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Convergence rate toward planar stationary solution for the compressible Navier-Stokes equation in half space

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Abstract

The present paper concerns a large-time behavior of a solution to an isentropic model of the compressible Navier–Stokes equation in multi-dimensional half space. Precisely, we obtain a convergence rate toward a planar stationary wave for an outflow problem, where fluid blows out from a boundary, under the assumption that an initial perturbation and a boundary strength are sufficiently small. For a supersonic flow at spatial infinity, if the initial perturbation belongs to the algebraically weighted Sobolev space $H^s \cap L^2_\alpha$ for $s := [(n-1)/2] + 2$ and $\alpha \geq 0$, then the convergence rate is $t^{-\alpha/2-(n-1)/4}$ in L^∞-norm. For a transonic flow, due to a degenerate property of the stationary solution, we require a restriction on a weight exponent α to obtain an algebraic convergence rate. Namely if the initial perturbation belongs to the algebraically weighted Sobolev space H^s_α for $\alpha \in [0, \alpha^*)$ where α^* is a certain positive constant, then the convergence rate is $t^{-\alpha/4-(n-1)/4}$.

1 Introduction

We study an asymptotic behavior of a solution to the compressible Navier–Stokes equation in the multi-dimensional half space $\mathbb{R}^n_+ := \mathbb{R}_+ \times \mathbb{R}^{n-1}$ for $\mathbb{R}_+ := (0, \infty)$ and $n = 2, 3$:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1a}$$

$$\rho\{u_t + (u \cdot \nabla)u\} = \mu_1 \Delta u + (\mu_1 + \mu_2)\nabla(\operatorname{div} u) - \nabla p(\rho), \tag{1.1b}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ is a space variable, which is often abbreviated as $x = (x_1, x')$ with $x_1 \in \mathbb{R}_+$ and $x' := (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. Unknown functions $\rho = \rho(t, x)$ and $u = u(t, x) = (u_1(t, x), \ldots, u_n(t, x))$ stand for fluid density and fluid velocity, respectively. The function $p = p(\rho)$ means a pressure, which is explicitly given by $p(\rho) := K\rho^\gamma$ for constants $K > 0$ and $\gamma \geq 1$. The constants μ_1 and μ_2 are viscosity coefficients satisfying $\mu_1 > 0$ and $2\mu_1 + n\mu_2 \geq 0$. We prescribe an initial condition

$$(\rho, u)(0, x) = (\rho_0, u_0)(x) \tag{1.2}$$

and an outflow boundary condition

$$u(t,0,x') = (u_b, 0, \dots, 0), \tag{1.3}$$

where u_b is a negative constant. It is assumed that a spatial asymptotic state of the initial data in a normal direction x_1 is a constant. Precisely, a normal component u_0^1 of $u_0 = (u_0^1, \dots, u_0^n)$ tends to a certain constant u_+ and a tangential component $u_0' = (u_0^2, \dots, u_0^n)$ tends to 0:

$$\lim_{x_1\to\infty} \rho_0(x) = \rho_+, \quad \lim_{x_1\to\infty} (u_0^1, u_0')(x) = (u_+, 0). \tag{1.4}$$

It is also assumed that the initial density is uniformly positive:

$$\inf_{x\in\mathbb{R}^n_+} \rho_0(x) > 0, \quad \rho_+ > 0.$$

From the pioneering work [2] by Il'in and Oleĭnik, there have been many researches on the asymptotic stability of several kinds of nonlinear waves for the initial value problem to scalar viscous conservation laws. Especially, Kawashima, Matsumura and Nishihara in [6, 12] proved the asymptotic stability of viscous shock waves and obtained the convergence rate by using a weighted energy method, of which idea is also used to obtain the rate. The research by Liu, Matsumura and Nishihara in [8] started the analysis on the asymptotic stability of stationary waves for the initial and boundary value problem in one-dimensional half space.

For the isentropic model of the compressible Navier–Stokes equation, Matsumura in [10] gave classification of the possible asymptotic states for a one-dimensional half space problem and expected that one of asymptotic states for the outflow problem is a stationary solution. This fact for the special case was verified by Kawashima, Nishibata and Zhu in [7] under the smallness condition on the boundary strength and the initial perturbation. The convergence rate for this system was firstly obtained by Nakamura, Nishibata and Yuge in [14] by assuming that the initial perturbation belongs to a certain weighted Sobolev space. Let us note that this type of convergence rate for the system is first obtained by Nishikawa and Nishibata in [16], where a coupled system of Burgers Poisson equations is studied.

For the multi-dimensional problem for the isentropic model, Matsumura in [9] obtained a convergence rate $O(t^{-3/4})$ toward a constant state for a three-dimensional case under the smallness of H^3 norm of the initial data. The proof is based on time weighted energy estimates of solutions. For the half space problem in $\mathbb{R}^n_+$, Kagei and Kobayashi in [5] proved that the perturbation from a constant state decays with a convergence rate $O(t^{-n/2})$ for the impermeable problem if the initial perturbation in $H^s \cap L^1$ is sufficiently small where $s \geq [n/2]+1$. Kagei and Kawashima in [4] studied the outflow problem (1.1), (1.2) and (1.3) and showed that the planar stationary wave is time asymptotically stable by using the H^s energy method where $s \geq [n/2]+1$.

In the present paper, we study a convergence rate of the solution toward the planar stationary wave $(\tilde{\rho}(x_1), \tilde{u}(x_1))$, which is a solution to (1.1), independent of t and x', satisfying that $\tilde{u}$ is given by the form $\tilde{u} = (\tilde{u}_1, 0, \dots, 0)$. Therefore it satisfies

$$(\tilde\rho\tilde u_1)_{x_1} = 0, \tag{1.5a}$$
$$(\tilde\rho\tilde u_1^2 + p(\tilde\rho))_{x_1} = \mu\tilde u_{1x_1x_1}, \tag{1.5b}$$

where μ is a positive constant defined by $\mu := 2\mu_1 + \mu_2$. The stationary solution $(\tilde\rho(x_1), \tilde u_1(x_1))$ is supposed to satisfy the boundary condition (1.3) and the spatial asymptotic condition (1.4) as well as a positivity of the density:

$$\tilde u_1(0) = u_{\mathrm b}, \quad \lim_{x_1\to\infty}(\tilde\rho(x_1), \tilde u_1(x_1)) = (\rho_+, u_+), \quad \inf_{x_1\in\mathbb{R}_+}\tilde\rho(x_1) > 0. \tag{1.6}$$

The solvability of the one-dimensional boundary value problem (1.5) and (1.6) is discussed in the paper [7]. To summarize the existence result and the decay property of the stationary wave, we define sound speed c_+ and the Mach number M_+ at the spatial asymptotic state:

$$c_+ := \sqrt{p'(\rho_+)} = \sqrt{\gamma K\rho_+^{\gamma-1}}, \quad M_+ := \frac{|u_+|}{c_+}.$$

Moreover the quantity $\delta := |u_{\mathrm b} - u_+|$, which is called a boundary strength, plays an essential role in existence and stability analyses on the stationary wave.

Proposition 1.1 ([7]). *There exists a positive constant $w_{\mathrm c}$ such that the boundary value problem (1.5) and (1.6) has a unique solution $(\tilde\rho, \tilde u_1)$ if and only if the conditions $M_+ \geq 1$,*

$$u_+ < 0 \quad \text{and} \quad u_{\mathrm b} < w_{\mathrm c}u_+ \tag{1.7}$$

hold. Moreover the solution $(\tilde\rho, \tilde u_1)$ satisfies the following decay estimates.

(i) *If $M_+ > 1$, there exist positive constants c and C such that the stationary solution $(\tilde\rho, \tilde u_1)$ satisfies*

$$|\partial_1^k(\tilde\rho(x_1) - \rho_+, \tilde u_1(x_1) - u_+)| \leq C\delta e^{-cx_1} \quad \text{for } k = 0, 1, 2, \ldots. \tag{1.8}$$

(ii) *If $M_+ = 1$, there exist a positive constant C such that the stationary solution $(\tilde\rho, \tilde u_1)$ satisfies*

$$|\partial_1^k(\tilde\rho(x_1) - \rho_+, \tilde u_1(x_1) - u_+)| \leq C\frac{\delta^{k+1}}{(1+\delta x_1)^{k+1}} \quad \text{for } k = 0, 1, 2, \ldots. \tag{1.9}$$

The constant $w_{\mathrm c}$ in (1.7) is determined as a root of $H(w_{\mathrm c}) = 0$, where H is defined by

$$H(w_{\mathrm c}) := \frac{\rho_+u_+}{\mu}(w_{\mathrm c} - 1) + \frac{K\rho_+^\gamma}{\mu u_+}\left(w_{\mathrm c}^{-\gamma} - 1\right).$$

Namely, for the subsonic case $M_+ > 1$, $w_{\mathrm c}$ is a root of the equation

$$\rho_+u_+^2(w_{\mathrm c} - 1) + K\rho_+^\gamma(w_{\mathrm c}^{-\gamma} - 1) = 0$$

satisfying $w_{\mathrm c} > 1$. For the super sonic case $M_+ = 1$, $w_{\mathrm c}$ is equal to 1.

For the multi-dimensional half space problem, Kagei and Kawashima in [4] proved the asymptotic stability of the planar stationary wave $(\tilde\rho, \tilde u)$ under smallness assumptions on the initial perturbation and the boundary strength δ. In the paper [13], a convergence rate of the solution toward the planar stationary wave is obtained by assuming that the initial perturbation decays in the normal direction with the algebraic or the exponential rate. It is also required that the initial perturbation is sufficiently small in $H^s(\mathbb{R}^n_+)$ with $n = 2$ and 3. Here s is a positive integer defined by

$$s := \left[\frac{n-1}{2}\right] + 2, \tag{1.10}$$

where $[x]$ denotes the greatest integer which does not exceed x.

We summarize the results on convergence rates obtained in the paper [13]. In Theorem 1.2, the convergence rate for a supersonic case $M_+ > 1$ is proved.

Theorem 1.2 ([13]). *Let $n = 2$ or 3, and s be a positive integer defined by* (1.10). *Suppose that the conditions $M_+ > 1$,* (1.7) *and $\|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_{H^s} + \delta \leq \varepsilon_0$ hold for a certain positive constant ε_0. Moreover, if the initial data satisfies $(\rho_0 - \tilde{\rho}, u_0 - \tilde{u}) \in L^2_\alpha(\mathbb{R}^n_+)$ for a certain constant $\alpha \geq 0$, then the solution (ρ, u) to the initial boundary value problem* (1.1), (1.2) *and* (1.3) *satisfies the decay estimate*

$$\|(\rho, u)(t) - (\tilde{\rho}, \tilde{u})\|_{L^\infty} \leq C(1+t)^{-\alpha/2-(n-1)/4}. \tag{1.11}$$

The convergence rate $(1+t)^{-\alpha/2}$ in (1.11) holds since the two characteristics of the corresponding Euler equation

$$\begin{aligned} \rho_t + (\rho u_1)_{x_1} &= 0, \\ (\rho u_1)_t + (\rho u_1^2 + p(\rho))_{x_1} &= 0 \end{aligned}$$

over $\mathbb{R}_+$ are negative as $x_1 \to \infty$. On the other hand, the convergence rate $(1+t)^{-(n-1)/4}$ in (1.11) is obtained by using a dissipative property of the viscosity in the equation (1.1b). Therefore, the convergence rate (1.11) holds owing to both of the hyperbolicity and the parabolicity of the equations (1.1).

The next theorem shows the algebraic convergence for the transonic case $M_+ = 1$. In this case, owing to the degenerate property of the stationary wave, the convergence rate is worse than that for the supersonic case $M_+ > 1$. Moreover we need a upper restriction on the exponent α. Namely we assume that α is smaller than a certain positive constant α_* in (1.12) below. This kind of restriction is also required in the researches for scalar viscous conservation laws in [12, 17], in which the asymptotic stability of a degenerate traveling wave is proved.

Theorem 1.3 ([13]). *Let $n = 2$ or 3, and s be a positive integer defined by* (1.10). *Suppose that $M_+ = 1$ and* (1.7) *hold. Let α be a constant satisfying $\alpha \in [0, \alpha_*)$, where α_* is a positive constant defined by*

$$\alpha_* := \frac{2}{a}(1 + \sqrt{a+1}), \quad a := 1 + (n-1)\Big(\frac{\mu_1 + \mu_2}{2\mu_1 + \mu_2}\Big)^2. \tag{1.12}$$

Then there exists a certain positive constant ε_0 such that if $\|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_{H^s_\alpha} + \delta \leq \varepsilon_0$, then the solution (ρ, u) to the initial boundary value problem (1.1), (1.2) *and* (1.3) *satisfies the estimate*

$$\|(\rho, u)(t) - (\tilde{\rho}, \tilde{u})\|_{L^\infty} \leq C(1+t)^{-\alpha/4-(n-1)/4}. \tag{1.13}$$

Remark 1.4. For the case of $M_+ > 1$, we can also obtain the exponential convergence rate

$$\|(\rho,u)(t)-(\tilde{\rho},\tilde{u})\|_{L^\infty} \le Ce^{-\alpha t}$$

by assuming that the initial perturbation belongs to the exponentially weighted space $L^2_{\alpha,\exp}(\mathbb{R}^n_+) := \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n_+)\ ;\ e^{(\alpha/2)x_1}u \in L^2(\mathbb{R}^n_+)\}$.

Notations. Let $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_t := \frac{\partial}{\partial t}$. The operators $\nabla := (\partial_1,\ldots,\partial_n)$ and $\Delta := \sum_{i=1}^n \partial_i^2$ denote standard gradient and Palladian with respect to $x = (x_1,\ldots,x_n)$. The operators $\partial_{x'} := (\partial_2,\ldots,\partial_n)$ and $\Delta_{x'} := \sum_{i=2}^n \partial_i^2$ denote tangential gradient and Laplacian with respect to $x' = (x_2,\ldots,x_n)$. The expression $\partial_{x'}u$ is sometimes abbreviated to $u_{x'}$. For a non-negative integer k, we denote by ∇^k and $\partial_{x'}^k$ the totality of all k-th order derivatives with respect to x and x', respectively. For non-negative integers i, j and k, the operators $T_{j,k}$ and $\partial_{i,j,k}$ are defined by $T_{j,k} := \partial_{x'}^j\partial_t^k$ and $\partial_{i,j,k} := \partial_1^i\partial_{x'}^j\partial_t^k = \partial_1^i T_{j,k}$. For a domain $\Omega \subset \mathbb{R}^n_+$ and $1 \le p \le \infty$, the space $L^p(\Omega)$ denotes the standard Lebesgue space equipped with the norm $\|\cdot\|_{L^p(\Omega)}$. We sometimes abbreviate $L^p(\Omega)$ to L^p if $\Omega = \mathbb{R}^n_+$. We also use the notations $L^p_{x'} := L^p(\mathbb{R}^{n-1})$ and $L^p_{x_1} := L^p(\mathbb{R}_+)$. For a non-negative integer s, $H^s = H^s(\mathbb{R}^n_+)$ denotes the s-th order Sobolev space over $\mathbb{R}^n_+$ in the L^2 sense with the norm $\|\cdot\|_{H^s}$. We note $H^0 = L^2$ and $\|\cdot\| := \|\cdot\|_{L^2}$. For a constant $\alpha \in \mathbb{R}$, the space $L^2_\alpha(\mathbb{R}^n_+)$ denotes the algebraically weighted L^2 space in the normal direction defined by $L^2_\alpha(\mathbb{R}^n_+) := \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n_+)\ ;\ |u|_\alpha < \infty\}$ equipped with the norm

$$|u|_\alpha := \|u\|_{L^2_\alpha} := \Big(\int_{\mathbb{R}^n_+} (1+x_1)^\alpha |u(x)|^2\,dx\Big)^{1/2}.$$

The space $H^s_\alpha(\mathbb{R}^n_+)$ denotes the algebraic weighted H^s space corresponding to L^2_α defined by $H^s_\alpha(\mathbb{R}^n_+) := \{u \in L^2_\alpha\ ;\ \nabla^k u \in L^2_\alpha \text{ for } 0 \le k \le s\}$, equipped with the norm

$$\|u\|_{H^s_\alpha} := \Big(\sum_{k=0}^s |\nabla^k u|_\alpha^2\Big)^{1/2}.$$

2 A priori estimates for supersonic flow

In this section, we show key estimates of solutions, which are essential in the proof of the stability theorems. The proof is mainly based on the time and space weighted *a priori* estimates of the perturbation in H^s and weighted L^2 spaces. To this end, we employ the perturbation

$$(\varphi,\psi)(t,x) := (\rho,u)(t,x) - (\tilde{\rho},\tilde{u})(x_1)$$

from the stationary solution $(\tilde{\rho},\tilde{u})$. Owing to equations (1.1) and (1.5), the perturbation (φ,ψ) satisfies the system of equations

$$\varphi_t + u\cdot\nabla\varphi + \rho\,\mathrm{div}\,\psi = f, \tag{2.1a}$$
$$\rho\{\psi_t + (u\cdot\nabla)\psi\} - L\psi + p'(\rho)\nabla\varphi = g, \tag{2.1b}$$

where f, g and $L\psi$ are given by

$$f := -\operatorname{div}\tilde{u}\varphi - \nabla\tilde{\rho}\cdot\psi,$$
$$g := -\rho(\psi\cdot\nabla)\tilde{u} - \varphi(\tilde{u}\cdot\nabla)\tilde{u} - (p'(\rho) - p'(\tilde{\rho}))\nabla\tilde{\rho},$$
$$L\psi := \mu_1\Delta\psi + (\mu_1+\mu_2)\nabla\operatorname{div}\psi.$$

The initial and the boundary conditions for (φ,ψ) are derived from (1.2) and (1.3) as

$$(\varphi,\psi)(0,x) = (\varphi_0,\psi_0)(x) := (\rho_0,u_0)(x) - (\tilde{\rho},\tilde{u})(x_1), \tag{2.2}$$
$$\psi(t,0,x') = 0. \tag{2.3}$$

The perturbation is often abbreviated as

$$\Phi := (\varphi,\psi), \quad \Phi_0 := (\varphi_0,\psi_0).$$

To summarize the *a priori* estimate for (φ,ψ), we employ the following notations:

$$|||u|||_m^2 := \sum_{i=0}^{m} |[u]|_i^2, \quad |[u]|_m^2 := \sum_{k=0}^{[m/2]} \|\nabla^{m-2k}\partial_t^k u\|^2$$

and a time weighted norm $E(t)$ and a corresponding dissipative norm $D(t)$ defined by

$$E(t)^2 := \sum_{j=0}^{s-1}(1+t)^j |||\partial_{x'}^j\Phi(t)|||_{s-j}^2, \quad N(t) := \sup_{0\le\tau\le t} E(\tau),$$
$$D(t)^2 := \sum_{j=0}^{s-1}(1+t)^j\hat{D}_j(t)^2,$$
$$\hat{D}_j(t)^2 := \sum_{i=1}^{s-j} |[\partial_{x'}^j\Phi(t)]|_i^2 + |[\partial_{x'}^j\psi(t)]|_{s+1-j}^2 + \|\partial_{x'}^j\varphi(t,0,\cdot)\|_{L^2_{x'}}^2.$$

In addition, define spatial weighted norms $\tilde{E}_\alpha(t)$ and $\tilde{D}_\alpha(t)$ by

$$\tilde{E}_\alpha(t)^2 := E(t)^2 + |\Phi(t)|_\alpha^2, \quad \tilde{D}_\alpha(t)^2 := D(t)^2 + \alpha|\Phi(t)|_{\alpha-1}^2 + |\nabla\psi(t)|_\alpha^2.$$

We show a uniform bound of $\tilde{E}_\alpha(t)$, which is summarized in Proposition 2.2. To this end, we employ function spaces as

$$X(0,T) := \{(\varphi,\psi)\in C([0,T];H^s)\ ;\ \nabla\varphi\in L^2(0,T;H^{s-1}),\ \nabla\psi\in L^2(0,T;H^s)\},$$
$$X_\alpha(0,T) := \{(\varphi,\psi)\in X(0,T)\ ;\ (\varphi,\psi)\in C([0,T];L^2_\alpha),\ \nabla\psi\in L^2(0,T;L^2_\alpha)\}$$

for $T>0$ and $\alpha\ge 0$.

The following lemma shows the existence of the solution to (2.1), (2.2) and (2.3) locally in time, which can be proved by a standard iteration method with using the idea in [3].

Lemma 2.1. *Suppose that the initial data satisfies $(\varphi_0,\psi_0)\in H^s(\mathbb{R}^n_+)$ and a suitable compatibility condition. Then there exists a positive constant T depending on $\|(\varphi_0,\psi_0)\|_{H^s}$ such that the problem (2.1), (2.2) and (2.3) has a unique solution $(\varphi,\psi)\in X(0,T)$. Moreover, if the initial data satisfies $(\varphi_0,\psi_0)\in L^2_\alpha(\mathbb{R}^n_+)$, it holds $(\varphi,\psi)\in X_\alpha(0,T)$.*

The following proposition gives the algebraically weighted *a priori* estimates for the supersonic case $M_+ > 1$. From the algebraically weighted estimates (2.4) and (2.5), we see that the tangential derivatives of the solution verify better decay estimates than the normal derivatives.

Proposition 2.2. *Suppose that $M_+ > 1$ holds. Let $(\varphi, \psi) \in X_\alpha(0,T)$ be a solution to (2.1), (2.2) and (2.3) for certain $T > 0$ and $\alpha \geq 0$. Then there exist positive constants ε_1 and C independent of T such that if $N(T) + \delta \leq \varepsilon_1$, then the solution $\Phi = (\varphi, \psi)$ satisfies the following estimates for $t \in [0,T]$:*

$$(1+t)^\ell \tilde{E}_{\alpha-\ell}(t)^2 + \int_0^t (1+\tau)^\ell \tilde{D}_{\alpha-\ell}(\tau)^2 \, d\tau \leq C\big(|\Phi_0|_\alpha^2 + \|\Phi_0\|_{H^s}^2\big) \tag{2.4}$$

for an arbitrary integer $\ell = 0, \ldots, [\alpha]$ and

$$(1+t)^\xi \tilde{E}_0(t)^2 + \int_0^t (1+\tau)^\xi \tilde{D}_0(\tau)^2 \, d\tau \leq C\big(|\Phi_0|_\alpha^2 + \|\Phi_0\|_{H^s}^2\big)(1+t)^{\xi-\alpha} \tag{2.5}$$

for an arbitrary $\xi > \alpha$.

The proof of Proposition 2.2 is based on deriving the estimates in $L^2_\alpha(\mathbb{R}^n_+)$ and $H^s(\mathbb{R}^n_+)$. To obtain these estimates, we utilize an interpolation inequality and the Poincaré type inequality, which are summarized

Lemma 2.3. *Let $\Phi = (\varphi, \psi)$ be a solution to (2.1), (2.2) and (2.3).*

(i) *Let $2 < p \leq \infty$ and let j and m be integers satisfying*

$$0 \leq j + m \leq s, \quad \theta := \frac{n}{m}\left(\frac{1}{2} - \frac{1}{p}\right) \in (0,1).$$

Then the solution Φ satisfies

$$\|\partial_{x'}^j \Phi(t)\|_{L^p} \leq C \|\partial_{x'}^j \Phi(t)\|^{1-\theta} \|\nabla^m \partial_{x'}^j \Phi(t)\|^\theta \leq CE(t)(1+t)^{-j/2}. \tag{2.6}$$

The inequality (2.6) also holds for the cases $p = 2$, $m = 0$, $\theta = 0$ and $0 \leq j \leq s - 1$.

(ii) *Suppose that $M_+ > 1$ holds. Let $\tilde{u}$ be a stationary solution to (1.5) and (1.6) satisfying (1.8). Then Φ satisfies*

$$\int_{\mathbb{R}^n_+} |\nabla^k \tilde{u}| |\partial_{x'}^j \Phi(t)|^2 \, dx \leq C\delta\big(\|\nabla \partial_{x'}^j \Phi(t)\|^2 + \|\partial_{x'}^j \varphi(t,0,\cdot)\|_{L^2_{x'}}^2\big) \tag{2.7}$$

for integers $k \geq 1$ and $0 \leq j \leq s-1$.

Using the above lemma, we consider the derivation of the estimate of the perturbation (φ, ψ) in $L^2_\alpha(\mathbb{R}^n_+)$. To do this, we introduce an energy form $\mathcal{E}$, similarly as in [7]:

$$\mathcal{E} := K\tilde{\rho}^{\gamma-1}\omega\Big(\frac{\tilde{\rho}}{\rho}\Big) + \frac{1}{2}|\psi|^2, \quad \omega(r) := r - 1 - \int_1^r \eta^{-\gamma} \, d\eta.$$

Under the smallness assumption on $N(T)$, we have $\|\Phi(t)\|_{L^\infty} \ll 1$. Hence, the energy form $\mathcal{E}$ is equivalent to the square of the perturbation (φ, ψ):

$$c(\varphi^2 + |\psi|^2) \leq \mathcal{E} \leq C(\varphi^2 + |\psi|^2). \tag{2.8}$$

Moreover we have the uniform bounds of solutions as

$$0 < c \le \rho(t,x) \le C, \quad |u(t,x)| \le C, \quad -C \le u_1(t,x) \le -c < 0, \tag{2.9}$$

owing to $u_b < 0$ and $N(T) + \delta \ll 1$. Using the time and space weighted energy method, we obtain the energy inequality in L^2 framework.

Lemma 2.4. *Suppose that the same conditions as in Proposition 2.2 hold. Then there exists a positive constant ε_1 such that if $N(T) + \delta \le \varepsilon_1$, it holds*

$$\begin{aligned}(1+t)^\xi|\Phi(t)|_\beta^2 &+ \int_0^t (1+\tau)^\xi \big(\beta|\Phi(\tau)|_{\beta-1}^2 + |\nabla\psi(\tau)|_\beta^2 + \|\varphi(\tau,0,\cdot)\|_{L^2_{x'}}^2\big)\,d\tau \\ &\le C|\Phi_0|_\beta^2 + C\xi\int_0^t (1+\tau)^{\xi-1}|\Phi(\tau)|_\beta^2\,d\tau + C\delta\int_0^t (1+\tau)^\xi\|\nabla\varphi(\tau)\|^2\,d\tau \end{aligned} \tag{2.10}$$

for $t \in [0,T]$ and arbitrary constants $\beta \in [0,\alpha]$ and $\xi \ge 0$.

Next we show the estimates for higher order derivatives. Precisely we derive the time weighted energy estimate in $H^s(\mathbb{R}^n_+)$, which is summarized in the next proposition.

Proposition 2.5. *Suppose that the same conditions as in Proposition 2.2 hold. Then there exists a positive constant ε_1 such that if $N(T) + \delta \le \varepsilon_1$, it holds*

$$(1+t)^\xi E(t)^2 + \int_0^t (1+\tau)^\xi D(\tau)^2\,d\tau \le C\|\Phi_0\|_{H^s}^2 + C\xi\int_0^t (1+\tau)^{\xi-1}|||\Phi(\tau)|||_s^2\,d\tau \tag{2.11}$$

for an arbitrary $\xi \ge 0$.

Here we give a brief outline of the proof. (For the details, the readers are referred to the paper [13].) It is divided into several steps. We firstly discuss the derivation of estimates for tangential and time derivatives $T_{j,k}\Phi$ for $0 \le j + 2k \le s$. By using the parabolicity, we show estimates of $\nabla T_{j,k}\psi$ for $0 \le j+2k \le s-1$. Then we obtain estimates of x_1-derivatives of φ, i.e., $\partial_{i+1,j,k}\varphi$ for $0 \le i + j + 2k \le s - 1$. Finally we get estimates of second order x_1-derivatives of ψ by substituting the previously obtained estimates in the equation (2.1b). These computations give the desired estimate (2.11).

Next we discuss the derivation of the estimates (2.4) and (2.5). Adding (2.10) to (2.11) and then letting δ suitably small, we have

$$\begin{aligned}(1+t)^\xi \tilde{E}_\beta(t)^2 &+ \int_0^t (1+\tau)^\xi\big(\beta|\Phi(\tau)|_{\beta-1}^2 + \tilde{D}_\beta(\tau)^2\big)\,d\tau \\ &\le C(|\Phi_0|_\beta^2 + \|\Phi_0\|_{H^s}^2) + C\xi\int_0^t (1+\tau)^{\xi-1}\big(|\Phi(\tau)|_\beta^2 + |||\Phi(\tau)|||_s^2\big)\,d\tau. \end{aligned} \tag{2.12}$$

Substituting the inequality

$$|||\Phi(\tau)|||_s^2 = \|\Phi(\tau)\|^2 + \sum_{i=1}^s [[\Phi(\tau)]]_i^2 \le |\Phi(\tau)|_\beta^2 + \tilde{D}_\beta(\tau)^2$$

in the second term on the right hand side of (2.12), we get

$$(1+t)^{\xi}\tilde{E}_{\beta}(t)^2+\int_0^t(1+\tau)^{\xi}\big(\beta|\Phi(\tau)|^2_{\beta-1}+\tilde{D}_{\beta}(\tau)^2\big)\,d\tau$$
$$\le C(|\Phi_0|^2_{\beta}+\|\Phi_0\|^2_{H^s})+C\xi\int_0^t(1+\tau)^{\xi-1}\big(|\Phi(\tau)|^2_{\beta}+\tilde{D}_{\beta}(\tau)^2\big)\,d\tau.$$

Applying an induction with respect to β and ξ, of which idea is developed in [6] and [15], we obtain the desired estimates (2.4) and (2.5).

Finally, Theorem 1.2 is proved by using the interpolation inequality in L^{∞} norm:

$$\begin{aligned}\|\Phi\|_{L^{\infty}} &= \sup_{x_1\in\mathbb{R}_+}\|\Phi(x_1,\cdot)\|_{L^{\infty}_{x'}}\\ &\le C\sup_{x_1\in\mathbb{R}_+}\big(\|\Phi(x_1,\cdot)\|^{1-\theta}_{L^2_{x'}}\|\partial^{s-1}_{x'}\Phi(x_1,\cdot)\|^{\theta}_{L^2_{x'}}\big)\\ &\le C\|(\Phi,\nabla\Phi)\|^{1-\theta}\|\partial^{s-1}_{x'}(\Phi,\nabla\Phi)\|^{\theta}\quad\text{for}\quad\theta=\frac{n-1}{2(s-1)},\end{aligned}\tag{2.13}$$

which follows from the Gagliardo–Nirenberg inequality over $\mathbb{R}^{n-1}$ and the Sobolev inequality $\|v\|_{L^{\infty}(\mathbb{R}_+)}\le C\|v\|_{L^2(\mathbb{R}_+)}\|v_{x_1}\|_{L^2(\mathbb{R}_+)}$. Then substituting the decay estimates

$$\|(\Phi,\nabla\Phi)(t)\|\le C(1+t)^{-\alpha/2},\quad \|\partial^{s-1}_{x'}(\Phi,\nabla\Phi)(t)\|\le C(1+t)^{-(\alpha+s-1)/2},$$

which are direct consequences of (2.5), in the inequality (2.13), we get the desired decay estimate (1.11).

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