the maximal existence time. This question will be a future work.

This article is organized as follows: In Section 2 we carry out formal analysis for a solution of (1.5) that is obtained by perturbing the singular steady state. In section 3 we describe the sketch of proof of Theorem 1.

2 Formal expansion at a singular point

In this section, we consider the formal expansion of a solution $\varphi(z)$ of (1.5) satisfying (A1) and (A2). Assuming that the solution resembles the singular solution $\varphi_\infty(z)$ around 0, we may naturally expand $\varphi(z)$ as

$$\varphi(z) = Lr^{-m}\Big\{1 + \sum_{i=1}^{k} b_i(\omega)r^i + v(z)r^m\Big\}, \tag{2.1}$$

where

$$r = |z|, \quad \omega = \frac{z}{r} \in S^{N-1}, \quad k = [m],$$

and the remainder term v satisfies

$$v(z) = o(|z|^{-m}) \quad \text{as } |z| \to 0. \tag{2.2}$$

Substituting (2.1) into (1.5), and using

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}}$$

and the Taylor expansion, we compare the coefficients of r^{-m+i-2} for $i = 0, 1, \ldots, k$. Here $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1}. Then we obtain

$$r^{-m-2}\ ;\ (Lr^{-m})_{rr} + \frac{N-1}{r}(Lr^{-m})_r + (Lr^{-m})^p = 0,$$

$$r^{-m-1}\ ;\ \Delta_{S^{N-1}} b_1 + \{(-m+1)(N-m-1) + pm(N-m-2)\} b_1 = \frac{m}{2}(a\cdot\omega), \quad (2.3)$$

$$\begin{aligned} r^{-m}\ ;\ &\Delta_{S^{N-1}} b_2 + \{(-m+2)(N-m) + pm(N-m-2)\} b_2 \\ &= \frac{(m-1)}{2}(a\cdot\omega) b_1 + \frac{1}{2}\{a\cdot\nabla_{S^{N-1}} b_1 - (a\cdot\omega)(\nabla_{S^{N-1}} b_1 \cdot \omega)\} \\ &\quad - \frac{p(p-1)}{2} L^{p-1} b_1^2, \end{aligned} \quad (2.4)$$

$$\begin{aligned} r^{-m+i-2};\ &\Delta_{S^{N-1}} b_i + \{(-m+i)(N-m+i-2) + pm(N-m-2)\} b_i \\ &= G_i(\omega; b_1, b_2, \ldots, b_{i-1}, a) \quad (i = 3, 4, \ldots, k), \end{aligned} \quad (2.5)$$

where for each $i = 3, 4, \ldots, k$, the function $G_i(\omega; b_1, b_2, \ldots, b_{i-1}, a)$ on S^{N-1} is determined by $b_1, b_2, \ldots, b_{i-1}$ and a.

The equality for r^{-m-2} always holds by (1.2). From other equalities, we have the above system of inhomogeneous elliptic equations for b_i on S^{N-1}. By these equations, $b_1, b_2, \ldots$ are determined sequentially.

Let us consider the solvability of (2.3), (2.4) and (2.5). It is well known (see e.g. [1]) that for every $j = 0, 1, 2, \ldots$, the eigenvalues of $-\Delta_{S^{N-1}}$ are given by

$$\mu_j = j(N+j-2), \quad j = 0, 1, 2, \ldots,$$

and the eigenspace E_j associated with μ_j is given by

$$E_j = \{\, f|_{S^{N-1}} : f \text{ is a harmonic homogeneous polynomial of degree } j \,\}.$$

Therefore, unless (1.6) holds, the operators in the left-hand side of (2.3), (2.4) and (2.5) are invertible. Moreover, we consider $G_i(\omega; b_1, b_2, \ldots, b_{i-1}, a)$ in details and obtain next lemma.

Lemma 1. *If $p \notin \Lambda$, then for any $a \in \mathbb{R}^N$, there exist $b_1(\omega; a)$, $b_2(\omega; a)$, …, $b_k(\omega; a) \in C^\infty(S^{N-1})$ such that* (2.3), (2.4) *and* (2.5) *hold. Moreover,*

$$\|b_i(\cdot; a)\|_{C^\infty(S^{N-1})} \to 0 \quad \text{as } |a| \to 0 \quad (2.6)$$

for all $i = 1, \ldots, k$.

By this lemma, in order to show the existence of a solution of (1.5), it suffices to consider $v(z)$. By taking $b_i(\omega)$ as in Lemma 1, (1.5) is satisfied if $v(z)$ satisfies

$$\Delta v + \frac{z+a}{2}\cdot\nabla v + \frac{m}{2} v + F(v, z) = 0 \quad \text{on } \mathbb{R}^N, \quad (2.7)$$

where $F(v,z)$ is determined by $b_1,\ b_2,\dots,b_k$ and a. After tedious computations, we notice that

$$F(v,z)=\frac{pL^{p-1}}{r^2}v+o(r^{-2})\quad \text{as}\ \ z\to 0.$$

Therefore, as $a\to 0$, (2.7) reduces to

$$\Delta v+\frac{z}{2}\cdot\nabla v+\frac{m}{2}v+\frac{pL^{p-1}}{r^2}v=0\quad \text{on}\ \ \mathbb{R}^N, \tag{2.8}$$

In order to consider the existence of solutions of (2.7), we first consider the equation

$$\Delta v+\frac{z}{2}\cdot\nabla v+\frac{\mu}{2}v+\frac{l}{r^2}v=0\quad \text{on}\ \ \mathbb{R}^N \tag{2.9}$$

with parameters μ and l. We define $\lambda_1(l)$ and $\lambda_2(l)$ by

$$\lambda_1(l):=\frac{N-2-\sqrt{(N-2)^2-4l}}{2},$$
$$\lambda_2(l):=\frac{N-2+\sqrt{(N-2)^2-4l}}{2}.$$

By a similar method to [3, Lemma 3.1 (i)], we obtain the following lemma.

Lemma 2. *If*

$$0<l<\frac{(N-2)^2}{4}\quad \textit{and}\quad \lambda_1(l)<\mu<\lambda_2(l)+2,$$

then (2.9) *has a radial solution* $v(|z|;\mu,l)$ *with the following properties:*

(i) $\lim_{r\to 0}r^{\lambda_1(l)}v=1$ *and* $\lim_{r\to 0}(r^{\lambda_1(l)}v)_r=0$.

(ii) $v>0$ *and* $(r^{\lambda_1(l)}v)_r<0$ *for all* $r>0$.

(iii) *For each* $r_0>0$, *there exists* $c_-(r_0)>0$ *such that* $v(r)\geq c_-(r_0)r^{-\mu}$ *for* $r>r_0$.

(iv) *There exists* $c_+>0$ *such that* $v(r)\leq c_+r^{-\mu}$ *for all* $r>0$.

Applying Lemma 2, we see that there exists a positive radial solution $v(|z|)$ of (2.8) if

$$0 < pL^{p-1} < \frac{(N-2)^2}{4} \tag{2.10}$$

and

$$\lambda_1 < m < \lambda_2 + 2, \tag{2.11}$$

where λ_1 and λ_2 are defined by

$$\lambda_1 := \frac{N-2-\sqrt{(N-2)^2-4pL^{p-1}}}{2},$$
$$\lambda_2 := \frac{N-2+\sqrt{(N-2)^2-4pL^{p-1}}}{2}.$$

We note that for $N \geq 3$ and $p_{sg} < p < p_*$, the constants $\lambda_1 < \lambda_2$ are positive roots of

$$\lambda^2 - (N-2)\lambda + pL^{p-1} = 0.$$

Since the gradient term in (2.7) and the higher order term of F do not affect the well-posedness for small $|a|$, we must assume (2.10) and (2.11) for the solvability of (2.7). The inequalities (2.10) hold if and only if p satisfies $p_{sg} < p < p_*$ for $N \geq 3$ or

$$p > p_{JL} := \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}$$

for $N > 10$. Here the exponent p_{JL} was first introduced by Joseph-Lundgren [6] and is known to play an important role for the dynamics of solutions of (1.1). If $p > p_{JL}$, then $\lambda_1 < m$ does not hold so that (2.2) may not be true. Hence we exclude the case $p > p_{JL}$. On the other hand, in the case $p_{sg} < p < p_*$, (2.11) holds if and only if (1.7) holds.

Based on the above formal analysis, we will focus on the case (1.7).

3 Sketch of Proof of Theorem 1

In this section, taking into account of the formal analysis in the previous section, we describe the sketch of proof of Theorem 1.

The sketch of proof of Theorem 1 is divided into three steps. Roughly speaking, we first construct a suitable supersolution and subsolution of (1.5) satisfying (A2). Next, we construct a sequence of approximate solutions and find a convergent subsequence. Then we show that the limiting function is indeed a solution of (1.5) satisfying (A1) and (A2), and the function u defined by (1.4) satisfies (1.1) in the distribution sense.

3.1 Construction of a supersolution and a subsolution

In this subsection, we construct a supersolution and a subsolution of (1.5) satisfying (A2).

We first note that if $p \notin \Lambda$, then by Lemma 1,

$$b_1(\omega; a),\ b_2(\omega; a),\ \ldots,\ b_k(\omega; a) \in C^2(S^{N-1})$$

are obtained by solving (2.3), (2.4) and (2.5). If p satisfies (1.7), we can take l such that

$$0 < pL^{p-1} < l < \frac{(N-2)^2}{4}, \quad \lambda_1(l) < m < \lambda_2(l)+2, \quad [m-\lambda_1] = [m-\lambda_1(l)],$$

and replace k defined in Section 2 with $k := [m - \lambda_1]$. We set

$$M(a) := \sup_{\omega \in S^{N-1}} \{\max_i(|b_i(\omega; a)|, |\nabla_{S^{N-1}} b_i(\omega; a)|)\}.$$

By (2.6), we have $M(a) \to 0$ as $a \to 0$. We also take ϵ_0 so small that

$$0 < \epsilon_0 < l - pL^{p-1}.$$

Let B_R denote a ball centered at 0 with radius $R > 0$. First we construct a supersolution and a subsolution of (1.5) in B_R by using (2.7). By (2.1), we have

$$\Delta_z \varphi + \frac{z+a}{2} \cdot \nabla_z \varphi + \frac{m}{2}\varphi + \varphi^p = L\left\{\Delta v + \frac{z+a}{2} \cdot \nabla v + \frac{m}{2} v + F(v, z)\right\}.$$

Hence

$$\overline{\varphi}(z) = Lr^{-m}\left\{1 + \sum_{i=1}^{k} b_i(\omega; a) r^i + \overline{v}(z) r^m\right\}$$

is a supersolution of (1.5) if and only if $\overline{v}$ is a supersolution of (2.7). Similarly,

$$\underline{\varphi}(z) = Lr^{-m}\left\{1 + \sum_{i=1}^{k} b_i(\omega; a) r^i + \underline{v}(z) r^m\right\}$$

is a subsolution of (1.5) if and only if $\underline{v}$ is a subsolution of (2.7).

We will show that $\overline{v} := C_1 v(|z|; m, l)$ is a supersolution of (2.7) on B_{R_1} for some $R_1 = R_1(C_1, a) > 0$. We take R_1 such that

$$L^{p-1} r^{-m-2} \Bigg[\Big\{ 1 + \sum_{i=1}^{k} b_i(\omega; a) r^i + C_1 v(|z|; m, l) r^m \Big\}^p - 1 - \sum_{j=1}^{k} \Big\{ r^j \sum_{l=1}^{j} \sum_{i_1+\cdots+i_l=j,\ i_1,\ldots,i_l \geq 1} A(p,j) b_{i_1}(\omega; a) \cdots b_{i_l}(\omega; a) \Big\} \Bigg] \leq C_1 \Big(pL^{p-1} + \frac{1}{2}\epsilon_0 \Big) r^{-2} v(|z|; m, l) \quad \text{in } B_{R_1},$$

and

$$R_1 \to \infty \quad \text{as } |a| \to 0,\ C_1 \to 0.$$

Since it follows from tedious calculation that $\overline{v} = C_1 v(|z|; m, l)$ is a supersolution of (2.7) in B_{R_1} for small $|a|$,

$$\overline{\varphi}_{in} := L r^{-m} \Big\{ 1 + \sum_{i=1}^{k} b_i(\omega; a) r^i + C_1 v(|z|; m, l) \Big\}$$

is a supersolution of (1.5) on B_{R_1} for small $|a|$.

We will construct a subsolution as follows. For sufficiently large $C_2 > 0$, there exist a domain Ω^- and a constant $R_2 = R_2(C_2, a) > 0$ such that

$$0 \in \Omega^- \subset B_{R_2}, \qquad R_2 \to 0 \quad \text{as } |a| \to 0,\ C_2 \to \infty$$

and

$$1 + \sum_{i=1}^{k} b_i(\omega; a) r^i - C_2 r^{m-\lambda_1(l)} \geq 0 \quad \text{in } \Omega^-,$$

$$1 + \sum_{i=1}^{k} b_i(\omega; a) r^i - C_2 r^{m-\lambda_1(l)} = 0 \quad \text{on } \partial\Omega^-,$$

$$L^{p-1} r^{-m-2} \Bigg[\Big\{ 1 + \sum_{i=1}^{k} b_i(\omega; a) r^i - C_2 r^{m-\lambda_1(l)} \Big\}^p - 1 - \sum_{j=1}^{k} \Big\{ r^j \sum_{l=1}^{j} \sum_{i_1+\cdots+i_l=j,\ i_1,\ldots,i_l \geq 1} A(p,j) b_{i_1}(\omega; a) \cdots b_{i_l}(\omega; a) \Big\} \Bigg] \geq -C_2 \Big(pL^{p-1} + \frac{1}{2}\epsilon_0 \Big) r^{-\lambda_1(l)-2} \quad \text{in } \Omega^-.$$

Since it follows from tedious calculation that $\underline{v} = -C_2 r^{-\lambda_1(l)}$ is a subsolution of (2.7) on Ω^- for small $|a|$ and large C_2,

$$\underline{\varphi}_{in} := Lr^{-m}\Big\{1 + \sum_{i=1}^{k} b_i(\omega; a) r^i - C_2 r^{m-\lambda_1(l)}\Big\}$$

is a subsolution of (1.5) on Ω^- for small $|a|$ and large C_2.

Next, we construct a supersolution and a subsolution near infinity. By direct calculation, we see that

$$\overline{\varphi}_{out} := Lr^{-m} + C_3 r^{-q}$$

is a supersolution of (1.5) on $\mathbb{R}^N \setminus B_{R_3}$ for some $R_3 = R_3(C_3, a) > 0$. Moreover, we may assume

$$R_3 \to R_* := \left\{\frac{2(q-\lambda_1)(q-\lambda_2)}{q-m}\right\}^{1/2} \quad \text{as} \quad |a| \to 0, \ C_3 \to 0.$$

Clearly $\underline{\varphi} \equiv 0$ is a subsolution of (1.5) on $\mathbb{R}^N$.

Finally, we connect these supersolutions and subsolutions in the intermediate region. We first assume $a = 0$. Then, from Lemma 2 (i), (ii), (iv), if C_1, C_3 and C_1/C_3 are sufficiently small, we can take $R_3 < R_4 < R_1$ such that $\overline{\varphi}_{in} < \overline{\varphi}_{out}$ for $r < R_4$ and $\overline{\varphi}_{in} > \overline{\varphi}_{out}$ for $r > R_4$. Hence,

$$\overline{\varphi} := \min\{\overline{\varphi}_{in}, \overline{\varphi}_{out}\}$$

is a supersolution of (1.5) with $a = 0$.

By the continuity and Lemma 2 (i), for each small $|a|$, there exists Ω^+ such that $B_{R_3} \subset \Omega^+ \subset B_{R_1}$ and

$$\begin{aligned} &\overline{\varphi}_{in} < \overline{\varphi}_{out} \quad \text{if} \ z \in \Omega^+ \ \text{is near} \ \partial\Omega^+, \\ &\overline{\varphi}_{in} > \overline{\varphi}_{out} \quad \text{if} \ z \notin \Omega^+ \ \text{is near} \ \partial\Omega^+. \end{aligned}$$

Then

$$\overline{\varphi} := \begin{cases} \overline{\varphi}_{in} & \text{if} \ z \in \Omega^+, \\ \overline{\varphi}_{out} & \text{if} \ z \notin \Omega^+ \end{cases}$$

is a supersolution of (1.5) for small $|a|$. Clearly,

$$\underline{\varphi} := \begin{cases} \underline{\varphi}_{in} & \text{if} \ z \in \Omega^-, \\ 0 & \text{if} \ z \notin \Omega^- \end{cases}$$

is a subsolution of (1.5) for small $|a|$.

3.2 Construction of approximate solutions

In this subsection, by using the supersolution and subsolution given in the previous subsection, we construct a series of approximate solutions that is convergent in an appropriate function space.

Define a sequence of annular domains

$$A_n := \left\{ z \in \mathbb{R}^N : \frac{1}{n} < |z| < n \right\} \qquad (n = 1, 2, \ldots).$$

For each n, let $\varphi_n(z)$ be a classical solution of

$$\begin{cases} \Delta\varphi_n + \dfrac{z+a}{2} \cdot \nabla\varphi_n + \dfrac{1}{p-1}\varphi_n + \varphi_n^p = 0 & \text{in } A_n, \\ \varphi_n = \underline{\varphi} & \text{on } \partial A_n. \end{cases}$$

Then, by the standard elliptic theory [4], the Ascoli-Arzelà theorem and a diagonal procedure, we obtain a subsequence $\{\varphi_n^{(n)}\}_n$ such that

$$\varphi_n^{(n)} \to \varphi \quad \text{uniformly in } A_j \text{ as } n \to \infty$$

for each j, and the limiting function $\varphi(z)$ satisfies

$$\varphi \in C(\mathbb{R}^N \setminus \{0\}), \qquad \underline{\varphi} \le \varphi \le \overline{\varphi} \text{ in } \mathbb{R}^N \setminus \{0\}.$$

3.3 Completion of the proof

In this subsection, we show that the limiting function $\varphi(z)$ obtained as above is indeed a solution of (1.5) satisfying (A1) and (A2), and the function u defined by (1.4) satisfies (1.1) in the distribution sense.

First, by $\underline{\varphi} \le \varphi \le \overline{\varphi}$ and the Lebesgue theorem, we can show that the function φ satisfies (1.5) in the distribution sense. Next, by $\underline{\varphi} \le \varphi \le \overline{\varphi}$ and the standard elliptic theory [4], the function φ has the desired properties (A1) and (A2). Therefore, it is shown that the function φ is the solution of (1.5) satisfying (A1) and (A2).

Since $\varphi(z)$ satisfies (1.5) in the distribution sense and (A1), it follows from the definition of u that u satisfies (1.1) in $\mathbb{R}^N \times (0,\infty) \setminus \bigcup_{0<t<\infty}\{((t+1)^{1/2}a, t)\}$. Thus, by $\underline{\varphi} \le \varphi \le \overline{\varphi}$ and simple calculation, we can show that the function u satisfies (1.1) in the distribution sense.

∎

Acknowledgments

The author was supported by the Global COE Program "Weaving Science Web beyond Particle-Matter Hierarchy" at the Graduate School of Science, Tohoku University, from the Ministry of Education, Culture, Sports, Science and Technology.

References

[1] M. Berger, P. Gauduchon and E. Mazet, Le spectre d'une variété riemanniennes, Lecture Notes in Math. **194**, Springer-Verlag, 1971.

[2] C.-C. Chen and C.-S. Lin, Existence of positive weak solutions with a prescribed singular set of semilinear elliptic equations, J. Geometric Analysis **9** (1999), 221–246.

[3] M. Fila, J. R. King, M. Winkler and E. Yanagida, Optimal lower bound of the grow-up rate for a supercritical parabolic equation, J. Differential Equations **228** (2006), 339–356.

[4] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer, Berlin, 1998.

[5] A. Haraux and F. B. Weissler, Non-uniqueness for a semilinear initial value problem, Indiana Univ. Math. J. **31** (1982), 167–189.

[6] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rat. Mech. Anal. **49** (1973), 241–269.

[7] O. Kavian, Remarks on the large time behavior of a nonlinear diffusion equation, Annal. Institut Henri Poincaré-Analyse Nonlinéaire **4** (1987), 423–452.

[8] T. Kawanago, Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity, Annal. Institut Henri Poincaré-Analyse Nonlinéaire **13** (1996), 1–15.

[9] Y. Naito, An ODE approach to the multiplicity of self-similar solutions for semi-linear heat equations, Proceedings of the Royal Society of Edinburgh **136A** (2006), 807–835.

[10] Y. Naito and T. Suzuki, Radial symmetry of self-similar solutions for semilinear heat equations, J. Differential Equations **163** (2000), 407–428,

[11] S. Sato and E. Yanagida, Solutions with a Moving Singularities for a Semilinear Parabolic Equation, J. Differential Equations **246** (2009), 724–748.

[12] S. Sato and E. Yanagida, Forward Self-Similar Solution with a Moving Singularity for a Semilinear Parabolic Equation, preprint.

[13] Ph. Souplet and F. B. Weissler, Regular self-similar solutions of nonlinear heat equation with initial data above the singular steady state, Annal. Institut Henri Poincaré-Analyse Nonlinéaire **20** (2003), 213–235.

[14] L. Véron, Singularities of solutions of second order quasilinear equations, Pitman Research Notes in Mathematics Series **353**, Longman, Harlow, 1996.