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On the Three-Dimensional Orthogonal Drawing of Outerplanar Graphs (Extended Abstract)

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Abstract

It has been known that every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing, while it has been open whether every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing. We show in this paper that every outerplanar 5-graph has a 1-bend 3-D orthogonal drawing.

1 Introduction

We consider the problem of generating orthogonal drawings of graphs in the space. The problem has obvious applications in the design of 3-D VLSI circuits and optoelectronic integrated systems [3, 5].

Throughout this paper, we consider simple connected graphs G with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex v in G, and by $\Delta(G)$ the maximum degree of a vertex of G. G is called a k-graph if $\Delta(G) \leq k$. The connectivity of a graph is the minimum number of vertices whose removal results in a disconnected graph or a single vertex graph. A graph is said to be k-connected if the connectivity of the graph is at least k.

It is well-known that every graph can be drawn in the space so that its edges intersect only at their ends. Such a drawing of a graph G is called a *3-D drawing* of G. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a *2-D drawing* of G.

A *3-D orthogonal drawing* of a graph G is a 3-D drawing such that each edge is drawn by a sequence of contiguous axis-parallel line segments. Notice that a graph G has a 3-D orthogonal drawing only if $\Delta(G) \leq 6$. A 3-D orthogonal drawing with no more than b bends per edge is called a b-bend 3-D orthogonal drawing.

Eades, Symvonis, and Whitesides [2], and Papakostas and Tollis [6] showed that every 6-graph has a 3-bend 3-D orthogonal drawing. Eades, Symvonis, and Whitesides [2] also posed an interesting open question of whether every 6-graph has a 2-bend 3-D orthogonal drawing. Wood [8] showed that every 5-graph has a 2-bend 3-D orthogonal drawing. Tayu, Nomura, and Ueno [7] showed that every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing. Moreover, Nomura, Tayu, and Ueno [4] showed that every outerplanar 6-graph has a 0-bend 3-D orthogonal drawing if and only if it contains no triangle as a subgraph, while Eades, Stirk, and Whitesides [1] proved that it is NP-complete to decide if a given 5-graph has a 0-bend 3-D orthogonal drawing. Tayu, Nomura, and Ueno [7] also posed an interesting open question of whether every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing.

We shown in this paper the following theorem.

Theorem I *Every outerplanar 5-graph has a 1-bend 3-D orthogonal drawing.*

The proof of Theorem 1 is constructive and provides a polynomial time algorithm to generate such a drawing for an outerplanar 5-graph. It is still open whether every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing.

2 Preliminaries

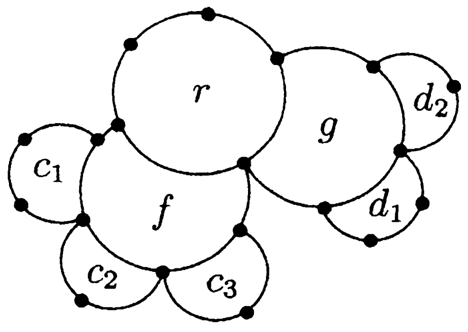
A 2-D drawing of a planar graph G is regarded as a graph isomorphic to G, and referred to as a plane graph. A plane graph partitions the rest of the plane into connected regions. A *face* is a closure of such a region. The unbounded region is referred to as the *external face.* We denote the boundary of a face f of a plane graph Γ by $b(f)$. If Γ is 2-connected then $b(f)$ is a cycle of Γ.

Given a plane graph Γ, we can define another graph Γ^* as follows: corresponding to each face f of Γ there is a vertex f^* of Γ^*, and corresponding to each edge e of Γ there is an edge e^* of Γ^*; two vertices f^* and g^* are joined by the edge e^* in Γ^* if and only if the edge e in Γ lies on the common boundary of faces f and g of Γ. Γ^* is called the *(geometric-)dual* of Γ.

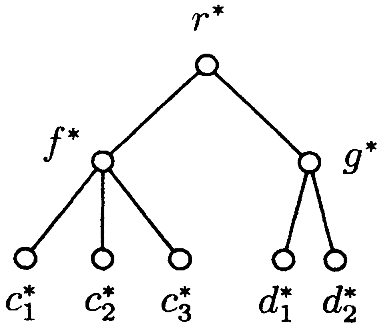
A graph is said to be *outerplanar* if it has a 2-D drawing such that every vertex lies on the boundary of the external face. Such a drawing of an outerplanar graph is said to be *outerplane.* It is well-known that an outerplanar graph is a series-parallel graph. Let Γ be an outerplane graph with the external face f_o, and $\Gamma^* - f_o^*$ be a graph obtained from Γ^* by deleting vertex f_o^* together with the edges incident to f_o^*. It is easy to see that if Γ is an outerplane graph then $\Gamma^* - f_o^*$ is a forest. In particular, an outerplane graph Γ is 2-connected if and only if $\Gamma^* - f_o^*$ is a tree.

3 2-Connected Outerplanar Graphs

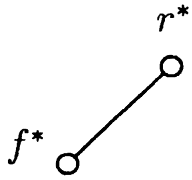
We first consider the case when G is 2-connected. Let G be a 2-connected outerplanar 5-graph and Γ be an outerplane graph isomorphic to G. Since Γ is 2-connected, $T^* = \Gamma^* - f_o^*$ is a tree. A vertex r^* of T^* is designated as a root, and T^* is considered as a rooted tree. If l^* is a leaf of T^* then l is called a *leaf face* of Γ. If g^* is a child of f^* in T^* then f is called the *parent face* of g, and g is called a *child face* of f in Γ. The unique edge in $b(f) \cap b(g)$ is called the *base* of g. We choose r^* so that $b(r) \cap b(f_o) \neq \emptyset$, and any edge in $b(r) \cap b(f_o)$ is defined as the base of r. Let S^* be a rooted subtree of T^* with root r^*. If S^* is consisting of just r^* then S^* is denoted by r^*. $\Gamma[S^*]$ is a subgraph of Γ induced by the vertices on boundaries of faces of Γ corresponding to the vertices of S^*. It should be noted that $\Gamma[S^*]$ is a 2-connected outerplane graph. Let f^* be a vertex in $V(T^*) - V(S^*)$ which is a child of $p^* \in V(S^*)$. $S^* + f^*$ is a subtree of T^* obtained from S^* by adding f^* and edge (f^*, p^*). Let $\overline{S^*}$ be a rooted subtree of T^* with root r^*



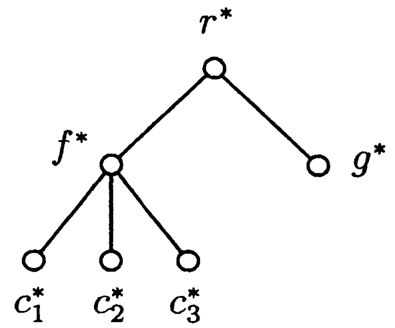
(a) Γ.



(b) T^*.



(c) S^*.



(d) $\overline{S^*}$.

Figure 1: Example of an outerplanae graph Γ, rooted tree T^*, subtrees S^* and $\overline{S^*}$ of T^*.

induced by the vertices of S^* and the children of the vertices of S^*. Fig. 1 shows an example of an outerplane graph Γ, rooted tree T^*, and rooted subtrees S^* and $\overline{S^*}$.

For any face f of Γ, $b(f)$ is a cycle since Γ is 2-connected. Let

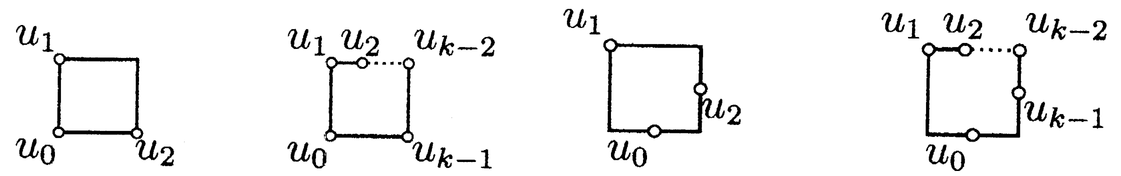
$$\begin{aligned} V(b(f)) &= \{u_i \mid 0 \le i \le k-1\}, \\ E(b(f)) &= \{e_0 = (u_0, u_{k-1})\} \cup \{e_{i+1} = (u_i, u_{i+1}) \mid 0 \le i \le k-2\} \end{aligned}$$

where e_0 is the base of f. A 1-bend 3-D orthogonal drawing of $b(f)$ is said to be *canonical* if $b(f)$ is drawn as one of the following four configurations.

Configuration 1 (Rectangle-1) : If $k = 3$ then only e_2 has a bend as shown in Fig. 2(a). If $k \ge 4$ then every edge has no bend, and $u_1, u_2, \dots, u_{k-2}$ are drawn on a side of a rectangle as shown in Fig. 2(b).

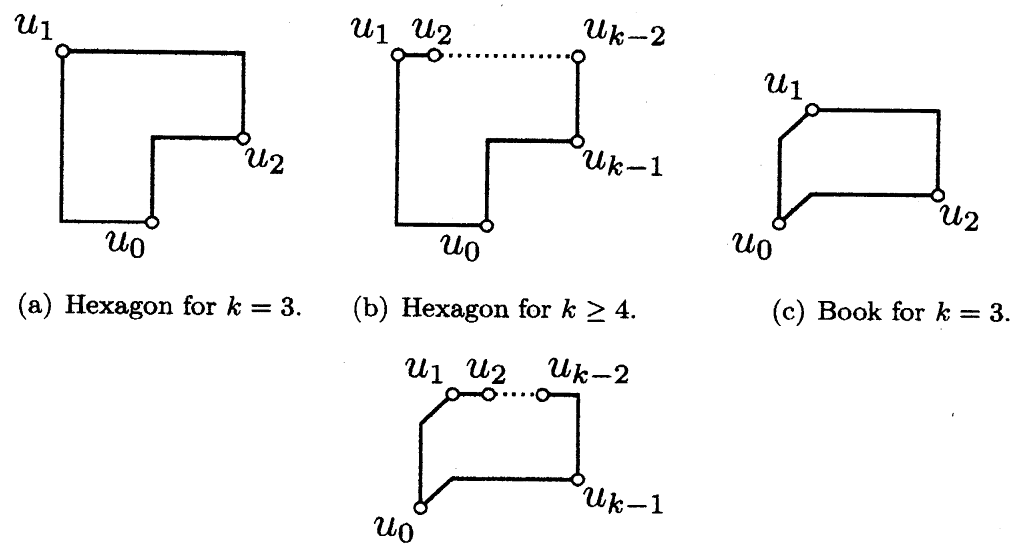
Configuration 2 (Rectangle-2) : If $k = 3$ then every edge has a bend, and u_1 is at a corner of a rectangle as shown in Fig. 2(c). If $k \ge 4$ then only e_0 and e_1 have a bend, $u_1, u_2, \dots, u_{k-2}$ are drawn on a side of a rectangle, and u_0 and u_{k-1} are on another different sides of the rectangle as shown in Fig. 2(d).

Configuration 3 (Hexagon) : If $k = 3$ then every edge has a bend as shown in Fig. 3(a). If $k \ge 4$ then only e_0 and e_1 have a bend, and $u_1, u_2, \dots, u_{k-2}$ are on a side of a hexagon as shown in Fig. 3(b).



(a) Rectangle-1 for $k = 3$. (b) Rectangle-1 for $k \geq 4$. (c) Rectangle-2 for $k = 3$. (d) Rectangle-2 for $k \geq 4$.

Figure 2: Rectangles-1 and -2.



(a) Hexagon for $k = 3$. (b) Hexagon for $k \geq 4$. (c) Book for $k = 3$.

(d) Book for $k \geq 4$.

Figure 3: Hexagon and Book.

Configuration 4 (Book) : A *book* is obtained from a rectangle by bending at a line segment, called the *spine*, parallel to a side of the rectangle. If $k = 3$ then every edge has a bend as shown in Fig. 3(c). If $k \geq 4$ then only e_0, e_1, and e_{k-1} have a bend, and $u_1, u_2, \ldots, u_{k-2}$ are on a side of a book as shown in Fig. 3(d).

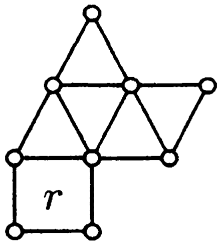
A drawing of Γ is said to be *canonical* if every face is drawn canonically. Fig. 4 shows an example of an outerplane graph Γ, and a 1-bend 3-D orthogonal canonical drawing of Γ.

Roughly speaking, we will show that if $\Gamma[\overline{S^*}]$ has a 1-bend 3-D orthogonal canonical drawing then $\Gamma[\overline{S^* + f^*}]$ also has a 1-bend 3-D orthogonal canonical drawing, where f^* is a leaf of $\overline{S^*}$. The following theorem immediately follows by induction.

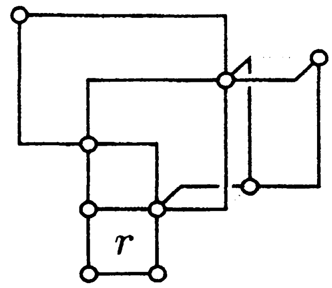
Theorem II *A 2-connected outerplanar 5-graph has a 1-bend 3-D orthogonal drawing.*

3.1 Proof of Theorem II

For a grid point $p = (p_x, p_y, p_z)$ and a vector $\boldsymbol{v} = (v_x, v_y, v_z)$, let $p + \boldsymbol{v}$ be the grid point $(p_x + v_x, p_y + v_y, p_z + v_z)$. For a unit vector $\boldsymbol{d}$, we denote $-\boldsymbol{d} = \overline{\boldsymbol{d}}$. Define that $\boldsymbol{e}_x = (1,$



(a) Γ.



(b) 1-bend 3-D orthogonal canonical drawing of Γ.

Figure 4: Example of Γ and 1-bend 3-D orthogonal canonical drawing of Γ.

$0, 0)$, $\boldsymbol{e}_y = (0, 1, 0)$, $\boldsymbol{e}_z = (0, 0, 1)$, and $D = \{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z, \overline{\boldsymbol{e}_x}, \overline{\boldsymbol{e}_y}, \overline{\boldsymbol{e}_z}\}$. Every vector in D is called a *direction*.

A 3-D orthogonal drawing of a plane graph Γ can be regarded as a pair $\langle\phi, \rho\rangle$ of one-to-one mappings $\phi : V(\Gamma) \to \mathbb{Z}^3$ and ρ which maps edges (u, v) to internally disjoint paths on the 3-D grid $\mathcal{G}$ connecting $\phi(u)$ and $\phi(v)$. For a direction $\boldsymbol{d} \in D$ and a vertex $v \in V(\Gamma)$, $\langle\phi, \rho\rangle$ is said to be ***d**-free* at $\phi(v)$ if $\rho(e)$ does not contain the edge of $\mathcal{G}$ connecting $\phi(v)$ and $\phi(v) + \boldsymbol{d}$.

Let Γ be a 2-connected outerplane graph, and $\langle\phi, \rho\rangle$ be a 3-D orthogonal canonical drawing of Γ. Let f be a leaf face of Γ, and

$$\begin{aligned} V(b(f)) &= \{u_i \mid 0 \le i \le k-1\}, \\ E(b(f)) &= \{(u_0, u_{k-1})\} \cup \{(u_i, u_{i+1}) \mid 0 \le i \le k-2\}, \end{aligned}$$

where (u_0, u_{k-1}) is the base of f. We define three unit vectors $\boldsymbol{d}_0(f, u_0)$, $\boldsymbol{d}_1(f, u_0)$, and $\boldsymbol{d}_2(f, u_0)$ as follows:

- If f is drawn as a rectangle-1, we define that $\boldsymbol{d}_0(f, u_0)$ is the unit vector directed from $\phi(u_{k-1})$ to $\phi(u_0)$, $\boldsymbol{d}_1(f, u_0) = \overline{\boldsymbol{d}_0(f, u_0)}$, and $\boldsymbol{d}_2(f, u_0)$ is a unit vector orthogonal to the rectangle.

- If f is drawn as a rectangle-2, let p be the bend of base (u_0, u_{k-1}). We define that $\boldsymbol{d}_1(f, u_0)$ is a unit vector orthogonal to the rectangle, and $\boldsymbol{d}_0(f, u_0)$ is the unit vector directed from $\phi(u_0)$ to p.

- If f is drawn as a hexagon, let p be the bend of base (u_0, u_{k-1}). We define that $\boldsymbol{d}_0(f, u_0)$ is the unit vector directed from p to $\phi(u_0)$, $\boldsymbol{d}_1(f, u_0)$ is the unit vector directed from p to $\phi(u_{k-1})$. and $\boldsymbol{d}_2(f, u_0)$ is a unit vector orthogonal to the hexagon.

- If f is drawn as a book, let p be the bend of base (u_0, u_{k-1}). We define that $\boldsymbol{d}_0(f, u_0)$ is the unit vector directed from $\phi(u_{k-1})$ to p, $\boldsymbol{d}_1(f, u_0)$ is the unit vector directed from $\phi(u_0)$ to p, and $\boldsymbol{d}_2(f, u_0)$ is the unit vector directed from the bend q of edge (u_{k-2}, u_{k-1}) to $\phi(u_{k-1})$.

Let $\langle\phi, \rho\rangle$ be a 1-bend 3-D orthogonal canonical drawing of Γ. $\langle\phi, \rho\rangle$ is said to be *addmissible* for a leaf face f of Γ if $\langle\phi, \rho\rangle$ satisfies one of the following conditions for f.

Condition 1 : f is drawn as a rectangle-1 or hexagon, and

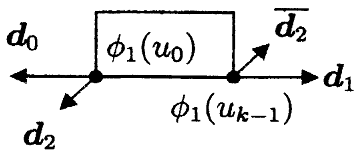
- if $d_\Gamma(u_0) \leq 4$ then $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_0(f, u_0)$-free or $\boldsymbol{d}_2(f, u_0)$-free at $\phi(u_0)$,
- if $d_\Gamma(u_{k-1}) \leq 4$ then $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_1(f, u_0)$-free or $\overline{\boldsymbol{d}_2(f, u_0)}$-free at $\phi(u_{k-1})$; (See Fig. 5(a) and (c).)

Condition 2 : f is drawn as a rectangle-2, and

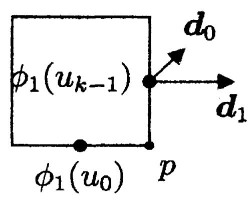
- $d_\Gamma(u_0) = 5$,
- $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_0(f, u_0)$-free at $\phi(u_{k-1})$,
- if $d_\Gamma(u_{k-1}) \leq 3$ then $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_1(f, u_0)$-free at $\phi(u_{k-1})$. (See Fig. 5(b).)

Condition 3 : f is drawn as a book, and

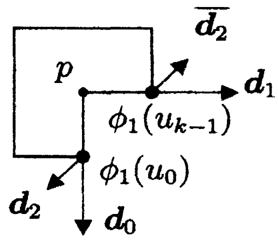
- if $d_\Gamma(u_0) \leq 4$ then $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_0(f, u_0)$-free or $\overline{\boldsymbol{d}_1(f, u_0)}$-free at $\phi(u_0)$,
- if $d_\Gamma(u_{k-1}) \leq 4$ then $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_1(f, u_0)$-free or $\overline{\boldsymbol{d}_0(f, u_0)}$-free at $\phi(u_{k-1})$,
- if $d_\Gamma(u_0) \leq 4$, $d_\Gamma(u_{k-1}) \leq 4$, $\langle \phi, \rho \rangle$ is not $\boldsymbol{d}_0(f, u_0)$-free at $\phi(u_0)$, and $\langle \phi, \rho \rangle$ is not $\boldsymbol{d}_1(f, u_0)$-free at $\phi(u_{k-1})$ then $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_2(f, u_0)$-free at $\phi(u_{k-1})$, and $d_\Gamma(u_{k-1}) = 4$,
- spine except for their ends is not used in the drawing; (See Fig. 5(d).)



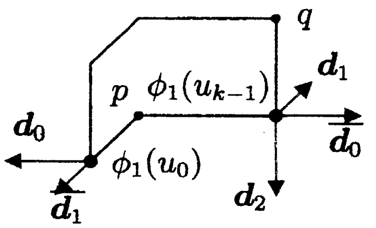
(a) Rectangle-1.



(b) Rectangle-2.



(c) Hexagon.



(d) Book.

Figure 5: Directions for draiwng of face f.

$\langle \phi, \rho \rangle$ is called a *1-bend 3-D orthogonal τ-drawing* of Γ if $\langle \phi, \rho \rangle$ is addmissible for every leaf face of Γ. In order to prove Theorem II, it suffices to prove the following.

Theorem III *A 2-connected outerplanar 5-graph has a 1-bend 3-D orthogonal τ-drawing.*

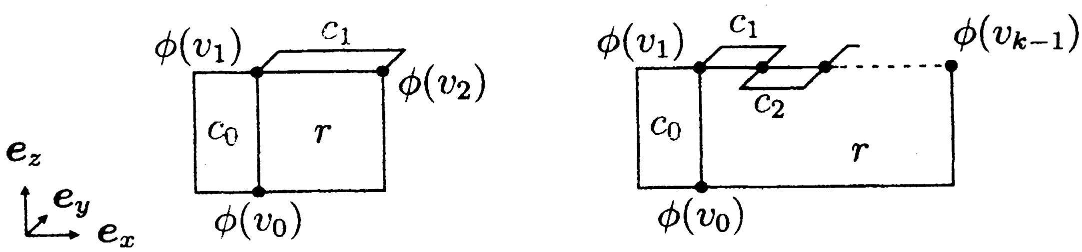
Proof. Let G be a 2-connected outerplanar 5-graph, Γ be a outerplane graph isomorphic to G, and $T^* = \Gamma^* - f_o^*$ be a tree rooted at r^*. We prove the theorem by induction. The basis of the induction is stated as follows.

Lemma 1 *$\Gamma[\overline{r^*}]$ has a 1-bend 3-D orthogonal τ-drawing.*

Proof of Lemma 1 Let

$$\begin{aligned} V(b(r)) &= \{v_i \mid 0 \leq i \leq k-1\}, \\ E(b(r)) &= \{(v_0, v_{k-1})\} \cup \{(v_i, v_{i+1}) \mid 0 \leq i \leq k-2\}, \end{aligned}$$

where (v_0, v_{k-1}) is the base of r. Let c_i be a child face of r with base (v_i, v_{i+1}) for $0 \leq i \leq k-2$, if any. Let $\langle \phi, \rho \rangle$ be a 1-bend 3-D orthogonal canonical drawing of $\Gamma[\overline{r^*}]$ as shown in Fig. 6, where c_i is drawn as rectangle-1, if any. Since $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_0(c_0, v_0)$-free at $\phi(v_0)$ and $\boldsymbol{d}_1(c_0, v_0)$-free at $\phi(v_1)$, $\langle \phi, \rho \rangle$ satisfies Condition 1 for c_0. If $k = 3$, by taking $\boldsymbol{d}_2(c_1, v_1) = \boldsymbol{e}_z$, $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_2(c_1, v_1)$-free at $\phi(v_1)$ and $\boldsymbol{d}_1(c_1, v_1)$-free at $\phi(v_2)$. Therefore, $\langle \phi, \rho \rangle$ also satisfies Condition 1 for c_1, and we conclude that $\langle \phi, \rho \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{r^*}]$. If $k \geq 4$, by taking $\boldsymbol{d}_2(c_i, v_i) = \boldsymbol{e}_z$ for $1 \leq i \leq k-3$, $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_2(c_i, v_i)$-free at $\phi(v_i)$ and $\boldsymbol{d}_2(c_i, v_i)$-free at $\phi(v_{i+1})$. Thus, $\langle \phi, \rho \rangle$ satisfies Condition 1 for c_i $(1 \leq i \leq k-3)$. Similarly, by taking $\boldsymbol{d}_2(c_{k-2}, v_{k-2}) = \boldsymbol{e}_z$, $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_2(c_{k-2}, v_{k-2})$-free at $\phi(v_{k-2})$ and $\boldsymbol{d}_2(c_{k-2}, v_{k-2})$-free at $\phi(v_{k-1})$. Thus, $\langle \phi, \rho \rangle$ satisfies Condition 1 for c_{k-2}. So, we conclude that $\langle \phi, \rho \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{r^*}]$. □



(a) $k = 3$. (b) $k \geq 4$.

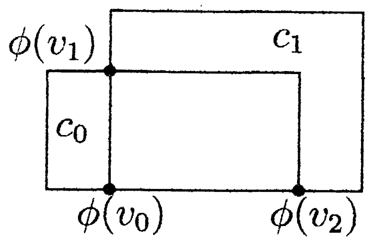
Figure 6: Drawing of initial case.

Let S^* be a rooted subtree of T^* with root r^*. The following lemma is used to prove the inductive step. The proof is immediate from the definition of the 1-bend 3-D orthogonal τ-drawing.

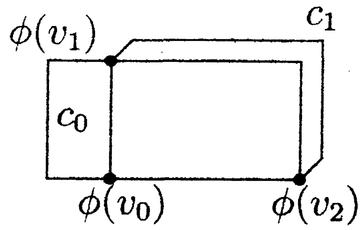
Lemma 2 *Let $\langle \phi_1, \rho_1 \rangle$ be a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^*}]$, f^* be a leaf of $\overline{S^*}$, and $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal drawing of $\Gamma[\overline{S^* + f^*}]$, obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of the child faces of f. If $\langle \phi_2, \rho_2 \rangle$ is admissible for every child face of f then $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal drawing of $\Gamma[\overline{S^* + f^*}]$.* □

The inductive step is stated as follows.

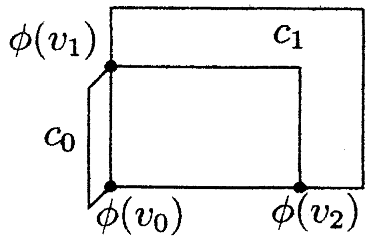
Lemma 3 *If $\Gamma[\overline{S^*}]$ has a 1-bend 3-D orthogonal τ-drawing then $\Gamma[\overline{S^* + f^*}]$ also has a 1-bend 3-D orthogonal τ-drawing, where f^* is a leaf of $\overline{S^*}$ in T^*.*



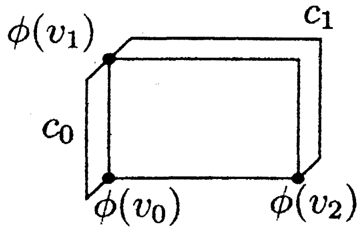
(a) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0$-free at $\phi(v_0)$ and $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$.



(b) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$ and not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$.



(c) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.



(d) $\langle\phi_1,\rho_1\rangle$ is not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ or $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

Figure 7: Drawing of child faces of f in Case 1-1.

Proof of Lemma 3 (scketch) Let $\Lambda_1 = \Gamma[\overline{S^*}]$ and $\Lambda_2 = \Gamma[\overline{S^* + f^*}]$, and let $\langle\phi_1,\rho_1\rangle$ be a 1-bend 3-D orthogonal τ-drawing of Λ_1. We will construct a 1-bend 3-D orthogonal τ-drawing $\langle\phi_2,\rho_2\rangle$ of Λ_2. Let

$$\begin{aligned} V(b(f)) &= \{v_i \mid 0 \le i \le k-1\}, \\ E(b(f)) &= \{e_0 = (v_0, v_{k-1})\} \cup \{e_{i+1} = (v_i, v_{i+1}) \mid 0 \le i \le k-2\}, \end{aligned}$$

where $e_0 = (v_0, v_{k-1})$ is the base of f. We distinguish four cases depending on the configuration of f by $\langle\phi_1,\rho_1\rangle$.

Case 1. f is drawn as a rectangle-1:
Without loss of generality, we assume that $\boldsymbol{d}_0(f,v_0) = \overline{\boldsymbol{e}_x}$, $\boldsymbol{d}_2(f,v_0) = \overline{\boldsymbol{e}_y}$, and z-coordinate of $\phi_1(v_1)$ is larger than that of $\phi_1(v_0)$. Let c_i be a child face of f with base (v_i, v_{i+1}) for $0 \le i \le k-2$, if any. We further distinguish three cases.

Case 1-1. $k = 3$:
Since $\langle\phi_1,\rho_1\rangle$ is a 1-bend 3-D orthogonal τ-drawing, we distinguish four cases depending on free directions.

Case 1-1-1. $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0(f,v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f,v_0)$-free at $\phi_1(v_2)$:
Since $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0(f,v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f,v_0)$-free at $\phi_1(v_2)$, canonical drawings of c_0 and c_1 can be added to $\langle\phi_1,\rho_1\rangle$ as shown in Fig. 7(a), if any. Let $\langle\phi_2,\rho_2\rangle$ be the resultant 1-bend 3-D orthogonal canonical drawing. If c_0 exists and $d_{\Lambda_2}(v_0) \le 4$ then $\langle\phi_2,\rho_2\rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\boldsymbol{e}_y$-free at $\phi_2(v_0)$, since $d_{\Lambda_2}(v_0) \le 4$ (see Fig. 7(a)). Since $\boldsymbol{d}_0(c_0,v_0) = \overline{\boldsymbol{e}_z}$ by definition, by taking $\boldsymbol{d}_2(c_0,v_0) = \boldsymbol{e}_y$, $\langle\phi_2,\rho_2\rangle$ is $\boldsymbol{d}_0(c_0,v_0)$-free or $\boldsymbol{d}_2(c_0,v_0)$-free at $\phi_2(v_0)$, and $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_0. If c_1 exists and $d_{\Lambda_2}(v_2) \le 4$ then $\langle\phi_2,\rho_2\rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\overline{\boldsymbol{e}_y}$-free at $\phi_2(v_2)$. Since $\boldsymbol{d}_1(c_1,v_1) = \overline{\boldsymbol{e}_z}$ by definition, by taking $\boldsymbol{d}_2(c_1,v_1) = \boldsymbol{e}_y$, $\langle\phi_2,\rho_2\rangle$ is $\boldsymbol{d}_1(c_1,v_1)$-free or $\boldsymbol{d}_2(c_1,v_1)$-free at $\phi_2(v_2)$. Therefore, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_1, since

$\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{d}_2(c_1, v_1)$-free at $\phi_2(v_1)$. Thus by Lemma 2, we conclude that $\langle\phi_2, \rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

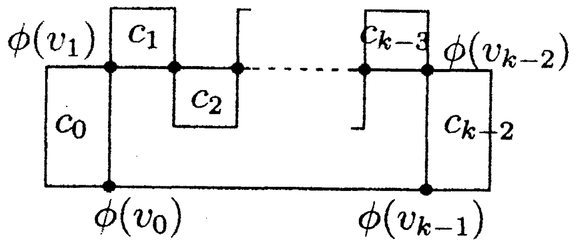
Case 1-1-2. $\langle\phi_1, \rho_1\rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$: In this case, $\langle\phi_1, \rho_1\rangle$ is $\overline{\boldsymbol{d}_2(f, v_0)}$-free at $\phi_1(v_2)$, since $\langle\phi_1, \rho_1\rangle$ satisfies Condition 1 for f. Let $\langle\phi_2, \rho_2\rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle\phi_1, \rho_1\rangle$ by adding canonical drawings of c_0 and c_1 as shown in Fig. 7(b), if any. By the similar arguments to Case 1-1-1, we can see that $\langle\phi_2, \rho_2\rangle$ satisfies Condition 1 for c_0, if any. If c_1 exists and $d_{\Lambda_2}(v_2) \leq 4$ then $\langle\phi_2, \rho_2\rangle$ is $\overline{\boldsymbol{e}_z}$-free at $\phi_2(v_2)$, since $\langle\phi_1, \rho_1\rangle$ is not $\boldsymbol{e}_x$-free at $\phi_1(v_2)$ and $d_{\Lambda_2}(v_2) \leq 4$. Also, $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{e}_z$-free at $\phi_2(v_1)$. Since $\boldsymbol{d}_0(c_1, v_1) = \boldsymbol{e}_z$ and $\boldsymbol{d}_1(c_1, v_1) = \boldsymbol{e}_x$ by definition, $\langle\phi_2, \rho_2\rangle$ satisfies Condition 3 for c_1. Thus by Lemma 2, we conclude that $\langle\phi_2, \rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 1-1-3. $\langle\phi_1, \rho_1\rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$: In this case, $\langle\phi_1, \rho_1\rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$, since $\langle\phi_1, \rho_1\rangle$ satisfies Condition 1 for f. Let $\langle\phi_2, \rho_2\rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle\phi_1, \rho_1\rangle$ by adding canonical drawings of c_0 and c_1 as shown in Fig. 7(c), if any. If $d_{\Lambda_2}(v_0) \leq 4$ and c_0 exists then $\langle\phi_2, \rho_2\rangle$ is $\overline{\boldsymbol{e}_z}$-free at $\phi_2(v_0)$, since $\langle\phi_1, \rho_1\rangle$ is not $\overline{\boldsymbol{e}_x}$-free at $\phi_1(v_0)$ and $d_{\Lambda_2}(v_0) \leq 4$. Then by taking $\boldsymbol{d}_2(c_0, v_0) = \boldsymbol{e}_x$, $\langle\phi_2, \rho_2\rangle$ satisfies Condition 1 for c_0, since $\langle\phi_2, \rho_2\rangle$ is $\overline{\boldsymbol{d}_2(c_0, v_0)}$-free at $\phi_2(v_1)$. Also, by the similar arguments to Case 1-1-1, we can see that $\langle\phi_2, \rho_2\rangle$ satisfies Condition 1 for c_1, if any. Thus by Lemma 2, we conclude that $\langle\phi_2, \rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

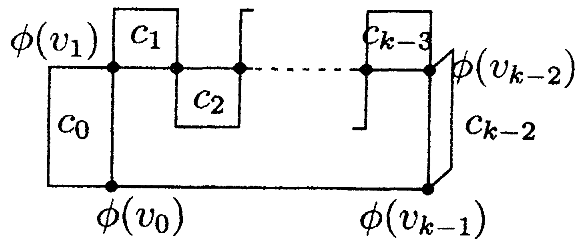
Case 1-1-4. $\langle\phi_1, \rho_1\rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ nor $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$: In this case, $\langle\phi_1, \rho_1\rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$ and $\overline{\boldsymbol{d}_2(f, v_0)}$-free at $\phi_1(v_2)$, since $\langle\phi_1, \rho_1\rangle$ satisfies Condition 1 for f. Let $\langle\phi_2, \rho_2\rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle\phi_1, \rho_1\rangle$ by adding canonical drawings of c_0 and c_1 as shown in Fig. 7(d), if any. Then by similar arguments to Case 1-1-3 and Case 1-1-2, we can see that $\langle\phi_2, \rho_2\rangle$ satisfies Conditions 1 and 3 for c_0 and c_1, respectively. Thus by Lemma 2, we conclude that $\langle\phi_2, \rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 1-2. $k \geq 4$, k is even: Since $\langle\phi_1, \rho_1\rangle$ is a 1-bend 3-D orthogonal τ-drawing, we distinguish four cases depending on free directions.

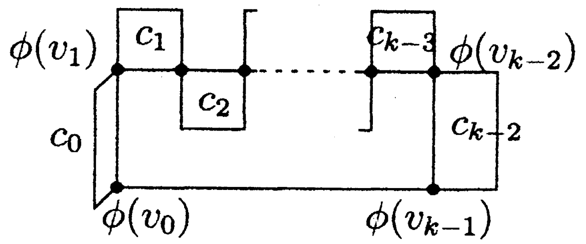
Case 1-2-1. $\langle\phi_1, \rho_1\rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_{k-2})$-free at $\phi_1(v_{k-1})$: Since $\langle\phi_1, \rho_1\rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_{k-1})$, canonical drawings of c_i $(0 \leq i \leq k-2)$ can be added to $\langle\phi_1, \rho_1\rangle$ as shown in Fig. 8(a), if any. Let $\langle\phi_2, \rho_2\rangle$ be the resultant 1-bend 3-D orthogonal canonical drawing. If c_0 exists and $d_{\Lambda_2}(v_0) \leq 4$ then $\langle\phi_2, \rho_2\rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\boldsymbol{e}_y$-free at $\phi_2(v_0)$, since $d_{\Lambda_2}(v_0) \leq 4$ (see Fig. 8(a)). Since $\boldsymbol{d}_0(c_0, v_0) = \overline{\boldsymbol{e}_z}$ by definition, by taking $\boldsymbol{d}_2(c_0, v_0) = \boldsymbol{e}_y$, $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{d}_0(c_0, v_0)$-free or $\boldsymbol{d}_2(c_0, v_0)$-free at $\phi_2(v_0)$. Also, $\langle\phi_2, \rho_2\rangle$ is $\overline{\boldsymbol{d}_2(c_0, v_0)}$-free at $\phi_2(v_1)$. Therefore, $\langle\phi_2, \rho_2\rangle$ satisfies Condition 1 for c_0. If c_{k-2} exists and $d_{\Lambda_2}(v_{k-1}) \leq 4$ then $\langle\phi_2, \rho_2\rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\overline{\boldsymbol{e}_y}$-free at $\phi_2(v_{k-1})$. Since $\boldsymbol{d}_1(c_{k-2}, v_{k-2}) = \overline{\boldsymbol{e}_z}$ by definition, by taking $\boldsymbol{d}_2(c_{k-2}, v_{k-2}) = \boldsymbol{e}_y$, $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{d}_1(c_{k-2}, v_{k-2})$-free or $\overline{\boldsymbol{d}_2(c_{k-2}, v_{k-2})}$-free at $\phi_2(v_{k-1})$. Thus, $\langle\phi_2, \rho_2\rangle$ satisfies Condition 1 for c_{k-2}, since $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{d}_2(c_{k-2}, v_{k-2})$-free at $\phi_2(v_{k-2})$. Since $\langle\phi_2, \rho_2\rangle$ is $\overline{\boldsymbol{e}_y}$-free at $\phi_2(v_i)$ and $\boldsymbol{e}_y$-free at $\phi_2(v_{i+1})$ $(1 \leq i \leq k-2)$, $\langle\phi_2, \rho_2\rangle$ satisfies Condition 1 for c_i with $1 \leq i \leq k-2$,



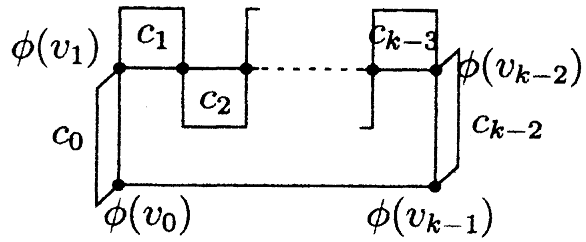
(a) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0$-free at $\phi(v_0)$ and $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$.



(b) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$ and not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$.



(c) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.



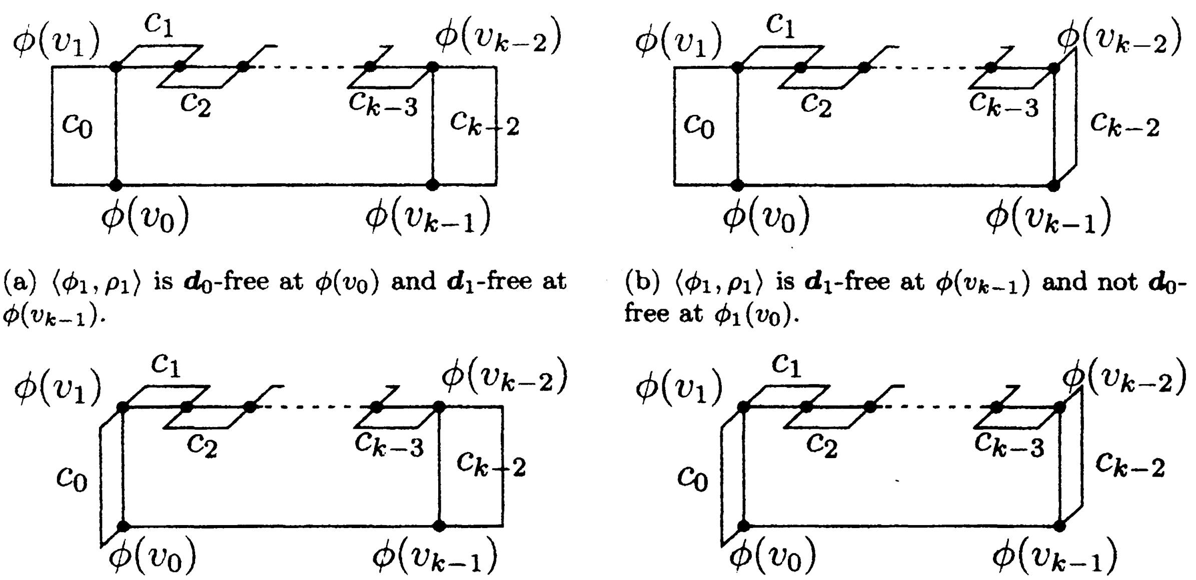
(d) $\langle\phi_1,\rho_1\rangle$ is not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ or $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

Figure 8: Drawing of child faces of f in Case 1-2.

if any. Therefore, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle\phi_2,\rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^*+f^*]$.

Case 1-2-2. $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0(f,v_0)$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1(f,v_{k-2})$-free at $\phi_1(v_{k-1})$: In this case, $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_2(f,v_0)$-free at $\phi_1(v_{k-1})$, since $\langle\phi_1,\rho_1\rangle$ satisfies Condition 1 for f. Let $\langle\phi_2,\rho_2\rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle\phi_1,\rho_1\rangle$ by adding canonical drawings of c_i $(0 \le i \le k-2)$ as shown in Fig. 8(b), if any. If c_{k-2} exists and $d_{\Lambda_2}(v_{k-1}) \le 4$ then $\langle\phi_2,\rho_2\rangle$ is $\overline{\boldsymbol{e}_z}$-free at $\phi_2(v_{k-1})$, since $\langle\phi_1,\rho_1\rangle$ is not $\boldsymbol{e}_x$-free at $\phi_1(v_{k-1})$ and $d_{\Lambda_2}(v_{k-1}) \le 4$. Also, $\langle\phi_2,\rho_2\rangle$ is $\boldsymbol{e}_x$-free at $\phi_2(v_{k-2})$. Thus by taking $\boldsymbol{d}_2(c_{k-2},v_{k-2}) = \boldsymbol{e}_x$, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_{k-2}. Also, by the similar arguments to Case 1-2-1, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_i with $0 \le i \le k-3$. Therefore, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle\phi_2,\rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^*+f^*]$.

Case 1-2-3. $\langle\phi_1,\rho_1\rangle$ is not $\boldsymbol{d}_0(f,v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f,v_{k-2})$-free at $\phi_1(v_{k-1})$: In this case, $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_2(f,v_0)$-free at $\phi_1(v_0)$, since $\langle\phi_1,\rho_1\rangle$ satisfies Condition 1 for f. Let $\langle\phi_2,\rho_2\rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle\phi_1,\rho_1\rangle$ by adding canonical drawings of c_i $(0 \le i \le k-2)$ as shown in Fig. 8(c), if any. If $d_{\Lambda_2}(v_0) \le 4$ and c_0 exists then $\langle\phi_2,\rho_2\rangle$ is $\overline{\boldsymbol{e}_z}$-free at $\phi_2(v_0)$, since $\langle\phi_1,\rho_1\rangle$ is not $\overline{\boldsymbol{e}_x}$-free at $\phi_1(v_0)$ and $d_{\Lambda_2}(v_0) \le 4$. Then by taking $\boldsymbol{d}_2(c_0,v_0) = \boldsymbol{e}_x$, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_0, since $\langle\phi_2,\rho_2\rangle$ is $\boldsymbol{d}_2(c_0,v_0)$-free at $\phi_2(v_1)$. By the similar arguments to Case 1-2-1, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_i with $1 \le i \le k-1$, if any. Therefore, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle\phi_2,\rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of



(a) $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0$-free at $\phi(v_0)$ and $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$.

(b) $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$ and not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$.

(c) $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

(d) $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ or $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

Figure 9: Drawing of child faces of f in Case 1-3.

$\Gamma[\overline{S^* + f^*}]$.

Case 1-2-4. $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ nor $\underline{\boldsymbol{d}_1(f, v_{k-2})}$-free at $\phi_1(v_{k-1})$: In this case, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$ and $\overline{\boldsymbol{d}_2(f, v_0)}$-free at $\phi_1(v_{k-1})$, since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_i $(0 \le i \le k-2)$ as shown in Fig. 8(d), if any. Then by similar arguments to Case 1-2-3 and Case 1-2-2, we can see that $\langle \phi_2, \rho_2 \rangle$ satisfies Conditions 1 for c_0 and c_{k-2}, respectively. Also, by the similar arguments to Case 1-2-1, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_i with $1 \le i \le k-3$. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 1-3. $k \ge 5$, k is odd:
Since $\langle \phi_1, \rho_1 \rangle$ is a 1-bend 3-D orthogonal τ-drawing, we distinguish four cases depending on free directions.

Case 1-3-1. $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_{k-2})$-free at $\phi_1(v_{k-1})$: Since $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_{k-1})$, canonical drawings of c_i $(0 \le i \le k-2)$ can be added to $\langle \phi_1, \rho_1 \rangle$ as shown in Fig. 9(a), if any. Let $\langle \phi_2, \rho_2 \rangle$ be the resultant 1-bend 3-D orthogonal canonical drawing. If c_0 exists and $d_{\Lambda_2}(v_0) \le 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\boldsymbol{e}_y$-free at $\phi_2(v_0)$, since $d_{\Lambda_2}(v_0) \le 4$ (see Fig. 9(a)). Since $\boldsymbol{d}_0(c_0, v_0) = \overline{\boldsymbol{e}_z}$ by definition, by taking $\boldsymbol{d}_2(c_0, v_0) = \boldsymbol{e}_y$, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_0(c_0, v_0)$-free or $\boldsymbol{d}_2(c_0, v_0)$-free at $\phi_2(v_0)$. Also, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_1(c_0, v_0)$-free at $\phi_2(v_1)$. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_0. If c_{k-2} exists and $d_{\Lambda_2}(v_{k-1}) \le 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\overline{\boldsymbol{e}_y}$-free at $\phi_2(v_{k-1})$. Since $\boldsymbol{d}_1(c_{k-2}, v_{k-2}) = \overline{\boldsymbol{e}_z}$ by definition, by taking $\boldsymbol{d}_2(c_{k-2}, v_{k-2}) = \boldsymbol{e}_y$, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_1(c_{k-2}, v_{k-2})$-free or $\overline{\boldsymbol{d}_2(c_{k-2}, v_{k-2})}$-free at $\phi_2(v_{k-1})$. Thus, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_{k-2}, since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_2(c_{k-2}, v_{k-2})$-free at $\phi_2(v_{k-2})$. Since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{e}_z$-free at $\phi_2(v_i)$ for $1 \le i \le k-2$

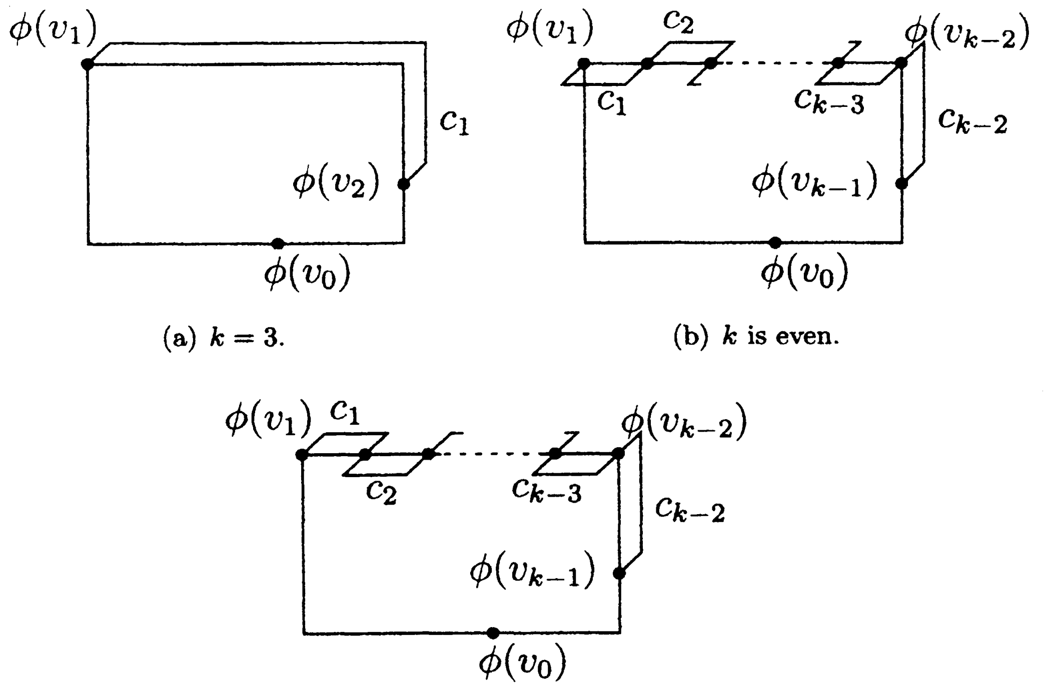
and $\overline{e_y}$-free at $\phi_2(v_i)$ for $2 \le i \le k-3$, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_i with $1 \le i \le k-2$, if any. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 1-3-2. $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1(f, v_{k-2})$-free at $\phi_1(v_{k-1})$: In this case, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_{k-1})$, since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_i $(0 \le i \le k-2)$ as shown in Fig. 9(b), if any. If c_{k-2} exists and $d_{\Lambda_2}(v_{k-1}) \le 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{e_z}$-free at $\phi_2(v_{k-1})$, since $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{e}_x$-free at $\phi_1(v_{k-1})$ and $d_{\Lambda_2}(v_{k-1}) \le 4$. Also, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{e}_x$-free at $\phi_2(v_{k-2})$. Thus by taking $\boldsymbol{d}_2(c_{k-2}, v_{k-2}) = \boldsymbol{e}_x$, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_{k-2}. Also, by the similar arguments to Case 1-3-1, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_i with $0 \le i \le k-3$. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 1-3-3. $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_{k-2})$-free at $\phi_1(v_{k-1})$: In this case, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$, since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_i $(0 \le i \le k-2)$ as shown in Fig. 9(c), if any. If $d_{\Lambda_2}(v_0) \le 4$ and c_0 exists then $\langle \phi_2, \rho_2 \rangle$ is $\overline{e_z}$-free at $\phi_2(v_0)$, since $\langle \phi_1, \rho_1 \rangle$ is not $\overline{e_x}$-free at $\phi_1(v_0)$ and $d_{\Lambda_2}(v_0) \le 4$. Then by taking $\boldsymbol{d}_2(c_0, v_0) = \boldsymbol{e}_x$, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_0, since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_2(c_0, v_0)$-free at $\phi_2(v_1)$. By the similar arguments to Case 1-3-1, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_i with $1 \le i \le k-1$, if any. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 1-3-4. $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ nor $\boldsymbol{d}_1(f, v_{k-2})$-free at $\phi_1(v_{k-1})$: In this case, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_{k-1})$, since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_i $(0 \le i \le k-2)$ as shown in Fig. 9(d), if any. Then by similar arguments to Case 1-3-3 and Case 1-3-2, we can see that $\langle \phi_2, \rho_2 \rangle$ satisfies Conditions 1 for c_0 and c_{k-2}, respectively. Also, by the similar arguments to Case 1-3-1, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_i with $1 \le i \le k-3$. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 2. f is drawn as a rectangle-2:
Without loss of generality, we assume that x- and z-coordinates of $\phi_1(v_{k-1})$ are larger than those of $\phi_1(v_0)$, and that $\boldsymbol{d}_0(f, v_{k-1}) = \boldsymbol{e}_y$. It should be noted that $d_{\Lambda_1}(v_0) = 5$, since f is drawn as a rectangle-2. So, there is no child face of f with base (v_0, v_1). Let c_i be a child face of f with base (v_i, v_{i+1}) for $1 \le i \le k-2$, if any. Since $\langle \phi_1, \rho_1 \rangle$ is a 1-bend 3-D orthogonal τ-drawing, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_{k-1})$, i.e., $\boldsymbol{e}_y$-free at $\phi_1(v_{k-1})$. Therefore, canonical drawings of c_i $(1 \le i \le k-2)$ can be added to $\langle \phi_1, \rho_1 \rangle$ as shown in Fig. 10, if any. Let $\langle \phi_2, \rho_2 \rangle$ be the resultant 1-bend 3-D orthogonal canonical drawing. We further distinguish



(a) $k = 3$.

(b) k is even.

(c) k is odd.

Figure 10: Drawing of child faces of f in Case 2.

two cases.

Case 2-1. $k = 3$:
Since $\boldsymbol{d}_0(c_1, v_1) = \boldsymbol{e}_y$, $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{e}_y$-free at $\phi_2(v_1)$. Therefore, $\langle\phi_2, \rho_2\rangle$ satisfies Condition 3 for c_0 if $d_{\Lambda_2}(v_0) = 4$. Assume that $d_{\Lambda_2}(v_0) \leq 4$. Then, $d_{\Lambda_1}(v_0) = d_{\Lambda_2}(v_0) - 1 \leq 3$, and so $\langle\phi_1, \rho_1\rangle$ is $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$. This implies that $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{e}_x$-free at $\phi_2(v_2)$. Therefore, $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$, and $\langle\phi_2, \rho_2\rangle$ satisfies Condition 3 for c_1. Thus by Lemma 2, we conclude that $\langle\phi_2, \rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 2-2. $k \geq 4$:
If c_{k-2} exists and $d_{\Lambda_2}(v_{k-1}) \leq 4$, then $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{e}_x$-free at $\phi_2(v_{k-1})$, since $d_{\Lambda_1}(v_{k-1}) = d_{\Lambda_2}(v_{k-1}) - 1 \leq 3$ and $\langle\phi_1, \rho_1\rangle$ satisfies Condition 2 for f. Therefore, if c_{k-2} exists, $\langle\phi_2, \rho_2\rangle$ satisfies Condition 1 for c_{k-2} since $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{d}_0(c_{k-2}, v_{k-2})$-free at v_{k-2}. By definition, $\langle\phi_2, \rho_2\rangle$ is $\overline{\boldsymbol{e}_x}$-free at $\phi_2(v_1)$, $\boldsymbol{e}_y$-free at $\phi_2(v_i)$ for $1 \leq i \leq k-2$, $\overline{\boldsymbol{e}_y}$-free at $\phi_2(v_i)$ for $2 \leq i \leq k-3$, and $\boldsymbol{e}_x$-free at $\phi_2(v_{k-2})$. Therefore, by taking $\boldsymbol{d}_2(c_i, v_i) = \overline{\boldsymbol{e}_y}$, we have the following: $\langle\phi_2, \rho_2\rangle$ is $\boldsymbol{d}_0(c_1, v_1)$-free at v_1, $\overline{\boldsymbol{d}_2(c_i, v_i)}$-free at v_{i+1} for $2 \leq i \leq k-2$, and $\boldsymbol{d}_2(c_i, v_i)$-free at v_i for $2 \leq i \leq k-3$. So, $\langle\phi_2, \rho_2\rangle$ satisfies Condition 1 for c_i for $1 \leq i \leq k-3$. Therefore, $\langle\phi_2, \rho_2\rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle\phi_2, \rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 3. f is drawn as a hexagon:
Without loss of generality, we assume that $\boldsymbol{d}_0(f, v_0) = \overline{\boldsymbol{e}_z}$, $\boldsymbol{d}_1(f, v_0) = \boldsymbol{e}_x$, and $\boldsymbol{d}_2(f, v_0) = \overline{\boldsymbol{e}_y}$. Let c_i be a child face of f with base (v_i, v_{i+1}) for $0 \leq i \leq k-2$, if

any. We further distinguish three cases.

Case 3-1. $k = 3$:

Since $\langle \phi_1, \rho_1 \rangle$ is a 1-bend 3-D orthogonal τ-drawing, we distinguish four cases depending on free directions.

Case 3-1-1. $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:

Since $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$, canonical drawings of c_0 and c_1 can be added to $\langle \phi_1, \rho_1 \rangle$ as shown in Fig. 11(a), if any. Let $\langle \phi_2, \rho_2 \rangle$ be the resultant 1-bend 3-D orthogonal canonical drawing. If c_0 exists and $d_{\Lambda_2}(v_0) \leq 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{e}_x$-free or $\boldsymbol{e}_y$-free at $\phi_2(v_0)$, since $d_{\Lambda_2}(v_0) \leq 4$ (see Fig. 11(a)). Since $\boldsymbol{d}_0(c_0, v_0) = \boldsymbol{e}_x$ by definition, by taking $\boldsymbol{d}_2(c_0, v_0) = \boldsymbol{e}_y$, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_0(c_0, v_0)$-free or $\overline{\boldsymbol{d}_2(c_0, v_0)}$-free at $\phi_2(v_0)$, and $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_0, since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_2(c_0, v_0)$-free at $\phi_2(v_1)$. If c_1 exists and $d_{\Lambda_2}(v_2) \leq 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\overline{\boldsymbol{e}_y}$-free at $\phi_2(v_2)$. Then by taking $\boldsymbol{d}_2(c_1, v_1) = \boldsymbol{e}_y$, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_1(c_1, v_1)$-free or $\overline{\boldsymbol{d}_2(c_1, v_1)}$-free at $\phi_2(v_2)$. So, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_1, since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_2(c_1, v_1)$-free at $\phi_2(v_1)$. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for the child faces of f. Thus by Lemma 2, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 3-1-2. $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:

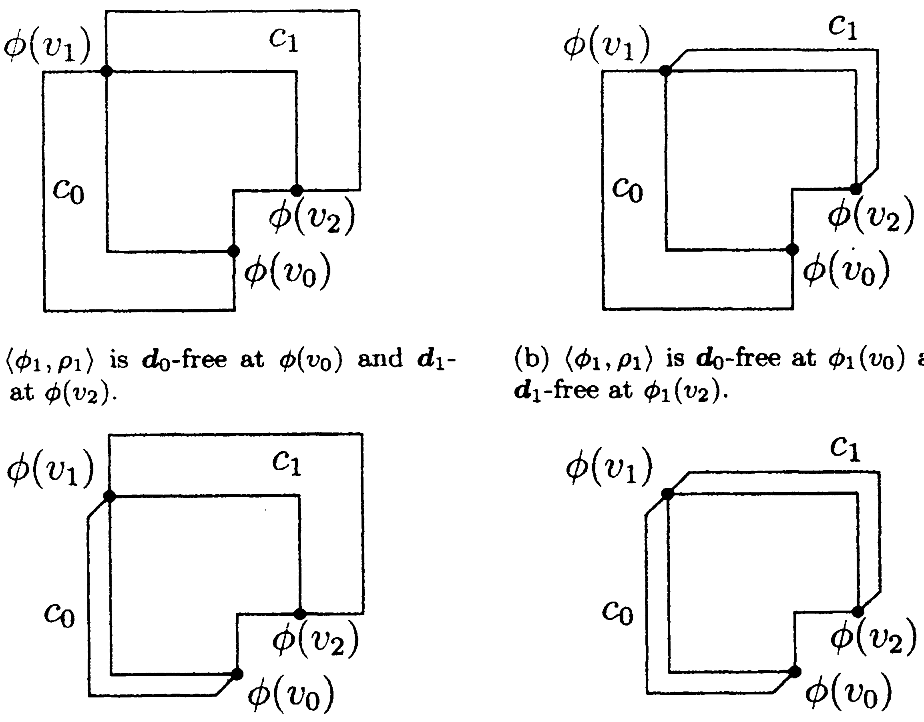
In this case, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{e}_y$-free at $\phi_1(v_2)$, since $d_{\Lambda_1}(v_2) = d_{\Lambda_2}(v_2) - 1 \leq 4$ and $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_0 and c_1 as shown in Fig. 11(b), if any. By the similar arguments to Case 3-1-1, we can see that $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_0, if any. If c_1 exists and $d_{\Lambda_2}(v_2) \leq 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free at $\phi_2(v_2)$, since $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{e}_x$-free at $\phi_1(v_2)$ and $d_{\Lambda_2}(v_2) \leq 4$. Also, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{e}_z$-free at $\phi_2(v_1)$. By definition, $\boldsymbol{d}_0(c_1, v_1) = \boldsymbol{e}_z$ and $\boldsymbol{d}_1(c_1, v_1) = \boldsymbol{e}_x$. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 3 for c_1, since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_0(c_1, v_1)$-free at $\phi_2(v_1)$. Thus by Lemma 2, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 3-1-3. $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:

In this case, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{e}_x$-free at $\phi_1(v_0)$, since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_0 and c_1 as shown in Fig. 11(c), if any. If $d_{\Lambda_2}(v_0) \leq 4$ and c_0 exists then $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{e}_x$-free at $\phi_2(v_0)$, since $\langle \phi_1, \rho_1 \rangle$ is not $\overline{\boldsymbol{e}_z}$-free at $\phi_1(v_0)$ and $d_{\Lambda_2}(v_0) \leq 4$. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 3 for c_0, since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_1(c_0, v_0)$-free at $\phi_2(v_1)$. Also, by the similar arguments to Case 3-1-1, we can see that $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_1, if any. Thus by Lemma 2, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 3-1-4. $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ nor $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:

In this case, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$ and $\overline{\boldsymbol{d}_2(f, v_0)}$-free at $\phi_1(v_2)$, since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_0 and c_1 as shown in Fig. 11(d), if any. Then by similar arguments to Case 3-1-3 and Case 3-1-2, we can see that $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 3 for c_0 and c_1, respectively. Thus by Lemma 2, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.



(a) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0$-free at $\phi(v_0)$ and $\boldsymbol{d}_1$-free at $\phi(v_2)$.

(b) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1$-free at $\phi_1(v_2)$.

(c) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_1$-free at $\phi(v_2)$ and not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$.

(d) $\langle\phi_1,\rho_1\rangle$ is not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ or $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

Figure 11: Drawing of child faces of f in Case 3-1.

Case 3-2. $k \geq 4$, k is even:

In this case, child faces of f can be drawn as shown in Fig. 9, if any. We can prove that the resultant drawing $\langle\phi_2,\rho_2\rangle$ is admissible for every child face, and $\langle\phi_2,\rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing by Lemma 2. The details are omitted in the extended abstract due to space limitation.

Case 3-3. $k \geq 5$, k is odd:

In this case, child faces of f can be drawn as shown in Fig. 9, if any. We can prove that the resultant drawing $\langle\phi_2,\rho_2\rangle$ is admissible for every child face, and $\langle\phi_2,\rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing by Lemma 2. The details are omitted in the extended abstract due to space limitation.

The following remaining case can be proved similarly. The proof is omitted in the extended abstract due to space limitation.

Case 4. f is drawn as a book. □

4 General Outerplanar Graphs

In this section, we shall complete the proof of Theorem I. We assume without loss of generality that G is a connected outerplanar 5-graph. Let $G_1, G_2, \ldots, G_m$ be 2-connected components of G. It is well-known that $E(G)$ can be partitioned into $E(G_1), E(G_2), \ldots,$ and $E(G_m)$. An *adjacent graph* A_G of G is defined as follows: $V(A_G) = \{G_1, G_2, \ldots, G_m\}$, and $(G_i, G_j) \in E(A_G)$ if and only if $V(G_i) \cap V(G_j) \neq \emptyset$. It is easy to see that A_G is connected. Suppose $(G_1, G_2, \ldots, G_m)$ is a preorder of $V(A_G)$ obtained by applying DFS on A_G. Then a subgraph H_i of G induced by the vertices in $\bigcup_{k=1}^{i} V(G_k)$ is connected for $1 \leq i \leq m$. We prove Theorem I by induction on i. Since $H_1 = G_1$ is a 2-connected outerplanar 5-graph, we know by Theorem II that H_1 has a 1-bend 3-D orthogonal drawing. The inductive step is stated as follows.

Lemma 4 *For $1 \leq i \leq m-1$, if H_i has a 1-bend 3-D orthogonal drawing then H_{i+1} has a 1-bend 3-D orthogonal drawing.* □

This proves Theorem I since $H_m = G$. The proof of Lemma 4 is omitted in the extended abstract due to space limitation.

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