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Lines pinning lines*

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1 Introduction

A *line transversal* to a family of convex objects in $\mathbb{R}^d$ is a line intersecting each member of the family. There is a rich theory of geometric transversals, see for instance the surveys of Danzer et al. [3], Eckhoff [4], Goodman et al. [5] and Wenger [7].

A classic result is Hadwiger's transversal theorem: a family $\mathcal{F}$ of (pairwise) disjoint compact convex sets in the plane has a line transversal if and only if there is an ordering of $\mathcal{F}$ such that each triple has a line transversal consistent with that ordering. The idea of Hadwiger's original proof is to uniformly shrink the sets until a triple has a unique transversal ℓ. The line ℓ is then necessarily an isolated transversal of the triple, that is, any perturbation of ℓ will miss a member of the triple.

Cheong et al. [2] recently proved a similar Hadwiger-type theorem: a family $\mathcal{F}$ of (pairwise) disjoint unit balls in $\mathbb{R}^d$ has a line transversal if and only if there is an ordering of $\mathcal{F}$ such that each $2d$-tuple has a line transversal consistent with that ordering. Again, the idea of the proof is to shrink the balls uniformly until some $2d$-tuple has a unique transversal.

We call an isolated transversal ℓ to a family $\mathcal{F}$ a *pinned* transversal, and the family $\mathcal{F}$ a *pinning* of ℓ. If $\mathcal{F}$ pins ℓ and no proper subset of $\mathcal{F}$ does, then $\mathcal{F}$ is a *minimal* pinning of ℓ. A minimal pinning by compact convex figures in the plane consists of exactly three objects; we say that convex figures have *pinning number* three. Cheong et al. [2] show that disjoint unit balls have pinning number $2d-1$, this result can be generalized to disjoint balls of arbitrary radius [1].

The existence of a bounded pinning number can be interpreted as a Helly-type theorem: a family $\mathcal{F}$ of disjoint balls does not pin a line ℓ if and only if every subfamily of size at most $2d-1$ does not pin ℓ. The version of Hadwiger's theorem for disjoint balls is rather exceptional in the sense that no such generalization holds for line transversals to disjoint convex sets in dimension greater than two, not even for disjoint translates of a convex set [6]. In particular, there are ordered families $\mathcal{F}$ of n convex polytopes in $\mathbb{R}^3$

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without a line transversal, but such that any subfamily of size $n-1$ has a line transversal consistent with the ordering.

In this paper, we show that, nevertheless, convex polytopes in $\mathbb{R}^3$ have bounded pinning number, at least under a mild assumption: If a family $\mathcal{F}$ of convex polytopes pins a line ℓ and ℓ touches each polytope in an edge, then there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of at most eight polytopes that already pins ℓ.

This implies that a bounded pinning number is a strictly weaker property than a Hadwiger-type theorem for transversals: while the existence of a line transversal is not necessarily determined by a bounded subfamily, *locally* this is the case. It will be interesting to see how far the bounded pinning number generalizes. So far, we know of no class of convex objects that does not have a bounded pinning number.

The assumption that ℓ may not touch a polytope $P \in \mathcal{F}$ in a facet is necessary for our arguments, since we reduce the family of polytopes to a family of oriented lines, as follows:

Given two non-parallel oriented lines ℓ_1 and ℓ_2 with direction vectors d_1 and d_2, we say that ℓ_2 *passes to the right of* ℓ_1 if ℓ_2 can be translated by a positive multiple of $d_1 \times d_2$ to meet ℓ_1.

Consider a line ℓ_0 that is pinned by a family of convex polytopes. Any polytope whose interior intersects ℓ_0 cannot contribute to the pinning, and so we can assume that all polytopes are tangent to ℓ_0. The restriction we will make in this paper is that we assume that ℓ_0 touches each polytope P in an edge e. This means that we can orient the supporting line g of e in such a way that for lines ℓ in a neighborhood of ℓ_0, ℓ intersects P if and only if ℓ meets g or passes to the right of g.

This observation fails if we allow ℓ_0 to touch P in a facet. On the other hand, touching in a vertex can be handled by considering *two* edges, as long as ℓ_0 is not coplanar with a facet incident to the vertex.

The observation allows us to discard the original family of convex polytopes, and to speak about lines pinned by a family $\mathcal{F}$ of oriented lines. Formally, a line ℓ_0 is pinned by a family $\mathcal{F}$ of oriented lines if there is a neighborhood N of ℓ_0, such that for every $\ell' \neq \ell_0$ in N, there is a line $g \in \mathcal{F}$ such that ℓ' passes to the left of g.

We will exclude families $\mathcal{F}$ where two lines of $\mathcal{F}$ are simultaneously concurrent and coplanar with ℓ_0. (Indeed, this cannot happen if the lines come from edges of polytopes pinning ℓ_0.)

2 Preliminaries

Without loss of generality, we identify ℓ_0 with the positively oriented z-axis. We are interested in lines in a neighborhood of ℓ_0. Let $\mathfrak{L}$ denote the set of lines (all our lines are oriented) whose direction makes positive dot-product with $(0,0,1)$. Any line $\ell \in \mathfrak{L}$ intersects the two planes $z = 0$ and $z = 1$, and $\mathfrak{L}$ is homeomorphic to $\mathbb{R}^4$, where $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ corresponds to the line $\ell(u)$ passing through $(u_1, u_2, 0)$ and $(u_3, u_4, 1)$. We have $\ell_0 = \ell(0)$, and any sufficiently small neighborhood of ℓ_0 corresponds to a neighborhood of 0 in $\mathbb{R}^4$.

Consider now the family $\mathfrak{C}$ of (oriented) lines crossing ℓ_0, that is, meeting ℓ_0 without being parallel to it. A line $g \in \mathfrak{C}$ meets ℓ_0 in a point $(0,0,\lambda)$, makes a slope δ with the plane

$z = \lambda$, and its projection on the xy-plane makes an angle α with the positive x-axis. Let $g(\lambda, \alpha, \delta)$ be this line. It passes through the points $(0, 0, \lambda)$ and then $(\cos\alpha, \sin\alpha, \lambda + \delta)$.

For a line $g = g(\lambda, \alpha, \delta) \in \mathfrak{C}$, we define the function $\sigma_g : \mathbb{R}^4 \to \mathbb{R}$ as

$$\begin{aligned}\sigma_g(u) := &\begin{vmatrix} u_1 & u_3 & 0 & \cos\alpha \\ u_2 & u_4 & 0 & \sin\alpha \\ 0 & 1 & \lambda & \lambda+\delta \\ 1 & 1 & 1 & 1 \end{vmatrix} \\ = &\ \delta(u_2u_3 - u_1u_4) + \eta(\lambda, \alpha) \cdot u,\end{aligned}$$

where

$$\eta(\lambda, \alpha) := \begin{pmatrix} (1-\lambda)\sin\alpha \\ -(1-\lambda)\cos\alpha \\ \lambda\sin\alpha \\ -\lambda\cos\alpha \end{pmatrix}.$$

We note that $\sigma_g(u)$ is zero if $\ell(u)$ meets g, positive if $\ell(u)$ passes to the right of g, and negative if $\ell(u)$ passes to the left of g.

The function σ_g is quadratic, so the set $\mathcal{S}_g := \{u \in \mathbb{R}^4 \mid \sigma_g(u) = 0\} \subset \mathbb{R}^4$ is a quadric through the origin. However, when g is parallel to the xy-plane, then $\delta = 0$, the quadratic term vanishes, and $\mathcal{S}_g$ becomes the hyperplane $\eta(\lambda, \alpha) \cdot u = 0$ through the origin. By the above, the line $\ell(u)$ meets g if and only if $u \in \mathcal{S}_g$.

We can now define $\mathcal{U}_g := \{u \in \mathbb{R}^4 \mid \sigma_g(u) \geq 0\} \subset \mathbb{R}^4$. The volume $\mathcal{U}_g$ is bounded by $\mathcal{S}_g$. The line ℓ_0 is pinned by a family $\mathcal{F} \subset \mathfrak{C}$ if and only if $N \cap \bigcap_{g\in\mathcal{F}} \mathcal{U}_g = \{0\}$ for some neighborhood N of the origin 0.

Let $\mathfrak{C}(\lambda, \alpha) := \{g(\lambda, \alpha, \delta) \mid \delta \in \mathbb{R}\} \subset \mathfrak{C}$, and consider $g \in \mathfrak{C}(\lambda, \alpha)$. The function σ_g is smooth and its gradient at the origin is $\eta(\lambda, \alpha)$. The vector $\eta(\lambda, \alpha)$ is therefore the normal to the surface $\mathcal{S}_g$ at the origin, pointing into the interior of $\mathcal{U}_g$. Since $\eta(g) := \eta(\lambda, \alpha)$ does not depend on δ, all surfaces $\mathcal{S}_g$ for lines $g \in \mathfrak{C}(\lambda, \alpha)$ share the same tangent hyperplane $h(\lambda, \alpha) = \{u \in \mathbb{R}^4 \mid \eta(\lambda, \alpha) \cdot u = 0\}$ at the origin. The line $g^{\perp}(\lambda, \alpha) := g(\lambda, \alpha, 0)$ is the only line $g \in \mathfrak{C}(\lambda, \alpha)$ for which $\mathcal{S}_g$ coincides with $h(\lambda, \alpha)$.

Given a family $\mathcal{F} \subset \mathfrak{C}$, we call $\mathcal{F}$ *generic* if the normal vectors η_g corresponding to every four lines in $\mathcal{F}$ are linearly independent.

For families of lines all orthogonal to ℓ_0, this is a natural definition: If $\mathcal{F}$ is generic, then the common intersection of any four hyperplanes $\mathcal{S}_g$ is exactly the origin $\{0\}$. This is the same as saying that any four lines in $\mathcal{F}$ have no infinite family of transversals (in fact, they have only the single transversal ℓ_0).

3 Pinning by orthogonal lines

In this section, we assume that $\mathcal{F}$ is a family of (oriented) lines all orthogonal to the line ℓ_0. This implies that the surfaces $\mathcal{S}_g$ are hyperplanes through the origin, and the volumes $\mathcal{U}_g$ are half-spaces. We can therefore use Helly's dimensional theorem to obtain our main result:

Theorem 1. *If a line ℓ_0 is pinned by a family $\mathcal{F}$ of oriented lines all orthogonal to ℓ_0, then there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of size at most eight that already pins ℓ_0. Furthermore, if $\mathcal{F}$ is generic, then the size of $\mathcal{F}'$ is at most five.*

Proof. The assumption of pinning implies $\bigcap_{g\in\mathcal{F}} \mathcal{U}_g = \{0\}$. By the dimensional Helly theorem, the intersection of n convex sets in $\mathbb{R}^4$ is a single point if and only if they all intersect and eight of them intersect in a single point. This implies that there is a subfamily $\mathcal{F}'$ of eight oriented lines that already pin ℓ_0.

Assume now that $\mathcal{F}$ is generic, and for each $g \in \mathcal{F}$, let the open halfspace $\mathcal{U}'_g$ be the interior of $\mathcal{U}_g$. We have $\bigcap_{g\in\mathcal{F}} \mathcal{U}'_g = \emptyset$, and so by Helly's theorem there is subfamily $\mathcal{F}' \subset \mathcal{F}$ of at most five halfspaces with $\bigcap_{g\in\mathcal{F}'} \mathcal{U}'_g = \emptyset$. Since any four normals are linearly independent, this implies that $\bigcap_{g\in\mathcal{F}'} \mathcal{U}_g = \{0\}$, proving the theorem. □

The number eight in the theorem is not tight, since we excluded pairs of lines that are concurrent and simultaneously coplanar with ℓ_0. In the full paper, we characterize all possible configurations.

4 Pinning by generic lines

We will start by discussing what it means for a family $\mathcal{F} \subset \mathfrak{C}$ to be generic.

Consider three lines $g_1, g_2, g_3 \in \mathfrak{C}$ with $g_i \in \mathfrak{C}(\lambda_i, \alpha_i)$. If all three lie in a common plane then $\alpha_1 = \alpha_2 = \alpha_3$. In this case, any line lying in this common plane will intersect all three lines, and the intersection $\mathcal{S}_{g_1} \cap \mathcal{S}_{g_2} \cap \mathcal{S}_{g_3}$ is a two-plane. In all other cases, $\mathcal{S}_{g_1} \cap \mathcal{S}_{g_2} \cap \mathcal{S}_{g_3}$ is a one-dimensional curve, corresponding to lines in $\mathfrak{L}$ that meet all three lines g_1, g_2, g_3. The union of these transversal lines is a doubly-ruled quadric $\mathcal{B}(g_1, g_2, g_3)$. The first family of rulings consists of the transversals to g_1, g_2, g_3, the second family of rulings contains the lines g_1, g_2, g_3. The line ℓ_0 is a line of the first family. For every point $(0, 0, \lambda) \in \ell_0$, there is exactly one line from the second family of rulings through $(0, 0, \lambda)$. This means that there is an α and a δ such that $g(\lambda, \alpha, \delta)$ is contained in $\mathcal{B}(g_1, g_2, g_3)$. The tangent plane of $\mathcal{B}(g_1, g_2, g_3)$ at the point $(0, 0, \lambda)$ is the plane spanned by ℓ_0 and $g(\lambda, \alpha, \delta)$.

We can now characterize what it means for the normal vectors $\eta(g)$ to be linearly dependent.

Lemma 2. *Let $g_1, g_2, g_3, g_4 \in \mathfrak{C}$ with $g_i \in \mathfrak{C}(\lambda_i, \alpha_i)$, and let $\eta_i := \eta(g_i)$.*

The normals η_1 and η_2 are linearly dependent if ℓ_0, g_1 and g_2 are concurrent and coplanar. (We excluded this possibility.)

The normals η_1, η_2, and η_3 are linearly dependent if and only if some two of them are, or if ℓ_0, g_1, g_2, and g_3 are concurrent or *coplanar.*

The normals η_1, η_2, η_3, and η_4 are linearly dependent if and only if some three of them are, or if g_4 lies in the tangent plane to $\mathcal{B}(g_1, g_2, g_3)$ at $(0, 0, \lambda_4) = \ell_0 \cap g_4$.

For a family $\mathcal{F} \subset \mathfrak{C}$, we define the family $\mathcal{F}^\perp := \{g^\perp(\lambda, \alpha) \mid g(\lambda, \alpha, \delta) \in \mathcal{F}\}$ of "orthogonalized" lines. We have the following result:

Lemma 3. *Let $\mathcal{F} \subset \mathfrak{C}$. If $\mathcal{F}^\perp$ pins ℓ_0, then $\mathcal{F}$ pins ℓ_0.*

Proof. Assume $\mathcal{F}$ does not pin ℓ_0, and consider the intersection $\mathcal{U} := \bigcap_{g\in\mathcal{F}} \mathcal{U}_g$. Since $\mathcal{U}$ is a semialgebraic set, there is a smooth path starting at the origin 0 contained in $\mathcal{U}$. Let r be the direction of this path in 0. Then $r \cdot \eta(g) \geq 0$ for all $g \in \mathcal{F}$, implying $r \cdot \eta(g^\perp) \geq 0$ for all $g^\perp \in \mathcal{F}^\perp$. But then $\mathcal{F}^\perp$ does not pin ℓ_0. □

The reverse of Lemma 3 is not true—it fails for lines that are not generic. The following lemma, however, shows that the reverse holds for generic families of lines, and characterizes the case where it does not hold.

Lemma 4. *Let $\mathcal{F} \subset \mathfrak{C}$ such that $\mathcal{F}$ pins ℓ_0 but $\mathcal{F}^\perp$ does not pin ℓ_0. Then $\mathcal{F}$ is non-generic, and $\rho := \bigcap_{g^\perp \in \mathcal{F}^\perp} \mathcal{U}_{g^\perp}$ is a line through the origin or a half-line starting at the origin.*

Proof. If ρ has non-empty interior, then we can pick a direction r from 0 into the interior of ρ. The direction r must make a positive dot-product with all normals $\eta(g)$, $g \in \mathcal{F}$, and so ℓ_0 cannot be pinned by $\mathcal{F}$.

It follows that ρ has empty interior. Let S be the linear subspace spanned by ρ; it has dimension d at least one (since $\mathcal{F}^\perp$ does not pin ℓ_0) and at most three. But then S must be contained in $\mathcal{S}_{g^\perp}$ for at least $5-d$ lines $g^\perp$. The normals $\eta(g^\perp)$ must be orthogonal to the d-dimensional subspace S, implying that they lie in a $(4-d)$-dimensional subspace. But $5-d$ vectors in a $(4-d)$-subspace are linearly dependent, and so $\mathcal{F}$ is non-generic.

We now observe further that S cannot be three-dimensional, because that would imply two linearly dependent normals, and we excluded this case.

Assume that S is two-dimensional. This implies at least three linearly dependent normals. By Lemma 2, this means that there are three lines in $\mathcal{F}$ that all span the same plane with ℓ_0. This means that these lines restrict the transversal to this plane, and the remaining lines pin it inside the plane. But this property is then also true for $\mathcal{F}^\perp$, in contradiction to our assumption that $\mathcal{F}^\perp$ does not pin ℓ_0.

It follows that S is one-dimensional. Since ρ is an intersection of closed half-spaces, this means that either ρ is a line through the origin or a (closed) half-line starting at the origin. □

Lemma 4 implies that Theorem 1 immediately generalizes to generic families.

Theorem 5. *If the line ℓ_0 is pinned by a generic family $\mathcal{F} \subset \mathfrak{C}$ of oriented lines, then there is a subfamily $\mathcal{F}'$ of size at most five that already pins ℓ_0.*

Proof. By Lemma 4, $\mathcal{F}^\perp$ also pins ℓ_0. We can now use Theorem 1 to conclude that there is a subfamily $\mathcal{F}^{\perp\prime} \subset \mathcal{F}^\perp$ of at most five lines that already pin ℓ_0. By Lemma 3, the corresponding family $\mathcal{F}' \subset \mathcal{F}$ also pins ℓ_0, proving the theorem. □

5 When first-order approximation is not enough

The only case that remains to be considered is when $\mathcal{F}$ pins ℓ_0, but $\mathcal{F}^\perp$ does not. This really can happen, as the following example shows. The rest of this section is dedicated to showing that this example is really the only such situation.

Consider the surface $\mathcal{B}$ defined by the equation $y = xz$. Let g_0, g_1, g_2, g_3 be the lines in the intersection of $\mathcal{B}$ and the planes $z = 0, 1, 2, 3$, respectively. (That is, g_0 is the x-axis, g_1 is the line $(t, t, 1)$, g_2 is the line $(t, 2t, 2)$, and g_3 is the line $(t, 3t, 3)$, for $t \in \mathbb{R}$.)

We orient the four lines in alternating directions: g_0 and g_2 in the direction of increasing x-coordinates, g_1 and g_3 in the direction of decreasing x-coordinates. It is not difficult to see that any common transversal to the four lines must lie in the surface $\mathcal{B}$. This is because in order to pass on the correct side of the four lines $g_0, \dots, g_3$, a transversal not contained in $\mathcal{B}$ would have to intersect $\mathcal{B}$ at least three times (points of tangency counted twice), which is impossible as $\mathcal{B}$ is a quadric.

We now replace g_0 by g_0', by rotating the line slightly around the origin inside the plane $y = 0$ (the tangent plane to $\mathcal{B}$ at the origin). For concreteness, we chose g_0' to be the line $z = x/100$. This means that for points on g_0', we have $y - xz = 0 - x^2/100 < 0$ for $x \neq 0$. This implies that g_0' lies everywhere (except at the origin) below $\mathcal{B}$, which is the same as saying that lines in $\mathcal{B}$ pass g_0' on the left.

It follows that g_0', g_1, g_2, g_3 pin ℓ_0. Indeed, the only transversal lying inside $\mathcal{B}$ is ℓ_0, and the same argument as above implies that no other line can be a transversal.

If we set $\mathcal{F} = \{g_0', g_1, g_2, g_3\}$, then $\mathcal{F}$ pins ℓ_0, but $\mathcal{F}^\perp = \{g_0, g_1, g_2, g_3\}$ does not.

We start with a lemma that improves the characterization of Lemma 4.

Lemma 6. *Let $\mathcal{F} \subset \mathfrak{C}$ be a* minimal *family such that $\mathcal{F}$ pins ℓ_0 but $\mathcal{F}^\perp$ does not pin ℓ_0. Then $\rho := \bigcap_{g^\perp \in \mathcal{F}^\perp} \mathcal{U}_{g^\perp}$ is a line through the origin. In other words, there is a doubly-ruled quadric $\mathcal{B}$ containing ℓ_0 and all lines $g^\perp$, for $g \in \mathcal{F}$.*

We can show that in this case as well there is a constant number of lines pinning ℓ_0. The proof is omitted in this abstract, and we conclude with the main theorem.

Theorem 7. *Given a family $\mathcal{F} \subset \mathfrak{C}$ of lines that pin ℓ_0. Then there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of at most eight lines such that $\mathcal{F}'$ already pin ℓ_0. If $\mathcal{F}$ is generic, the size of $\mathcal{F}'$ is five.*

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