数理解析研究所講究録
第 1640 巻 2009 年 164-171

On instant blow-up for quasilinear parabolic equations with growing initial data

Noriaki Umeda
Graduate School of Mathematical Sciences,
University of Tokyo
3-8-1, Komaba, Meguro-ku, Tokyo 153-8914, Japan

We are interested in the existence of the solutions of the parabolic equations with initial data which are not bounded at space infinity.

In [4] Giga and the author considered a nonnegative blowing up solution of the semilinear parabolic equation of the form

$$u_t = \Delta u + f(u), \quad x \in \mathbf{R}^N, t > 0$$

with nonlinear terms f and nonnegative initial data u_0 satisfying that f is positive, nondecreasing and convex in $(0,\infty)$, $\int_1^\infty ds/f(s) < \infty$ and there are sequences $\{x_n\} \subset \mathbf{R}^N$ and $\{r_n\} \subset \mathbf{R}_+$ with $\lim_{n\to\infty} |x_n| = \infty$ and $\lim_{n\to\infty} r_n \geq 0$ such that

$$\lim_{n\to\infty} \frac{b_n}{r_n^2 f(b_n)} \text{ is small enough}$$

with $b_n = \inf\{u_0(x) : |x - x_n| \leq r_n\}$. They showed that the solutions do not exist even locally in time.

We consider the initial value problem for a quasilinear parabolic equation of the form

$$\begin{cases} u_t = \Delta u^m + u^p, & x \in \mathbf{R}^N, t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N. \end{cases} \tag{1}$$

Here we assume that $N \geq 1$, $1 \leq m < p$.

We are interested in the problem whether there is a local-in-time solution of (1) when an initial datum u_0 is continuous and grows at the space infinity, for example $\lim_{|x|\to\infty} u_0(x) = \infty$.

We consider the weak solution u in $\mathbf{R}^N \times [0,T)$ of (1) such that $u \in C(\mathbf{R}^n \times [0,\tau))$ for each $\tau \in (0,T)$, and for any bounded domain $\Omega \in \mathbf{R}^N$ with smooth boundary $\partial\Omega$, $0<\tau<T$ and nonnegative $\phi(x,t) \in C^{2,1}(\Omega \times [0,T))$ which vanishes on the boundary $\partial\Omega$,

$$\int_\Omega u(x,\tau)\phi(x,\tau)dx - \int_\Omega u(x,0)\phi(x,0)dx$$
$$= \int_0^\tau \int_\Omega \{u\partial_t\phi + u^m\Delta\phi + u^p\phi\}dxdt - \int_0^\tau \int_{\partial\Omega} u^m \partial_\nu \phi dSdt, \qquad (2)$$

where ν denote the outer unit normal to the boundary. Note that the solution of (1) may be nonunique. Define $T^* = T^*(u_0)$ as the supremum of all existence times of these solutions.

In this paper we shall prove that $T^* = 0$ when the initial data u_0 is growing at the space infinity. In other words there is even no local-in-time solution such that for any $\tau > 0$ the weak solution does not exist for $t \in (0,\tau)$. We say this phenomenon $T^* = 0$ an *instant blow-up*. We are able to prove that the instant blow-up occurs for more general initial data u_0.

Theorem. *Assume that $u_0 \in C(\mathbf{R}^N)$ is nonnegative. Assume that there are sequences $\{x_n\}_{n=1}^\infty \subset \mathbf{R}^N$ and $\{r_n\}_{n=1}^\infty \subset \mathbf{R}_+$ with $\lim_{n\to\infty}|x_n| = \infty$ and $\lim_{n\to\infty} r_n \geq 0$ such that*

$$\lim_{n\to\infty} r_n^2 b_n^{p-m} > \frac{1}{\varepsilon} \qquad (3)$$

for some $\varepsilon \in (0, 1/c)$, where $b_n = \inf\{u_0(x) : |x - x_n| \leq r_n\}$ and $c > 0$ is the first eigenvalue of $-\Delta$ in a unit ball with the Dirichlet boundary condition. Then $T^* = 0$, i.e., the instant blow-up occurs provided that only nonnegative solutions are considered.

The proof of Theorem depends on a classical Kaplan's argument [6] to show the existence of blow-up which uses principal eigenfunctions of the Laplace operator with the Dirichlet condition.

In [1] among other results there is one about a sufficient condition on initial data for nonexistence of a local-in-time nonnegative solution for $u_t = \Delta u^m + u^p/(1+|x|)^\alpha$ with $m \geq 1$, $p > 1$ and $\alpha \in \mathbf{R}$. In the case of $\alpha = 0$ the condition leads

$$\sup_{x\in\mathbf{R}^n} \int_{B(x,1)} u_0(y)dy = \infty. \qquad (4)$$

In [1] this is explicitly mentioned for $1 < p < m + 2/N$. However, their proof is still valid for all $p > 1$. By the way their main interest is the existence of solution; for example they proved the local existence when

$$\sup_{x\in\mathbf{R}^N}\int_{B(x,1)} u_0(y)dy < \infty$$

for $1 < p < m + 2/N$. The condition (3) is not included in the condition of their result for $p > m + 2/N$. In fact, if $u_0 \geq b_n$ on $B(x_n, r_n)$, then $\lim_{n\to\infty} b_n r_n^N = \infty$ is a sufficient condition for (4) (not a necessary condition). Our condition leads $\lim_{n\to\infty} r_n^2 b_n^{p-m}$ is large enough. This shows that our condition for $p > m + 2/N$ is not included in their condition.

In [1] they also prove the local existence for $p \geq m + 2/N$ when u_0 fulfills

$$\sup_{x\in\mathbf{R}^n}\int_{B(x,1)} u_0^q(y)dy < \infty$$

for some $q > N(p-m)/2$. In our nonexistence result u_0 satisfies

$$\sup_{x\in\mathbf{R}^n}\int_{B(x,1)} u_0^q(y)dy \geq \lim_{n\to\infty}\int_{B(x_n,1)} u_0^q(y)dy \geq \lim_{n\to\infty} \varepsilon^{-\frac{q}{p-m}} r_n^{N-\frac{2q}{p-m}} = \infty$$

for any $q > N(p-m)/2$, where ε is used in (3).

In [4] Theorem was proved in the case $m = 1$. They studied the instant blow-up by using not only the eigenfunction method in [6] same as this paper but also the energy method in [7] and [2].

In the rest of the paper Theorem will be proved by using the Kaplan's argument [6].

Lemma. *(c.f. [3, Lemma 4.2]) Let v be the solution of the integral equation of the form*

$$v(t) - v(0) = \int_0^t h(v(s))ds \tag{5}$$

in $[0, T_0)$ with h satisfying $h \in C^1[0,\infty)$ and $h' \geq 0$. Let $\tilde{v}$ be a nonnegative measurable function on $[0, T_0)$. Assume that $\tilde{v}$ satisfies

$$\tilde{v}(t) - \tilde{v}(t_0) \geq (\leq) \int_{t_0}^t h(v(s))(s)ds \quad \textit{for} \quad t_0, t \in [0, T_0) \quad \textit{with} \quad t_0 \leq t. \tag{6}$$

Assume that $\tilde{v}(0) \geq (\leq) v(0)$. Then

$$\tilde{v}(t) \geq (\leq) v(t) \quad \textit{for} \quad t \in [0, T_0).$$

Proof. We shall only prove the case $\tilde{v}(t) - \tilde{v}(t_0) \geq \int_{t_0}^{t} \tilde{v}^p(s)ds$ since the proof of the other case is parallel. Since $\tilde{v}(0) \geq v(0)$, the estimate (6) together with (5) yields

$$\tilde{v}(t) - v(t) \geq \int_0^t (h(\tilde{v}(s)) - h(v(s)))ds.$$

By the mean value theorem we observe that

$$\tilde{v}(t) - v(t) \geq \int_0^t c(s)\,(\tilde{v}(s) - v(s))\,ds,$$

where

$$c(s) = \int_0^1 h'\,(\theta v(s) + (1-\theta)\tilde{v}(s))\,d\theta.$$

We set $\psi_\epsilon(t) = \tilde{v}(t) - v(t) + \epsilon$ with $\epsilon > 0$, and observe that $\psi_\epsilon(t)$ satisfies

$$\psi_\epsilon \geq \int_0^t c(s)\psi_\epsilon(s)ds + \epsilon\left(1 - \int_0^t c(s)ds\right).$$

We set

$$t_1 = \sup\left\{t > 0; \int_0^t c(s)ds < \frac{1}{2}\right\}.$$

Then, for $t \in [0, t_1]$ we have

$$\psi_\epsilon(t) \geq \int_0^t c(s)\psi_\epsilon(s)ds + \frac{\epsilon}{2}. \tag{7}$$

We shall argue by contradiction to prove $\psi_\epsilon(t) \geq 0$. Suppose that $\psi_\epsilon(t) < 0$ for some $t \in [0, t_1]$. Then $\psi_\epsilon(\tau) = 0$ for

$$\tau = \inf\{t \in [0, t_1]; \psi_\epsilon < 0\}. \tag{8}$$

This τ must be positive. Indeed, since $\tilde{v}$ is nondecreasing by (6) and v is continuous, $\psi_\epsilon(0) > \epsilon$ implies $\tau > 0$.

Since $\int_0^\tau c(s)\psi_\epsilon(s)ds \geq 0$ and (8) imply $\psi_\epsilon(\tau) \leq 0$, we get a contradiction by (7). We thus proved that

$$\psi_\epsilon(t) \geq 0.$$

Since this holds for all $\epsilon > 0$, we get $\tilde{v}(t) \geq v(t)$ for $t \in [0, t_1]$. (If $\tilde{v}(t) < v(t)$ for some t, there exist $\epsilon > 0$ such that $\psi_\epsilon < 0$ for such t.)

Next, since $\tilde{v}(t) \geq v(t)$ for $t \in [0, t_1]$, we observe that

$$\psi_\epsilon \geq \int_{t_1}^t c(s)\psi_\epsilon(s)ds + \epsilon\left(1 - \int_{t_1}^t c(s)ds\right).$$

We set

$$t_2 = \sup\left\{t > t_0; \int_{t_1}^t c(s)ds < \frac{1}{2}\right\}$$

and observe that

$$\psi_\epsilon \geq \int_{t_1}^t c(s)\psi_\epsilon(s)ds + \frac{\epsilon}{2}$$

for $t \in [t_1, t_2]$. By the same argument one can prove $\psi_\epsilon \geq 0$ for all $\epsilon > 0$, and $\tilde{v}(t) \geq v(t)$ for $t \in [t_1, t_2]$.

We repeat this argument and conclude that

$$\tilde{v}(t) \geq v(t)$$

for all $t \in [0, T_0)$. By the same argument, we find if

$$\tilde{v}(t) - \tilde{v}(t_0) \leq \int_{t_0}^t \tilde{v}^p(s)ds \quad \text{for} \quad t_0, t \in [0, T_0) \quad \text{with} \quad t_0 \leq t,$$

then

$$\tilde{v}(t) \leq v(t) \quad \text{for} \quad t \in [0, T_0).$$

□

Proof of Theorem. Let $\{r_n\}_{n=1}^\infty$, $\{x_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be as in Theorem satisfying (3). Set $\lambda_n > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in $B(0, r_n)$, and let $\phi_n(x) \geq 0$ denote the corresponding positive eigenfunction normalized by $\int_{B(0,r_n)} \phi_n(x)dx = 1$. By scaling it is easy to observe that

$$\lambda_n = \frac{c}{r_n^2} \tag{9}$$

with c defined in Theorem. Define

$$G_n(t) = \int_{B(x_n,r_n)} u(x,t)\phi_n(x - x_n)dx.$$

Let $\nu_n(x)$ denote the outward unit normal to $B(0, r_n)$ at $x \in \partial B(0, r_n)$. By (2) and the fact that $\phi_n = 0$ and $\partial\phi_n/\partial\nu_n \leq 0$ on $\partial B(0, r_n)$ with the unit normal vector ν_n, we obtain

$$G_n(t) \geq G_n(0) + \int_0^t \int_{B(x_n, r_n)} (-\lambda_n u^m(x,s)\phi(x) + u^p(x,s)\phi(x))dxds.$$

Put

$$h_n(s) = \begin{cases} -\lambda_n s^m + s^p, & s \geq \left(\frac{m\lambda_n}{p}\right)^{\frac{1}{p-m}}, \\ -\lambda_n \left(\frac{m\lambda_n}{p}\right)^{\frac{m}{p-m}} + \left(\frac{m\lambda_n}{p}\right)^{\frac{p}{p-m}}, & 0 \leq s \leq \left(\frac{m\lambda_n}{p}\right)^{\frac{1}{p-m}}, \end{cases} \tag{10}$$

similarly as in [5]. Since h_n is convex, we obtain

$$G_n(t) \geq G_n(0) + \int_0^t h_n(G_n(s))ds. \tag{11}$$

by Jensen's inequality. Let us consider the system of ordinary differential equations

$$\begin{cases} g_n'(t) = h_n(g_n(t)), \\ g_n(0) = G_n(0) \geq b_n. \end{cases} \tag{12}$$

Define $T_{g_n} = \sup\{t \geq 0 : g_n(t) < \infty\}$ and $T_{G_n} = \sup\{t \geq 0 : G_n(t) < \infty\}$. Since g_n satisfies

$$g_n(t) = g_n(0) + \int_0^t h_n(g_n(s))ds,$$

and from Lemma, we obtain $G_n \geq g_n$ and $T_{g_n} \geq T_{G_n}$.

Consider the solutions of (1) with the initial data b_n. The maximal existence times of the solutions denoted by $T^*(b_n)$ is estimated as

$$T^*(b_n) = \int_{b_n}^{\infty} \frac{d\xi}{\xi^p}.$$

Note that $\lim_{n\to\infty} T^*(b_n) = 0$. From (3) we may assume that there exist $n_0 \geq 0$ such that

$$\frac{1}{r_n^2 b_n^{p-m}} < \varepsilon$$

for $n \geq n_0$ and $\varepsilon \in (0, 1/c)$. From (9) we see that

$$\lambda_n b_n^m < c\varepsilon b_n^p,$$

and

$$\lambda_n \xi^m < c\varepsilon \xi^p \tag{13}$$

for $\xi \geq b_n$ and $n \geq n_0$. Since $b_n \geq (m\lambda_n/p)^{1/(p-m)}$ by (13), we have

$$T_{g_n} = \int_{b_n}^{\infty} \frac{d\xi}{h_n(\xi)} = \int_{b_n}^{\infty} \frac{d\xi}{-\lambda_n \xi^m + \xi^p}$$

for $n \geq n_0$ by (13). Thus we see that

$$\frac{T^*(b_n)}{T_{g_n}} = \frac{\int_{b_n}^{\infty} d\xi/\xi^p}{\int_{b_n}^{\infty} d\xi/(-\lambda_n \xi^m + \xi^p)} > \frac{\int_{b_n}^{\infty} d\xi/\xi^p}{\int_{b_n}^{\infty} d\xi/\{(1-c\varepsilon)\xi^p\}} > 1 - c\varepsilon \tag{14}$$

for $n \geq n_0$. Thus we obtain

$$\lim_{m\to\infty} \frac{T^*(b_n)}{T_{g_n}} \geq 1 - c\varepsilon > 0.$$

Noting that $\lim_{n\to\infty} T^*(b_n) = 0$, we see that $\lim_{n\to\infty} T_{g_n} = 0$. Again we get $T_{G_n} \to 0$ as $n \to \infty$. By the definition of the weak solution we have $T^* = 0$. □

Acknowledgement. The author would like to thank Professor Ryuichi Suzuki for his useful discussions. Much of the work of the author was done while he visited the University of Tokyo during 2005-2009 as a postdoctoral fellow. Its hospitality is gratefully acknowledged as well as support from formation of COE "New Mathematical Development Center to Support Scientific Technology" during 2005-2008 and COE "The research and training center for new development in mathematics" in 2009, supported by JSPS.

References

[1] D. Andreucci and E. Di Benedetto, *On the Cauchy problem and initial trances for a class of evolution equations with strongly nonlinear sources*, Ann. Scuola Norm. Sup. Pisa, **18** (1991), 363-441.

[2] J. M. Ball, *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations*, Quart. J. Math. Oxford Ser. (2) **28** (1977), no. 112, 473–486.

[3] Y. Giga and N. Umeda, *On blow-up at space infinity for semilinear heat equations*, J. Math. Anal. Appl. **316** (2006), no. 2, 538–555.

[4] Y. Giga and N. Umeda, *On instant blow-up for semilinear heat equation with growing initial data*, Methods Appl. Anal. **15** (2008), no. 2, 185–196.

[5] T. Imai and K. Mochizuki, *On blow-up of solutions for quasilinear degenerate parabolic equation*, Publ. Res. Inst. Math. Soc. **27** (1991), 695–709.

[6] S. Kaplan, *On the growth of solutions of quasi-linear parabolic equations*, Comm. Pure Appl. Math. **16** (1963), 305–330.

[7] H. A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + \mathcal{F}(u)$, Arch. Rational Mech. Anal. **51** (1973), 371–386.