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Analyzing automorphism groups of oriented matroids by semidefinite programming

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Abstract

Many geometric realizability problems such as the realizability problem of oriented matroids can be reduced to *the Existential Theory of the Reals (ETR)*. The authors proposed a non-realizability certificate for *a symmetric realization problem* using semidefinite programming (SDP) [9]. In this paper, we review this study. In addition, we explain that this method can detect gaps between combinatorial symmetry and geometric symmetry by analyzing Richter-Gebert's example [12].

1 Introduction

Many geometric realizability problems can be reduced to *the Existential Theory of the Reals (ETR)*, the problem to decide whether a given polynomial equalities and inequalities system with integer coefficients has a real root or not. For example, the realizability problem of oriented matroids can be reduced to ETR. Therefore, one can analyze many geometric realizability problems in terms of polynomial system, which has been investigated largely in computational algebra. There are two main types of approaches to ETR. One is a type of symmbolic approaches such as cylindrical algebraic decomposition [4]. We obtain exact result by these approaches, but the time complexity of them are too large and we cannot apply these approaches to large instances. The other is a type of numerical approaches such as polynomial optimization using semidefinite programming [8, 11]. We can compute more fastly using these approaches, but they produce numerical errors and we cannot obtain exact results. On the other hand, many geometric realizability problems have the follwoing features. Firstly, their problem sizes often become large. Therefore, it is difficult to apply symbolic approaches directly to them. Sencondly, results containing numerical errors do not make sense. Therefore, we cannot apply numerical approach directly. To overcome this difficulty, Miyata, Moriyama and Imai [9] proposed a numerical method to prove infeasibility of polynomial system exactly, which was based on the SDP relaxation proposed by Lasserre [8] and Parrilo [11], and analyzed the realizability problem of oriented matroids using this method.

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In [10], we proposed a method to analyze automorphims groups of oriented matroids using semidefinite programming. Automorphism groups of vector configurations had been studied well, but there was little study on those of oriented matroids. We proposed to analyze automorphism groups of oriented matroids by measuring gaps between automorphism groups of oriented matroids and those of vector configurations. It can be formulated as a kind of geometric realizability problems: "Can a given oriented matroid be realized as a vector configuration whose symmetry group coincides with that of the oriented matroid?", which we call *a symmetric realization problem.* This problem can be reduced to ETR, and we analyzed it by semidefinite programming. This problem seems difficult to be analyzed by the existing non-realizability certificates of oriented matroids such as non-Euclidean [6], non-HK* [7] and a biquadratic final polynomial method [2], and only two examples with such gaps are known today [14, 12]. Actually, geometric non-realizability certificates such as non-Euclidean and non-HK* seem difficult to be applied to this problem, and a biquadratic final polynomial method canont be used to detect gaps between combinatrial symmetry and geometric symmetry by the following theorem.

Theorem 1.1 ([3]) *For any subgroup $G \subset Aut(M)$ we have: An oriented $M = (E, \chi)$ admits a bi-quadratic final polynomial, if and only if it admits a symmetric bi-quadratic final polynomial with respect to G.*

We applied our method to oriented matroids of rank 4 on up to 8 elements and those of rank 3 on up to 9 elements. Our method with relaxation order 1 could find oriented matroids of rank 3 on up to 8 elements or those of rank 4 on up 9 elements with such a gap, but we observed that there were gaps between the feasibility of our method wihtout the conditions of symmetry and those with the conditions of symmetry. In this paper, we explain that our method with higher relaxation order can detect gaps between geometric symmetry and combinatorial symmetry by analyzing Richter-Gebert's example [12].

2 Preliminaries

In this section, we explain symmetry of oriented matroids and that of vector configurations. For details about oriented matroids, see [1].

First, we explain automorphism groups of vector configurations.

Definition 2.1 *Let $V = (v_1, ..., v_n) \in \mathbb{R}^{n \times r}$ be a vector configuration. If there exist an inversible linear map $A : \mathbb{R}^r \to \mathbb{R}^r$ such that*

$$A \cdot v_i = v_{\pi(i)} \text{ for all } i = 1, ..., n,$$

for a permutation π on $\{1, ..., n\}$, we say that V has a symmetry π and A is an affine automorphism of V.

Definition 2.2 *The group formed by all symmetries of a vecotor configuration V is called the automorphism group of V, and we denote it by Aut(V).*

We immediately notice the following proposition.

Proposition 2.3 *Let A be an affine automorphism of a vector configuration V. Then,* $\det(A) = 1$ *or* $\det(A) = -1$.

Proof: Asume that $A \cdot v_i = v_{\pi_i}$. Because $Aut(V)$ is a finite group, there exists $m \in \mathbb{N}$ such that $\pi^m = id$, where id is the identity. Thus, $A^m(v_1, ..., v_n) = (v_1, ..., v_n)$. Then, we choose a maximal linear independent set $\{v_{i_1}, ..., v_{i_r}\}$ of V and make a inversible matrix $M_V = (v_{i_1}, ..., v_{i_r})$. Then, it satisfies $A^m M_V = M_V$ and we can conclude that $A^m = I$, where I is the identity matrix. Therefore, $\det(A)^m = 1$. It implies $\det(A) = 1$ or $\det(A) = -1$.

An automorphism whose determinant is -1 is called a reflection, and that whose determinant is 1 is called a rotation. Considering the above discussion, symmetry of oriented matroids is defined as follows.

Definition 2.4 *Let $M = (E, \chi)$ be an n-element rank-r oriented matroid.*

$$Aut(M) := \{\sigma \in S_n \mid \sigma \cdot \chi = \chi \text{ or } \sigma \cdot \chi = -\chi\}$$

is called the automorphism group of M.

An automorphism σ of M such that $\sigma \cdot \chi = \chi$ is called a rotation of M and we denote the set of all rotations of M by $Rot(M)$. Similarly, an automorphism σ of M such that $\sigma \cdot \chi = -\chi$ is called a reflection of M and we denote the set of all reflections of M by $Ref(M)$.

3 Analyzing gaps between combinatorial symmetry and geometric symmetry by SDP

In this section, we review a formulation of a non-realizability certificate of the following problem by semidefinite programming [10] in order to analyze automorphism groups of oriented matroids.

Problem 1 *Let $M = (E, \chi)$ be an oriented matroid. Then, is there a vector configuration V such that V is a realization of M and $Aut(V) = Aut(M)$?*

Let $M = (E, \chi)$ be an oriented matroid of rank r where $|E| = n$. Suppose that M can be realized as a vector configuration whose automorphism group is $Aut(M)$. Then there exists a vector configuration $V = (v_i)_{i=1}^n \in \mathbb{R}^{r \times n}$ such that for $1 \le i_1 < ... < i_r \le n$, $1 \le j_1 < ... < j_r \le n$, $\sigma \in Rot(M)$ and $\tau \in Ref(M)$,

$$\begin{cases} [i_1 ... i_r][j_1 ... j_r] - \sum_{k=1}^r [j_k i_2 ... i_r][j_1 ... j_{k-1} i_1 j_{k+1} ... j_r] = 0, \\ \qquad\qquad\qquad \text{(Grassmann-Plücker relations)} \\ \operatorname{sign}([i_1 ... i_r]) \;=\; \chi(i_1, ..., i_r), \\ \qquad\qquad\qquad \text{(Conditions of chirotope)} \\ [i_1 ... i_r] = [\sigma(i_1) ... \sigma(i_r)], \\ [i_1 ... i_r] = -[\tau(i_1) ... \tau(i_r)], \\ \qquad\qquad\qquad \text{(Conditions of symmetry)} \end{cases}$$

where $[i_1 \dots i_r] := \det(v_{i_1}, \dots, v_{i_r})$. Then, we rewrite the above system as follows. For $1 \le i_1 < \dots < i_r \le n$ and $1 \le j_1 < \dots < j_r \le n$,

$$\begin{cases} x_{(i_1,\dots,i_r),(j_1,\dots,j_r)} - \sum_{k=1}^{r} x_{(j_k i_2 \dots i_r),(j_1 \dots j_{k-1} i_1 j_{k+1} \dots j_r)} = 0, \\ \mathrm{sign}(x_{(i_1,\dots,i_r),(j_1,\dots,j_r)}) = \chi(i_1,\dots,i_r)\cdot\chi(j_1,\dots,j_r), \\ \mathrm{sign}(x_{(i_1,\dots,i_r)}) = \chi(i_1,\dots,i_r), \\ x_{(i_1,\dots,i_r)} = x_{(\sigma(i_1),\dots,\sigma(i_r))}, \\ x_{(i_1,\dots,i_r)} = -x_{(\tau(i_1),\dots,\tau(i_r))}, \\ x_{(i_1,\dots,i_r),(j_1,\dots,j_r)} = [i_1 \dots i_r][j_1 \dots j_r], \\ x_{(i_1,\dots,i_r)} = [i_1 \dots i_r] \end{cases}$$

where $x_{(i_{\sigma'(1)},\dots,i_{\sigma'(r)}),(j_{\tau'(1)},\dots,j_{\tau'(r)})}$ denotes $\mathrm{sgn}(\sigma')\cdot\mathrm{sgn}(\tau')\cdot x_{(i_1,\dots,i_r),(j_1,\dots,j_r)}$ for $1 \le i_1 < \dots < i_r \le n$ and $1 \le j_1 < \dots < j_r \le n$, and σ' and τ' are permutations on $\{1,\dots,r\}$.

Here we consider a vector u_1 given by listing 1 and elements of $\Lambda(n,r)$ in lexicographic order:

$$u_1 := (1, [1,2,\dots,r], [1,2,\dots,r-1,r+1], \dots, [n-r, n-r+1, \dots, n])^T.$$

Then, under the constraints $x_{(i_1,\dots,i_r),(j_1,\dots,j_r)} = [i_1 \dots i_r][j_1 \dots j_r]$ and $x_{(i_1,\dots,i_r)} = [i_1 \dots i_r]$ for $1 \le i_1 < \dots < i_r \le n$ and $1 \le j_1 < \dots < j_r \le n$,

$$X := \begin{pmatrix} 1 & x_{(1,2,\dots,r)} & \cdots & x_{(n-r,n-r+1,\dots,n)} \\ x_{(1,2,\dots,r)} & x_{(1,2,\dots r),(1,2,\dots,r)} & \cdots & x_{(1,2,\dots,r),(n-r,\dots,n)} \\ x_{(1,2,\dots,r-1,r+1)} & x_{(1,2,\dots,r-1,r+1),(1,2,\dots,r)} & \cdots & x_{(1,2,\dots,r-1,r+1),(n-r,\dots,n)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(n-r,\dots,n)} & x_{(n-r,\dots,n),(1,2,\dots,r)} & \cdots & x_{(n-r,\dots,n),(n-r,\dots,n)} \end{pmatrix}$$

$(= u_1 u_1^T)$ should be a positive semidefinite symmetric matrix. Therefore if M is realizable, the following system (SDP A) is feasible: for $1 \le i_1 < \dots < i_r \le n$ and $1 \le j_1 < \dots < j_r \le n$,

$$\begin{cases} x_{(i_1,\dots,i_r),(j_1,\dots,j_r)} - \sum_{k=1}^{r} x_{(j_k i_2 \dots i_r),(j_1 \dots j_{k-1} i_1 j_{k+1} \dots j_r)} = 0, \\ \mathrm{sign}(x_{(i_1,\dots,i_r),(j_1,\dots,j_r)}) = \chi(i_1,\dots,i_r)\cdot\chi(j_1,\dots,j_r), \\ \mathrm{sign}(x_{(i_1,\dots,i_r)}) = \chi(i_1,\dots,i_r), \\ x_{(i_1,\dots,i_r)} = x_{(\sigma(i_1),\dots,\sigma(i_r))}, \\ x_{(i_1,\dots,i_r)} = -x_{(\tau(i_1),\dots,\tau(i_r))}, \\ X \text{ is a positive semidefinite symmetric matrix,} \end{cases} \qquad \text{(SDP A)}$$

where $x_{(i_{\sigma'(1)},\dots,i_{\sigma'(r)}),(j_{\tau'(1)},\dots,j_{\tau'(r)})}$ denotes $\mathrm{sgn}(\sigma')\cdot\mathrm{sgn}(\tau')\cdot x_{(i_1,\dots,i_r),(j_1,\dots,j_r)}$ for all $i_1,\dots,i_r,j_1,\dots,j_r$ for $1 \le i_1 < \dots < i_r \le n$ and $1 \le j_1 < \dots < j_r \le n$, and σ' and τ' are permutations on $\{1,\dots,r\}$.

Remark1. The relaxation explained above corresponds with the SDP relaxation of a polynomial system with relaxation order 1 which was introduced in [8]. If we use the higher order SDP relaxation, we would obtain a more powerful certificate. However, the problem size will become very large. For detail about the higher order SDP relaxation,

see [8].
Remark2. We considered all Grassmann-Plücker relations as constraints in the above discussion, but Grassmann-Plücker relations have redundancy. One way to deal with this issue is considering a minmal generating set of Grassmann-Plücker ideal. For a minimal generating set of Grassmann-Plücker ideal, see [9].

4 Experimental results

In [10], we applied our method to oriented matroids on up to 8 elements using Finschi and Fukuda's database of oriented matroids [5] to look up oriented matroids and SeDuMi [13, 15] to solve SDPs.

rank \ elements	2	3	4	5	6	7	8	9	10
2	1	1	1	1	1	1	1	1	1
3		1	2	4	17	143	4890	461053	95052532
4			1	3	12	206	181472		
5				1	4	25	6029		
6					1	5	50	508321	
7						1	6	91	
8							1	7	164
9								1	8
10									1

Table 1: The number of oriented matroids [5] (reorientation class)

We decided non-realizability of 797 oriented matroids of rank 4 on 8 elements and 9 oriented matroids of rank 3 on 9 elements but all of them were non-realizable even without the conditions of symmetry. On the other hand, we observed that feasibility of SDPs without the symmetry constriants and those of SDPs with the symmetry constraints were different. Actually, 357 oriented matroids of rank 4 on 8 elements and 9 oriented matroids of rank 3 on 9 elements cannot be decided to be non-realizable by our method without constraints of symmetry, but decided to be non-realizable with symmetry constraints. It does not occur for a biquadratic final polynomial method, and implies possibility of detecting gaps between geometric symmetry and combinatorial symmetry. We actually explain that our method can detect such gaps in the next section.

5 Analyzing Richter-Gebert's example

In [12], Richter-Gebert constructed an example of 14-element rank-3 oriented matroid with combinatorial symmetry which cannot be realized geometrically. In this section, we analyze this example using our method. Richter's example is an oriented matroid associated with the vector configuration $X = (x_1, ..., x_{14})$ defined by $x_1 = (1,0,0)$, $x_2 = (0,1,0)$, $x_3 = (1,0,1)$, $x_4 = (0,1,1)$, $x_5 = (1-t)x_3 + (1+t)x_4$, $x_6 = x_5x_2 \wedge x_1x_4$, $x_7 = x_5x_1 \wedge x_2x_3$, $x_8 = x_6x_3 \wedge x_5x_1$, $x_9 = x_7x_4 \wedge x_5x_2$, $x_{10} = x_3x_4 \wedge x_8x_2$, $x_{11} = x_3x_4 \wedge x_9x_1$,

$x_{12} = x_7x_{10} \wedge x_{11}x_2$, $x_{13} = x_6x_{11} \wedge x_{10}x_1$, $x_{14} = x_1x_3 \wedge x_2x_4$ for $t \in (-3+\sqrt{8}, 0) \cup (0, 3-\sqrt{8})$. Note that it has a combinatorial reflection symmetry π:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 2 & 1 & 4 & 3 & 5 & 7 & 6 & 9 & 8 & 11 & 10 & 13 & 12 & 14 \end{pmatrix},$$

and $[12, 13, 14]$ has a positive value. We prove the oriented matroid associated with X cannot be realized as a vector configuration with a geometric symmetry π.
First, because of homogeneousity of Grassmann-Plücker system, we can assume that

$$[123] = 1, [134] = -1.$$

By the conditions of symmetry, we obtain

$$[124] = 1, [234] = -1.$$

By applying the conditions of symmetry to the Grassmann-Plücker relation "[312][345]-[314][325]+[315][324]=0", we obtain the following relation:

$$[145] + [135] = 0.$$

Similarly, we obtain the following relation by focusing on the Grassman-Plücker relation "[123][145]-[124][135]+[125][134]=0":

$$[125] = 2[145].$$

Using the above conditions, we compute x_i for $i = 6, 7, ..., 14$.

$$\begin{aligned}
x_6 = x_5x_2 \wedge x_1x_4 \quad &= [145]x_2 + x_5, \\
x_7 = x_5x_1 \wedge x_2x_3 \quad &= -[145]x_1 - x_5, \\
x_8 = x_6x_3 \wedge x_5x_1 \quad &= [145]^2x_2 + 2[145]^2x_3 + [145]x_5, \\
x_9 = x_7x_4 \wedge x_5x_2 \quad &= [145]^2x_1 + 2[145]^2x_4 + [145]x_5, \\
x_{10} = x_3x_4 \wedge x_8x_2 \quad &= -[145]^2x_4 - 3[145]^2x_3, \\
x_{11} = x_3x_4 \wedge x_9x_1 \quad &= -3[145]^2x_4 - [145]^2x_3, \\
x_{12} = x_7x_{10} \wedge x_{11}x_2 \quad &= -8[145]^5x_1 + 18[145]^5x_3 + 6[145]^5x_4 - 8[145]^4x_5, \\
x_{13} = x_6x_{11} \wedge x_{10}x_1 \quad &= -8[145]^5x_2 - 6[145]^5x_3 - 18[145]^5x_4 - 8[145]^4x_5, \\
x_{14} = x_1x_3 \wedge x_2x_4 \quad &= -x_1 + x_3.
\end{aligned}$$

Then, we evaluate $[12, 13, 14]$ and obtain

$$[12, 13, 14] = -[12, 13, 1] + [12, 13, 3] = -96[145]^{10}.$$

Therefore, $[12, 13, 14]$ should be non-positive, which contradicts the assumption. Clearly, it can also be checked by semidefinite programming and thus we conclude that our method can detect a gap between combinatorial symmetry and geometric symmetry.

6 Conclusion and Future works

In this paper, we reviewed a non-realizability certificate for a symmetric realization problem by semidefinite programming [10]. We saw that our method with relaxation order 1 had not be able to find gaps between combinatrial symmetry and geometric symmetry for oriented matroids of rank 4 on up to 8 elements and those of rank 3 on up to 9 elements, but we explained that our method with higher relaxation order can detect such gaps by analyzing an example in [12].

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