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MATHEMATICAL APPROACH TO ASYMMETRIC BURGERS VORTICES AT LARGE CIRCULATIONS

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In this report we consider a stationary flow of a viscous incompressible fluid whose velocity field U takes the form

$$U(x_1, x_2, x_3) = u_\lambda(x_1, x_2, x_3) + u(x_1, x_2). \tag{0.1}$$

Here the velocity field u_λ expresses an asymmetric background straining flow and is given by

$$u_\lambda(x_1, x_2, x_3) = (-\frac{1+\lambda}{2}x_1, -\frac{1-\lambda}{2}x_2, x_3),$$

with a fixed parameter $\lambda \in [0, 1)$. The parameter λ represents the asymmetry of the background straining flow. The solenoidal velocity field u expresses a two dimensional perturbation flow:

$$u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2), 0), \quad \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0.$$

We assume that the velocity field U solves the stationary Navier-Stokes equations:

$$\text{(NS)} \qquad \begin{cases} -\Delta U + (U, \nabla)U + \nabla P = 0, \ x \in \mathbb{R}^3, \\ \nabla \cdot U = 0, \ x \in \mathbb{R}^3. \end{cases}$$

Here $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$, $(U, \nabla) = \sum_{i=1}^3 U_i \frac{\partial}{\partial x_i}$, and $\nabla \cdot U = \sum_{i=1}^3 \frac{\partial U_i}{\partial x_i}$. We write ∂_i instead of $\frac{\partial}{\partial x_i}$ for simplicity. The function P represents a pressure field of the fluid.

We are interested in the behavior of the vorticity field. Taking the rotation of $U = u_\lambda + u$, we see that the vorticity $\Omega = \nabla \times U$ is given by

$$\Omega(x_1, x_2, x_3) = (0, 0, \omega(x_1, x_2)) \tag{0.2}$$

where $\omega = \partial_1 u_2 - \partial_2 u_1$. If ω is smooth and integrable, and u decays at infinity, then the velocity field u is recovered from the vorticity field ω

via the Biot-Savart law:

$$u = K * \omega, \tag{0.3}$$

where the convolution kernel K is given by

$$K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}, \ x^\perp = (-x_2, x_1). \tag{0.4}$$

The value $\int_{\mathbb{R}^2} \omega(x)dx$ is called the total circulation and its absolute value is called the vortex Reynolds number. Let $\alpha \in \mathbb{R}$ be a given real number. We consider the vorticity field ω whose total circulation is α. Since U satisfies (NS), we see that ω solves the following equation

$$(B_{\lambda,\alpha}) \qquad \begin{cases} (\mathcal{L} + \lambda\mathcal{M})\omega - B(\omega,\omega) = 0, \ x \in \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} \omega(x)dx = \alpha, \end{cases}$$

where

$$\mathcal{L} = \Delta + \frac{x}{2}\cdot\nabla + 1, \tag{0.5}$$

$$\mathcal{M} = \frac{1}{2}(x_1\partial_1 - x_2\partial_2), \tag{0.6}$$

$$B(f,h) = (K * f, \nabla)h. \tag{0.7}$$

Conversely, if ω is a solution to $(B_{\lambda,\alpha})$, then we can show that $u_\lambda + K*\omega$ gives a solution to (NS) with a suitably determined pressure P. We call solutions to $(B_{\lambda,\alpha})$ the Burgers vortices.

Let G be the two dimensional Gauss kernel:

$$G(x) = \frac{1}{4\pi}e^{-\frac{|x|^2}{4}}. \tag{0.8}$$

Then by direct calculations, we see that G satisfies

$$\mathcal{L}G = 0, \ (K * G, \nabla)G = 0.$$

Thus αG solves $(B_{\lambda,\alpha})$ for $\lambda = 0$. This exact solution was found by Burgers [1], and it is called the axisymmetric Burgers vortex. When $\lambda \in (0,1)$ a solution to $(B_{\lambda,\alpha})$ is called the asymmetric Burgers vortex. In this case we can not expect the explicit representation as in the axisymmetric case $\lambda = 0$. So the existence of solutions to $(B_{\lambda,\alpha})$ itself is an important problem and this is the main interest here.

The Burgers vortices have been used as a model of concentrated vorticity fields in turbulence. It is numerically observed that the region of intense vorticity fields in three dimensional turbulence tends to form a lot of tube-like structures, and that each vortex-tube is well described by the Burgers vortices. From this physical point of view, they have been numerically studied mainly in the case of large vortex Reynolds numbers, since the vortex Reynolds number is considered to represent the magnitude of the vorticity; see Robinson-Saffman [18], Kida-Ohkitani [10], Moffatt-Kida-Ohkitani [12], Prochazka-Pullin [16], Prochazka-Pullin [17]. One of the most interesting features of their numerical results is that as the vortex Reynolds number is increasing, the Burgers vortices tend to be more circular even when the asymmetry parameter is not zero. Based on their numerical results, Robinson and Saffman conjectured in [18] that the asymmetric Burgers vortices would rigorously exist for any $\lambda \in [0,1)$ at least when $\frac{\lambda}{1+|\alpha|}$ is sufficiently small. In [12] the above property of the Burgers vortices is explained by obtaining a formal asymptotic expansion at large vortex Reynolds numbers. First mathematical approach to this problem was done by Gallay and Wayne in [6] and [7]. In [7] the existence of the Burgers vortices is proved in the Gaussian weighted L^2 space for any $|\alpha|$ if the asymmetry parameter λ is sufficiently small ($\lambda << \frac{1}{2}$). In [7] the asymptotic expansion at large $|\alpha|$ indicated by [12] is rigorously verified. For not sufficiently small $\lambda \in (0,1)$ the problem becomes more complicated because the operator $\lambda\mathcal{M}$ breaks the symmetry of the equation strongly. As far as the author knows, the only mathematical results in this case are the results by [6] in which it is proved that the Burgers vortices exist in the polynomial weighted L^2 space when $|\alpha|$ is sufficiently small depending on $\lambda \in [0,1)$. So the above conjecture by Robinson and Saffman was still open.

In this report some recent developments of mathematical studies for the Burgers vortices are introduced. Especially, in [14, 15] it is proved that the Burgers vortices exist for each asymmetry parameter $\lambda \in [0,1)$ and all circulation numbers α, which gives the affirmative answer to Robinson-Saffman's question. Moreover, Moffatt-Kida-Ohkitani's asymptotic expansion at large vortex Reynolds numbers ([12]) are rigorously verified for any $\lambda \in [0,1)$. The stability of the Burgers vortices is also an important question, but still it is not well understood mathmatically. We give a remark on these issues in Remark 2.

To state our results precisely, let us introduce function spaces. Let G_λ be the function defined by

$$G_\lambda(x) = \frac{1-\lambda}{4\pi}\exp\left(-\frac{1-\lambda}{4}|x|^2\right). \tag{0.9}$$

Let X_λ, Y_λ be the complex Hilbert spaces defined as follows.

$$(0.10)\, X_\lambda = \{w \in L^2(\mathbb{R}^2) \mid G_\lambda^{-\frac{1}{2}} w \in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} w dx = 0,$$

$$< w_1, w_2 >_{X_\lambda} = \int_{\mathbb{R}^2} G_\lambda^{-1}(x) w_1(x) \overline{w_2(x)} dx\},$$

$$(0.11)\, Y_\lambda = \{w \in X_\lambda \mid \partial_i w \in X_\lambda,\ i = 1, 2,$$

$$< w_1, w_2 >_{Y_\lambda} = \int_{\mathbb{R}^2} G_\lambda^{-1}(x) \big(w_1(x) \overline{w_2(x)} + \nabla w_1(x) \cdot \nabla \overline{w_2(x)} \big) dx\}.$$

We also define the subspace of X_λ

$$(0.12)\ W_\lambda = \{w \in X_\lambda \mid,\ G_\lambda^{-\frac{1}{2}} x_i w \in L^2(\mathbb{R}^2)\ i = 1, 2,$$

$$< w_1, w_2 >_{W_\lambda} = \int_{\mathbb{R}^2} G_\lambda^{-1}(x) \big(w_1(x) \overline{w_2(x)} + |x|^2 w_1(x) \overline{w_2(x)} \big) dx.\}$$

The space X_0 (and also Y_0) are used in [7], [13], and [14]. Let $\mathcal{G}_\lambda$ be the function given by

$$(0.13) \qquad \mathcal{G}_\lambda(x) = \frac{\sqrt{1-\lambda^2}}{4\pi} e^{-\frac{1+\lambda}{4} x_1^2 - \frac{1-\lambda}{4} x_2^2}.$$

Note that $(\mathcal{L} + \lambda \mathcal{M})\mathcal{G}_\lambda = 0$ and $\int_{\mathbb{R}^2} \mathcal{G}_\lambda(x) dx = 1$ hold. The first main result is as follows.

Theorem 1 (Existence of asymmetric Burgers vortices; [15]). *Let $\lambda \in [0, 1)$ and $\alpha \in \mathbb{R}$. Then there is a (real valued) solution $\omega_{\lambda,\alpha}$ to $(B_{\lambda,\alpha})$ such that $\omega_{\lambda,\alpha} - \alpha \mathcal{G}_\lambda \in Y_\lambda \cap W_\lambda$.*

The above theorem is proved in [15] by a suitable application of the Schauder fixed point theorem. The key idea is to reduce $(B_{\lambda,\alpha})$ to an evolution equation by introducing the scaling variables $x = \frac{\xi}{\sqrt{\tau}}$ and apply the results of Carlen-Loss [2] in order to obtain a priori L^p estimates for solutions. Its argument is not so complicated, but instead, we do not have detailed informations on the solutions. Especially, the method used in the proof of Theorem 1 is less helpful if one wants to explain why the asymmetric Burgers vortex tends to be circular when the vortex Reynolds number is large. So we need a completely different approach in the study of the Burgers vortices for large $|\alpha|$.

Let $n \in \mathbb{Z}$ and let $\mathbb{P}_n$ be the orthogonal projection defined by

$$\mathbb{P}_n w = w_n(r) e^{in\theta}, \qquad w_n(r) = \frac{1}{2\pi} \int_0^{2\pi} w(r\cos\theta, r\sin\theta) e^{-in\theta} d\theta.$$

We set

$$\mathbb{P}_n X_\lambda = \{\mathbb{P}_n w \mid w \in X_\lambda\}, \tag{0.14}$$

$$\mathbb{P}^e X_\lambda = \oplus_{n\in\mathbb{Z}} \mathbb{P}_{2n} X_\lambda. \tag{0.15}$$

It will be useful to define the subspace of all "non-radially" symmetric functions:

$$\mathbb{P}_0^\perp X_\lambda = \{\mathbb{P}_0^\perp w \mid w \in X_\lambda,\ \mathbb{P}_0^\perp = I - \mathbb{P}_0\}. \tag{0.16}$$

For a given $h \in Y_\lambda$ we define an integro-differential operator Λ_h on Y_λ as

$$\Lambda_h f = B(h, f) + B(f, h). \tag{0.17}$$

In fact, we can see that $\mathbb{P}^e X_\lambda$ is invariant under the action of Λ_h if h belongs to $\mathbb{P}^e X_\lambda$. Let $w_\infty \in Y_0 \cap W_0$ be the function which satisfies the equation

$$\mathcal{M}G = \Lambda_G w_\infty. \tag{0.18}$$

The existence of w_∞ is proved in [7]; see also [12]. Especially, w_∞ is uniquely determined in $\mathbb{P}_{-2}X_\lambda \cup \mathbb{P}_2 X_\lambda$.

The second result is the existence and the asymptotic behavior of the Burgers vortices for large vortex Reynolds numbers.

Theorem 2 (Asymptotics expansion at large circulations; [14, 15]). *Let $\lambda \in [0, 1)$. Then there is a positive number $\Theta_1 = \Theta_1(\lambda) \geq 0$ such that for any $\alpha \in \mathbb{R}$ with $|\alpha| \geq \Theta_1$ there exists a (real valued) solution $\omega_{\lambda,\alpha}$ of $(B_{\lambda,\alpha})$ satisfying $\omega_{\lambda,\alpha} - \alpha G \in \mathbb{P}^e X_\lambda$ and*

$$||\omega_{\lambda,\alpha} - \alpha G - \lambda w_\infty||_{Y_\lambda \cap W_\lambda} \leq \frac{\lambda M(\lambda)}{1 + |\alpha|}, \tag{0.19}$$

where the constant $M(\lambda)$ depends only on λ. The constants $\Theta_1(\lambda)$ and $M(\lambda)$ are taken as

$$\lim_{\lambda\to 1} \Theta_1(\lambda) = \lim_{\lambda\to 1} M(\lambda) = \infty. \tag{0.20}$$

When $|\alpha|$ is large, we also have the uniqueness around $\alpha G + \lambda w_\infty$ as follows.

Theorem 3 (Uniqueness at large circulations; [14, 15]). *Let $\lambda \in [0, 1)$. Then for any $\tau > 0$ there is a positive number $\Theta_2 = \Theta_2(\lambda, \tau) \geq \Theta_1$ such that for any α with $|\alpha| \geq \Theta_2$, there exists at most one solution of $(B_{\lambda,\alpha})$ in the ball*

$$\mathcal{B}_\tau = \{f \in L^2(\mathbb{R}^2) \mid f - \alpha G \in \mathbb{P}^e X_\lambda\ ,\ ||f - \alpha G - \lambda w_\infty||_{Y_\lambda \cap W_\lambda} \leq \tau\}.$$

For each $\lambda \in [0, 1)$ the constant $\Theta_2(\lambda, \tau)$ is taken as

$$\lim_{\tau\to\infty} \Theta_2(\lambda, \tau) = \infty. \tag{0.21}$$

Remark 1 (Large-Reynolds-number asymptotics). As stated previously, Moffatt, Kida, and Ohkitani indicated in [12] that the asymmetric Burgers vortex would be expanded around $\alpha G + \lambda w_\infty$ when $\frac{\lambda}{1+|\alpha|}$ is sufficiently small for any $\lambda \geq 0$. This expansion was rigorously recovered by Gallay and Wayne in [7] when λ is sufficiently small. Theorem 2 shows that for any $\lambda \in [0,1)$ there is a solution which satisfies the above expansion. Unfortunately, we do not know whether or not the solution constructed in Theorem 1 satisfies (0.19) and coincides with the solution obtained in Theorem 2.

In order to prove Theorem 2 and Theorem 3, we first expand $(B_{\lambda,\alpha})$ around $\alpha G + \lambda w_\infty$. Then we get the equation for $w = \omega - \alpha G - \lambda w_\infty$:

$$(\mathcal{L} - \alpha\Lambda_G + \lambda\mathcal{M})w = B(w,w) + \lambda\Lambda_{w_\infty} w + \lambda f_\lambda. \tag{0.22}$$

The function f_λ is defined as

$$f_\lambda = -\mathcal{L}w_\infty + \lambda\big(B(w_\infty, w_\infty) - \mathcal{M}w_\infty\big). \tag{0.23}$$

It is known that $f_\lambda \in Y_0 \cap W_0 \cap \mathbb{P}^e X_0$ and $\mathbb{P}_0 f_\lambda = 0$; see [14, Corollary 2.3]. In next section we will see that $\mathbb{P}^e X_\lambda$ is invariant under the equations (0.22).

Let us state the difficulty of this problem and rough idea to overcome it. The most important step to solve (0.22) is to consider the linearized problem

$$\mathcal{L}_{\lambda,\alpha} w := (\mathcal{L} - \alpha\Lambda_G + \lambda\mathcal{M})w = f. \tag{0.24}$$

The main difficulty comes from the operator $\lambda\mathcal{M}$, since it leads to a slow spatial decay in x_2 direction and also breaks the symmetry of the equation. If $\lambda < \frac{1}{2}$, we can find solutions of (0.24) in the Gaussian weighted L^2 space X_0 (X_λ for $\lambda = 0$) at least for large $|\alpha|$; see [7] and [14]. The only reason we can rigorously treat the equation (0.24) even for large $|\alpha|$ in X_0 is that Λ_G is skew-symmetric in X_0 which is discovered by Gallay and Wayne in [5]. The skew-symmetry of Λ_G enables us to give uniform (or better) estimates for linearized operators $(\mathcal{L} - \alpha\Lambda_G)^{-1}$ or $(\mathcal{L} - \alpha\Lambda_G + \lambda\mathcal{M})^{-1}$ at large $|\alpha|$. To explain this, let us recall the argument used in [5] or [7]; see also [13], [14]. Let $h \in X_0$ be the solution of the equation $(\mathcal{L} - \alpha\Lambda_G)h = f$ for $f \in X_0$. Then we have

$$\begin{aligned}
\mathrm{Re} < f, h >_{X_0} &= \mathrm{Re} < (\mathcal{L} - \alpha\Lambda_G)h, h >_{X_0} \\
&= -\mathrm{Re} < (-\mathcal{L})h, h >_{X_0} \\
&= -\|(-\mathcal{L})^{\frac{1}{2}} h\|_{X_0}^2 \\
&\leq -\frac{1}{2}\|h\|_{X_0}^2,
\end{aligned}$$

by the self-adjointness of $\mathcal{L}$ with $-\mathcal{L} \geq \frac{1}{2}$ (see [7]) and the skew-symmetry of Λ in X_0. Thus we have

$$||h||_{X_0} \leq 2||f||_{X_0}, \tag{0.25}$$

which gives the uniform estimate for $h = (\mathcal{L} - \alpha\Lambda_G)^{-1}f$. In fact, it seems to be quite difficult to obtain this uniform estimate directly without using the skew-symmetry of Λ_G.

If $\lambda \geq \frac{1}{2}$, we can no longer expect that solutions belong to X_0, because of the loss of a spatial decay by the operator $\lambda\mathcal{M}$. So we are forced to deal with the equations (0.24) in other function spaces which allow functions with slower spatial decays. However, in general, the operator Λ is not skew-symmetric in such spaces. Mathemaically, this causes serious difficulties to establish useful estimates for solutions of the linearized problem for not small α. Especially, we need to control the term $\alpha\Lambda_G$ without the skew-symmetry of Λ_G itself.

To overcome this difficulty, we look for a linear operator which makes Λ skew-symmetric by its right action.

Definition 0.1 (Definition of a right skew-symmetrizer). *Let X be a Hilbert space and A be a linear operator in X, $D(A) \subset X$. Then we call a linear operator T in X a right skew-symmetrizer of A if the operator AT, $D(AT) = \{f \in D(T) \mid Tf \in D(A)\}$ is skew-symmetric, i.e.,*

$$< ATf, h >_X + < f, ATh >_X = 0$$

for $f, h \in D(AT)$. We say A is right skew-symmetrizable in X if there is a right skew-symmetrizer T of A in X.

Then the following lemma is essential.

Lemma 0.1 ([15]). *There is a right skew-symmetrizer T of the operator Λ_G in $\mathbb{P}^e X_\lambda$. Moreover, T satisfies the following:*
(1) *$T - I$ is compact in $\mathbb{P}^e X_\lambda$,*
(2) *T is injective in $\mathbb{P}^e X_\lambda$,*

By the Fredholm alternative theorem, T has the bounded inverse on $\mathbb{P}^e X_\lambda$. So we consider $v = T^{-1}w$ instead of the solution w of (0.24) itself. From the relations

$$\begin{aligned} (\mathcal{L} + \lambda\mathcal{M})w &= (\mathcal{L} + \lambda\mathcal{M})v + (\mathcal{L} + \lambda\mathcal{M})(T - I)v, \\ \alpha\Lambda_G w &= \alpha\Lambda_G T v, \end{aligned}$$

we obtain the equation for v:

$$(\mathcal{L} + \lambda\mathcal{M} - \alpha\Lambda_G T)v = -(\mathcal{L} + \lambda\mathcal{M})(T - I)v + f. \tag{0.26}$$

By the skew-symmetry of $\Lambda_G T$ and the characterization of Ker $\Lambda_G T$, we can show that the linear problem $(\mathcal{L} - \alpha\Lambda_G T + \lambda\mathcal{M})v = f$ is uniquely solvable in $\mathbb{P}^e X_\lambda$ if $|\alpha|$ is sufficiently large. The term $-(\mathcal{L}+\lambda\mathcal{M})(T-I)v$ can be regarded as lower order, since $T-I$ is compact. Using these facts, we can show that the equation (0.26) is uniquely solvable in $\mathbb{P}^e X_\lambda$ and so is true for (0.24) by the relation $w = Tv$. Then the nonlinear problem (0.22) will be solved by perturbation arguments as in [14].

To solve (0.26) we need to investigate the linear operator $\mathcal{L}+\lambda\mathcal{M}-\alpha\Lambda_G T$. For this purpose, we decompose $\mathcal{L}$ as

$$\mathcal{L} = \mathcal{L}_\lambda + \lambda\mathcal{N}, \tag{0.27}$$

where

$$\mathcal{L}_\lambda = \Delta + \frac{1-\lambda}{2} x\cdot\nabla + 1 - \lambda, \qquad \mathcal{N} = \frac{x}{2}\cdot\nabla + 1. \tag{0.28}$$

The reason why we decompose $\mathcal{L}$ as above is that the operator $\mathcal{L}_\lambda$ is self-adjoint in X_λ with the spectrum $\sigma(\mathcal{L}_\lambda) = \{-\frac{(1-\lambda)n}{2} \mid n = 1, 2, \cdots\}$ and that both $\mathcal{L}_\lambda$ and $\mathcal{N}$ map $\mathbb{P}_n X_\lambda \cap D(\mathcal{L}_\lambda)$ to $\mathbb{P}_n X_\lambda$. Especially, we can see by direct calculations that each of $\mathcal{L}_\lambda$, $\mathcal{M}$, and $\mathcal{N}$ maps $\mathbb{P}^e X_\lambda \cap D(\mathcal{L}_\lambda)$ to $\mathbb{P}^e X_\lambda$.

In order to derive better properties of $\mathcal{L}_{\lambda,\alpha}$ or $\mathcal{L}+\lambda\mathcal{M}-\alpha\Lambda_G T$ for large $|\alpha|$, it is important to characterize the kernel of Λ_G or $\Lambda_G T$. By a simple observation, it turns out that the kernel of Λ_G or $\Lambda_G T$ in $\mathbb{P}^e X_\lambda$ coincides with the subspace consisting of all radially symmetric functions in X_λ, i.e.,

$$\mathrm{Ker}\ \Lambda_G = \mathrm{Ker}\ \Lambda_G T = \mathbb{P}_0 X_\lambda. \tag{0.29}$$

This is useful and essential in our proof, since the decomposition of solutions into radially symmetric parts and non-radially symmetric parts matches the structure of the symmetry-breaking term $\lambda\mathcal{M}v$ or the nonlinear term $B(v,v)$. For example, if v is radially symmetric, then $\mathcal{M}v$ belongs to $\mathbb{P}_0^\perp X_\lambda$ and $B(v,v) = 0$.

We can show that $\mathcal{L}_{\lambda,\alpha}$ is invertible for large α and its inverse has better estimates as $|\alpha|$ is increasing. More precisely, the operator norms of $\mathcal{L}_{\lambda,\alpha}^{-1}\mathbb{P}_0^\perp$ and $\mathbb{P}_0^\perp\mathcal{L}_{\lambda,\alpha}^{-1}$ are estimated as small for large $|\alpha|$, where $\mathbb{P}_0^\perp = I - \mathbb{P}_0$. The solution to (0.22) is constructed by decomposing it into the radially symmetric part ($\mathbb{P}_0 X_\lambda$) and the non-radially symmetric part ($\mathbb{P}_0^\perp X_\lambda$). Unfortunately, we do not have better estimates for $\mathbb{P}_0\mathcal{L}_{\lambda,\alpha}^{-1}\mathbb{P}_0$ even if $|\alpha|$ is large. But since the radially symmetric part of solutions to (0.22) is essentially expressed by the non-radially symmetric part of them, we can establish necessary a priori estimates for solutions to (0.22) when the vortex Reynolds number $|\alpha|$ is sufficiently large; see [15] for details.

Finally, we give a remark on the stability of the Burgers vortices.

Remark 2 (Mathematical results on the stability of Burgers vortices). Since the axisymmetric Burgers vortex αG gives the nontrivial exact solution to three dimensional Navier-Stokes equations, its stability problem has attracted many researchers. In Giga-Kambe [9] it is proved that if the L^1-norm of initial data is sufficiently small, then the solution of the non-stationary equation associated with $(B_{\lambda,\alpha})$ with $\lambda = 0$ converges to αG where α is the total circulation of initial vorticity (note that the total circulation is conserved under the equation $(B_{\lambda,\alpha})$). Their result is extended by Carpio [3] and Giga-Giga [8] in which the global stability of the axisymmetric Burgers vortex (with respect to two dimensional perturbations) is obtained when the vortex Reynolds number is sufficiently small. Although the global stability for not small vortex Reynolds numbers had remained open for years, the affirmative answer is given by Gallay-Wayne [5]. The rate of convergence is also discussed there. As indicated by [16], it is important to consider the influence on the stability by a fast rotation $|\alpha| >> 1$. In [13] the spectrum of $\mathcal{L} - \alpha\Lambda_G$ in X_0 is studied and the rate of convergence to axisymmetric Burgers vortices is improved when the vortex Reynolds number is sufficiently large.

As for the asymmetric Burgers vortices, as far as the author knows, the mathematical understanding of their stability has not yet been achieved much. Gallay-Wayne [7] proved the local stability of asymmetric Burgers vortices when λ is sufficiently small. In Gallay-Wayne [6] the local stability with respect to three dimensional perturbations is obtained for $\lambda \in [0, 1)$ when $|\alpha|$ is sufficiently small. In [14] it is proved that the asymmetric Burgers vortices are locally stable with respect to two dimensional perturbations when $\lambda \in [0, \frac{1}{2})$ and $|\alpha|$ is sufficiently large. However it is still open whether or not the local stability of asymmetric Burgers vortices holds in general. In particular, when $\lambda \in [\frac{1}{2}, 1)$ we do not know whether the asymmetric Burgers vortices obtained in Theorem 2 are locally stable or not even in the case of sufficiently large $|\alpha|$. Finally, the global stability is not obtained so far in any asymmetric case $\lambda \in (0, 1)$.

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