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Nonstandard arguments and the characterization of independence in generic structures

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Abstract

以前、generic 構造を調べるための新たな方法論として超準的手法を用いる方法を提示し、generic 構造の安定性の強さを調べることに応用した [1]. 安定な generic 構造における forking independence はより具体的な特徴付けが得られることがわかっている。今回は、Wagner [11], Verbovskiy-Yoneda [10] の定理 3.7 に、超準的な手法を用いた別証明を与える。

1 Preliminaries

1.1 Generic structures

We review basic definitions and facts in the study of generic structures. Let $L=\{R_i : i\in I\}$ be a relational language. Fix any $\alpha_i\in\mathbb{R}_{\geq 0}$ for each i. For a finite L-structure A, put $\delta(A)=|A|-\sum_i \alpha_i|R_i^A|$. Let $\mathbb{K}=\{A :$ finite L-structures $|\delta(A')\geq 0, \forall A'\subset A\}$ and $\overline{\mathbb{K}}$ be the set of L-structures whose any finite substructure belongs to $\mathbb{K}$. For $A, B\subset M\in\overline{\mathbb{K}}$, we say that A and B are free over $A\cap B$ if $R_i^{AB}=R_i^A\cup R_i^B$ for each i.

Lemma 1.1 *For finite A and B,*

1. *(monotonicity)* $\delta(AB)+\delta(A\cap B)\leq\delta(A)+\delta(B)$,
2. *(modularity equation)* $\delta(AB)+\delta(A\cap B)=\delta(A)+\delta(B)$ *if and only if A and B are free over $A\cap B$.*

We define $\delta(A/B)$ the relative predimension of A over B. For finite A, B, $\delta(A/B)=\delta(AB)-\delta(B)$. For finite A and arbitrary B, $\delta(A/B)=\inf\{\delta(A-B/B_0) : B_0\subset_{\text{fin}} B\}$. By monotonicity, it is easy to check that these two definitions have the same value in the case A and B are both finite.

Definition 1.2 Let A and B be any members of $\mathbb{K}$ with $A \subseteq B$. We say that A is closed in B (or A is a strong substructure of B) if $\delta(A) \leq \delta(B')$ for any $A \subseteq B' \subseteq B$,. If A is closed in B, we write $A \leq B$. For any members A and B of $\overline{\mathbb{K}}$ with $A \subseteq B$, we say that $A \leq B$ if $A \cap B' \leq B'$ for any finite $B' \subset B$.

By monotonicity, it is easy to check that these two definitions are the same in the case A and B are both finite.

Fact 1.3 *Let M be any member of $\overline{\mathbb{K}}$ and A be any subset of M. Then there exists the smallest closed superset $\overline{A}$ of A in M. We call $\overline{A}$ the closure of A in M.*

Note that in the case α_i is irrational, $\overline{A}$ is not necessarily finite even if A is finite.

Fact 1.4 *For any subset A of M, $\overline{A}$ is contained in the algebraic closure* $\mathrm{acl}(A)$ *of A.*

Definition 1.5 Let M be an L-structure. We say that M is a $\mathbb{K}$-generic structure if the following conditions are satisfied:

1. M is countable;
2. M is a member of $\overline{K}$;
3. For any members A and B of $\mathbb{K}$, if $A \leq B$ and $A \leq M$, then there is a copy B' of B over A with $B' \leq M$.

Example 1.6 *A countable graph G is called a random graph if it satisfies the following property: for each $m, n \in \omega$ and all $a_1, \ldots, a_m, b_1, \ldots, b_n \in G$, if $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ are disjoint, then there exists $c \in G$ such that $G \models \bigwedge_{1 \leq i \leq m} R(c, a_i) \wedge \bigwedge_{1 \leq j \leq n} \neg R(c, b_j)$. It is well known that a random graph is uniquely exist. Let $\mathbb{K}$ be the set of all finite graphs and $\delta(A) = |A|$ for any finite graph A. Then the random graph is the $\mathbb{K}$-generic structure.*

Let M be a member of $\overline{\mathbb{K}}$. Put $d(A) = \inf\{\delta(A') | A \subseteq A' \subset_{\text{fin}} M\}$ for any finite subset A of M. We call d a dimension function for M. Note that if $\overline{A}$ is finite, then $d(A) = \delta(\overline{A})$. We define $d(A/B)$, the relative dimension of A over B as follows. For finite subsets A and B of M, we define $d(A/B) = d(AB) - d(B)$. For finite A and arbitrary B, put $d(A/B) = \inf\{d(A - B/B_0) : B_0 \subset_{\text{fin}} B\}$. Monotonicity of d is proved in Section 3. Then we have that these two definitions have the same value in the case A and B are both finite.

1.2 Stability

Now, we study stability theory very shortly. A more detail explanation is recorded in many books of stability theory, for example [7].

Definition 1.7 Let T be a theory and κ be an infinite cardinal. Let $\mathcal{M}$ be a big saturated model of T.

1. We say that T is κ-stable if for any $A \subset \mathcal{M}$ with $|A| = \kappa$, the cardinality of the set of complete types over A is equal to κ.

2. We say that T is stable if T is κ-stable for some κ.

3. Let M be a L-structure. We say that M is (κ-)stable if $\mathrm{Th}(M)$ is (κ-)stable.

Definition 1.8 Let κ be an infinite cardinal and $(\bar{a}_i)_{i<\kappa}$ be a sequence of n-tuples in $\mathcal{M}$. We say that $(\bar{a}_i)_{i<\kappa}$ is an indiscernible sequence over A if for any $k < \omega$, $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$, we have $\mathrm{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_k}/A) = \mathrm{tp}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_k}/A)$.

Fact 1.9 *Suppose that T is stable. Let $(\bar{a}_i)_{i<\kappa}$ be an indiscernible sequence. For any $k < \omega$, if $i_1, \ldots, i_k$ are distinct and $j_1, \ldots, j_k$ are distinct, then we have* $\mathrm{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_k}/A) = \mathrm{tp}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_k}/A)$.

Definition 1.10 Suppose that T is stable. Let $\mathcal{M}$ be a big saturated model of T.

1. Let $\bar{a}$ be a finite tuple in $\mathcal{M}$ and $\varphi(x, \bar{a})$ be an $L(\bar{a})$-formula. We say that $\varphi(x, \bar{a})$ forks over A if there is an indiscernible sequence $(\bar{a}_i)_{i<\omega}$ over A with $\bar{a}_0 = \bar{a}$ such that $\{\varphi(x, \bar{a}_i) | i < \omega\}$ is inconsistent.

2. Let $A \subseteq B \subset \mathcal{M}$ and $\Gamma(x)$ be a set of $L(B)$-formulas. We say that $\Gamma(x)$ forks over A if $\varphi(x)$ forks over A for some $\varphi(x) \in \Gamma(x)$.

3. Let A, B, and C be subsets of $\mathcal{M}$. We say that A and B are forking independent over C and write $A \mathop{\perp}_C B$ if $\mathrm{tp}(\bar{a}/BC)$ does not fork over C for any $\bar{a} \in A$.

2 Nonstandard arguments

In this section, we review how to apply a nonstandard argument to the study of generic structures. Note that arguments in this section is essentially the same as in [1].

Let $M \in \overline{\mathbb{K}}$. We consider M to be a 3-sorted structure:

$$(M \cup P \cup \mathbb{R}; F, \in, \delta, \leq, d, \cdots)$$

where P is the powerset of M, F is a unary relation on P such that for any subset A of M, $F(A)$ holds if and only if A is a finite set, $\in$ is the membership relation on $M \times P$, and "$\cdots$" contains L and $(+, \cdot, <)$ in $\mathbb{R}$.

We define the nonstandard model M^* of M by a sufficiently saturated elementary extension of this structure:

$$(M \cup P \cup \mathbb{R}; F, \in, \delta, \leq, d, \cdots) \prec (M^* \cup P^* \cup \mathbb{R}^*; F^*, \in^*, \delta^*, \leq^*, d^*, \cdots).$$

Notation 2.1
- For any set variables X and Y, we define $X \subseteq^* Y$ as an abbreviation for $\forall x(x \in^* X \to x \in^* Y)$.
- Note that a function from P to P which maps each member of P to its closure is defined by some formula $\varphi(X, Y)$. Let A^* be any member of P^*. We write $\overline{A^*}$ for the realization of $\varphi(A^*, Y)$. We say that A^* is closed if $\overline{A^*} = A^*$ holds.
- We denote F^* (resp., $\in^*$, $\subseteq^*$, δ^*, $\leq^*$, d^*) simply by F (resp., $\in$, $\subseteq$, δ, $\leq$, d) if there is no confusion.
- Ler r and s be any elements of $\mathbb{R}^*$. We write $r \approx s$ if $-a < r - s < a$ holds in M^* for all positive real numbers a.

Definition 2.2
1. Let A^* be any member of P^*. We say that A^* is a hyperfinite set if $F^*(A^*)$ holds in M^*.
2. Let A be any subset of M and A^* be any hyperfinite set. We say that A^* is a hyperfinite extension of A and write $A^* \supset_{\mathrm{hf}} A$ if
   - $M^* \models a \in^* A^*$ for each member a of A, and
   - $M^* \models A^* \subseteq^* A$.

Remark 2.3 *For any subset A of M, there exists a hyperfinite extension of A.*

Proof: It is enough to prove that the following set of formulas is satisfiable:

$$\Gamma(X) = \{a \in X | a \text{ is a member of } A\} \cup \{X \subseteq A\} \cup \{F(X)\}.$$

But for any finite subset A_0 of A, A_0 realizes the following set of formulas:

$$\Gamma_0(X) = \{a \in X | a \text{ is a member of } A_0\} \cup \{X \subseteq A\} \cup \{F(X)\}.$$

So, by compactness, $\Gamma(X)$ is satisfiable.

Lemma 2.4 *For any real number r, finite tuple $\bar{a}$ in M, and subset A of M, the following are equivalent:*

1. $\delta(\bar{a}/A) = r$;
2. $\delta^*(\bar{a}/A^*) \approx r$ *for any hyperfinite extension A^* of A;*
3. $\delta^*(\bar{a}/A^*) \approx r$, *for some hyperfinite extension A^* of A.*

Proof: We may assume that $\bar{a} \cap A = \emptyset$.
(1 $\Rightarrow$ 2) By monotonicity of δ, for each $n < \omega$, there is a finite subset A_n of A such that

$$M \models \forall X(F(X) \wedge A_n \subseteq X \subseteq A \rightarrow r \leq \delta(\bar{a}/X) \leq r + 1/n).$$

The above formula holds also in M^* for each $n < \omega$. So if A^* is a hyperfinite extension of A, then we have

$$r \leq \delta^*(\bar{a}/A^*) \leq r + 1/n$$

for each $n < \omega$. So we have $\delta^*(\bar{a}/A^*) \approx r$.
(2 $\Rightarrow$ 3) Trivial.
(3 $\Rightarrow$ 1) We assume 3 and choose a witness A^*. Then $\delta^*(\bar{a}/A^*) \approx r$. Suppose 1 is not the case. Let $s = \delta(\bar{a}/A)$. Then $s \neq r$. By (1 $\Rightarrow$ 2), we have $\delta^*(\bar{a}/A^*) \approx s$. A contradiction.

Definition 2.5 Let A^* be a hyperfinite set.

1. Let ϵ be any positive real number. We say that A^* is ϵ-closed if $\delta(B^*/A^*) \geq -\epsilon$ for any hyperfinite set B^*.
2. We say that A^* is quasi-closed if $\delta(B^*/A^*) \gtrsim 0$ (that is, $\delta(B^*/A^*) \geq -\epsilon$ for all positive real numbers) for any hyperfinite set B^*.

Remark 2.6 *For any hyperfinite set A^*, A^* is quasi-closed if and only if $d(A^*) \approx \delta(A^*)$.*

Proof: Immediate.

Remark 2.7 *1. Let B be a closed subset of M. Then for any finite subset A of B and any positive real number ϵ, there is a finite ϵ-closed set B_ϵ with $A \subseteq B_\epsilon \subseteq B$.*

2. *Let A be a finite subset of M. Then for any $\epsilon > 0$ and any $A \subseteq B \subset_{\text{fin}} \overline{A}$, there exists $B \subseteq C \subset_{\text{fin}} \overline{A}$ such that $\delta(C) \leq d(A) + \epsilon$.*

3. *Let A be a finite subset of M. Suppose that $(B_n)_{n<\omega}$ is an increasing sequence of subsets of $\overline{A}$ such that $\bigcup_{n<\omega} B_n = \overline{A}$ and B_n is $\frac{1}{n}$-closed for each $n < \omega$. Then $\lim_{n\to\omega} \delta(B_n) = d(A)$.*

4. *Let $A \subseteq B \subset_{\text{fin}} \overline{A}$. Then $d(A) = d(B)$.*

Proof: 1. Otherwise, for any finite $A_0 \subseteq B$, there is a finite set $A_1 \subseteq B$ such that $\delta(A_1/A_0) < -\epsilon$. Iterating this, we have a sequence of finite sets $(A_i)_{i<\omega}$ such that $\delta(A_n/A_0 \dots A_{n-1}) < -\epsilon$ for each $n < \omega$. For sufficiently large $n < \omega$, we must have $\delta(A_0 \dots A_n) < 0$, a contradiction.

2. We may assume that $\overline{A}$ is infinite. By the definition of $d(A)$, there exists $A \subseteq C_0 \subset_{\text{fin}} \overline{A}$ such that $\delta(C_0) \leq d(A) + \epsilon$. Because $C_0 \not\leq \overline{A}$, there exists $C_0 \subseteq C_1 \subset_{\text{fin}} \overline{A}$ such that $\delta(C_1) < \delta(C_0)$. Iterating this, we can find C such that $B \subseteq C \subset_{\text{fin}} \overline{A}$ and $\delta(C) < \delta(C_0) \leq d(A) + \epsilon$.

3. Fix arbitrary $\epsilon > 0$. For any sufficiently large n, B_n is ϵ-closed. By 2, there exists $B_n \subseteq C_n \subset_{\text{fin}} \overline{A}$ such that $\delta(C_n) \leq d(A) + \epsilon$ for each n. Then $\delta(B_n) \leq d(A) + 2\epsilon$. So, we have $\lim_{n\to\omega} \delta(B_n) \leq d(A)$. The other direction is clear.

4. Immediate from 3.

Lemma 2.8 *Let A be a finite subset of M and B, C be any subsets of M. If $B \subseteq C$, then $d(A/B) \geq d(A/C)$.*

Proof: Immediate from 4 of the above remark.

This Lemma is called monotonicity of d. It shows that two definitions of $d(A/B)$ have the same value in the case A and B are finite.

Lemma 2.9 *For any real number r, finite tuple $\bar{a}$ in M and subset A of M, the following are equivalent:*

1. *$d(\bar{a}/A) = r$;*

2. $d(\bar{a}/A^*) \approx r$ *for any hyperfinite extension* A^* *of* A*;*

3. $d(\bar{a}/A^*) \approx r$*, for some hyperfinite extension* A^* *of* A*.*

Proof: By monotonicity of d, we can prove this lemma in the same way of the proof of Lemma 2.4.

Lemma 2.10 *For any subset* A *of* M*, the following conditions are equivalent:*

1. A *is closed;*

2. there is a quasi-closed hyperfinite extension of A*.*

Proof: $(1 \Rightarrow 2)$ By Remark 2.7.1.

$(2 \Rightarrow 1)$ Suppose A is not closed. Then there exists finite subset B of M such that $\delta(B/A \cap B) < 0$. Take any hyperfinite extension A^* of A. Because $A \cap B \subseteq^* A^* \subseteq^* A$, we have $\delta(B/A^*) \leq \delta(B/B \cap A) < 0$.

Definition 2.11 [11]

1. Let A and B be any finite subsets of M and C be arbitrary subset of M. Then we say that A and B are d-independent over C and write $A \mathop{\perp}^d_C B$ if the following conditions are satisfied:
   - $d(A/BC) = d(A/C)$, and
   - $\overline{AC} \cap \overline{BC} = \overline{C}$.
2. For arbitrary subsets A, B, and C of M, we say that A and B are d-independent over C if $A_0 \mathop{\perp}^d_C B_0$ for every finite subset A_0 of A and every finite subset B_0 of B.

Note that for closed sets A and B, A and B are d-independent over $A \cap B$ if and only if $d(A_0/B_0(A \cap B)) = d(A_0/A \cap B)$ for every finite subset A_0 of A and every finite subset B_0 of B.

Definition 2.12 Let A and B be closed subsets of M. Then we say that A and B are d^*-independent over $A \cap B$ if there exist a hyperfinite extension A^* of A and a hyperfinite extension B^* of B such that

- A^* and B^* are both quasi-closed and
- $d(A^*/B^*) = d(A^*/A^* \cap B^*)$.

Proposition 2.13 *Let A and B be closed subsets of M. Then the following are equivalent:*

1. *A and B are d-independent over $A \cap B$;*

2. *A and B are d^*-independent over $A \cap B$;*

3. *There exist a hyperfinite extension A^* of A and a hyperfinite extension B^* of B such that*

   - *A^* and B^* are both quasi-closed,*
   - *$d(A^*/B^*) = d(A^*/A^* \cap B^*)$, and*
   - *$(A^*B^*) \cap A \cap B = A^* \cap B^*$.*

Proof: (3 $\Rightarrow$ 2) Trivial.

(2 $\Rightarrow$ 1) Let A^*, B^* be a witness of d^*-independence. Take any finite subset A' of A and any finite subset B' of B. Then $d(A^*/B^*) \approx d(A^*/A^* \cap B^*)$. By transposition, $d(B^*/A^*) \approx d(B^*/A^* \cap B^*)$. By monotonicity, $d(B^*/A'(A^* \cap B^*)) \approx d(B^*/A^* \cap B^*)$. Again by transposition, $d(A'/B^*) \approx d(A'/A^* \cap B^*)$. Again by monotonicity, $d(A'/B'(A^* \cap B^*)) \approx d(A'/A^* \cap B^*)$. Finally, by Lemma 2.9, we have $d(A'/B'(A \cap B)) = d(A'/A \cap B)$.

(1 $\Rightarrow$ 3) Take $A^* \supset_{\mathrm{hf}} A$ and $B^* \supset_{\mathrm{hf}} B$ such that A^* and B^* are both closed and $(A^*B^*) \cap A \cap B = A^* \cap B^*$.

By compactness, it is enough to prove that for any finite subset A_0 of A, the following set of formulas are satisfiable:

1. $F(X)$

2. $X \subseteq A$

3. $A_0 \subseteq X$

4. "X is quasi-closed"

5. "$d(X/B^*) \approx d(X/X \cap B^*)$"

6. $(XB^*) \cap A \cap B = X \cap B^*$

Note that 4 and 5 are both expressed by an infinite set of formulas. We show that $A_0^* = \overline{A_0(A^* \cap B^*)} \cap A^*$ is a realization of the above set of formulas. 1, 2, 3, and 4 are clear.

5. First,

$$\begin{aligned} d(A_0^*/B^*) &= d(A_0^*B^*) - d(B^*) \\ &= d(A_0B^*) - d(B^*) \\ &= d(A_0/B^*) \\ &\approx d(A_0/B). \end{aligned}$$

Second,

$$\begin{aligned} d(A_0^*/A_0^* \cap B^*) &= d(A_0^*) - d(A_0^* \cap B^*) \\ &= d(A_0(A^* \cap B^*)) - d(A_0^* \cap B^*) \\ &\leq d(A_0(A^* \cap B^*)) - d(A^* \cap B^*) \\ &= d(A_0/A^* \cap B^*) \\ &\approx d(A_0/A \cap B) \end{aligned}$$

Finally, by d-independence of A and B, $d(A_0/B) = d(A_0/A \cap B)$.

Hence, $d(A_0^*/A_0^* \cap B^*) \lesssim d(A_0^*/B^*)$. The other direction is clear by monotonicity.

6. Note that$(A_0^*B^*) \cap A \cap B = (A_0^* \cap A \cap B) \cup (B^* \cap A \cap B) \subseteq (A^* \cap B) \cup (A^* \cap B^*) \subseteq A^* \cap B^* \subseteq A_0^* \cap B^*$.

3 Characterization of independence in generic structures

In stable generic structures, we can define two notions of independence: forking independence and d-independence. Wagner [11] showed that for closed sets A and B, if $A \cap B$ is algebraically closed, then A and B are forking independent over $A \cap B$ if and only if A and B are d-independent over $A \cap B$. He proved the result in the case that $\mathbb{K}$ satisfies finite closure condition. Verbovskiy and Yoneda [10] showed that the same result without assuming the finite closure condition. In this paper, we also does not assume the finite closure condition. To show the equivalence of forking independence and d-independence, we prove the following two statements:

1. for closed sets A and B, A and B are d-independent over $A \cap B$ if and only if A and B are free over $A \cap B$ and AB is closed;

2. for closed sets A and B, if $A \cap B$ is algebraically closed, then A and B are forking independent over $A \cap B$ if and only if A and B are free over $A \cap B$.

Wagner proved item 1 by using the epsilon-delta argument. In the first half of this section, we give a nonstandard proof of item 1 by using Proposition 2.13.

Both in [11] and [10], item 1 is used to prove item 2. Tsuboi [9] proved item 2 without using item 1. In the second half of this section, we give a nonstandard proof of item 2 by using the idea in [9].

Let $M \in \overline{\mathbb{K}}$. Next lemma is a nonstandard version of Lemma ??

Lemma 3.1 *For any subsets A and B of M, the following conditions are equivalent:*

1. *A and B are free over $A \cap B$;*
2. *For any $A^* \supset_{\mathrm{hf}} A$ and any $B^* \supset_{\mathrm{hf}} B$ with $(A^*B^*) \cap A \cap B = A^* \cap B^*$,*
$$\delta(A^*/A^* \cap B^*) \approx \delta(A^*/B^*);$$
3. *There exist $A^* \supset_{\mathrm{hf}} A$ and $B^* \supset_{\mathrm{hf}} B$ such that $(A^*B^*) \cap A \cap B = A^* \cap B^*$ and*
$$\delta(A^*/A^* \cap B^*) \approx \delta(A^*/B^*).$$

Proof: (1 $\Rightarrow$ 2) Suppose 2 is not the case. The following finite set of formulas are satisfiable:

- $F(X) \wedge F(Y)$
- $X \subseteq A \wedge Y \subseteq B$
- $(XY) \cap A \cap B = X \cap Y$
- $\delta(X/X \cap Y) > \delta(X/Y)$.

Let A_0 and B_0 be subsets of M satisfying the above formulas. Then A_0 and B_0 witness that A and B are not free over $A \cap B$.

(2 $\Rightarrow$ 3) Trivial.

(3 $\Rightarrow$ 1) Suppose 1 is not the case. Then there exist finite $A' \subset A$ and finite $B' \subset B$ such that

$$(A'B') \cap A \cap B = A' \cap B' \text{ and } \delta(A'/A' \cap B') < \delta(A'/B').$$

On the other hands,

$$\begin{aligned} M \models\ & \forall X, Y \in F[(\exists X' \subset_{\mathrm{fin}} X, \exists Y' \subset_{\mathrm{fin}} Y \\ & (X'Y') \cap X \cap Y = X' \cap Y' \wedge \delta(X'/X' \cap Y') > \delta(X'/Y')) \\ & \rightarrow \delta(X/X \cap Y) > \delta(X/Y)]. \end{aligned}$$

So, for arbitrary $A^* \supset_{\mathrm{hf}} A$ and $B^* \supset_{\mathrm{hf}} B$ with $(A^*B^*) \cap A \cap B = A^* \cap B^*$, we have
$$\delta(A^*/A^* \cap B^*) > \delta(A^*/B^*).$$

Proposition 3.2 *Let A and B be closed subsets of M. Then the following are equivalent:*

1. *AB is closed and A and B are free over $A \cap B$;*
2. *A and B are d^*-independent over $A \cap B$;*
3. *A and B are d-independent over $A \cap B$.*

Proof: (2 $\Leftrightarrow$ 3) By Proposition 2.13.

(1 $\Rightarrow$ 2) Take $D^* \supset_{\mathrm{hf}} AB$ such that D^* is quasi-closed and $\delta(D^*) = \delta(D^* \cap A) + \delta(D^* \cap B) - \delta(D^* \cap A \cap B)$. Then we have $d(D^*) = d(D^* \cap A) + d(D^* \cap B) - d(D^* \cap A \cap B)$.

(2 $\Rightarrow$ 1) Let A^*, B^* be a witness of d^*-independence. By Proposition 2.13, we may assume that $(A^*B^*) \cap A \cap B = A^* \cap B^*$. Then,
$$\begin{aligned}
\delta(A^*B^*) &\le \delta(A^*) + \delta(B^*) - \delta(A^* \cap B^*) \\
&\approx d(A^*) + d(B^*) - d(A^* \cap B^*) \\
&\approx d(A^*B^*).
\end{aligned}$$

The other direction is trivial. So, we have $\delta(A^*B^*) \approx d(A^*B^*)$ and $\delta(A^*B^*) \approx \delta(A^*) + \delta(B^*) - \delta(A^* \cap B^*)$. By Lemma 2.10 and Lemma 3.1, AB is closed and A and B are free over $A \cap B$.

Put $T = \mathrm{Th}(M)$. We assume that T is stable. Let $\mathcal{M}$ be a big model of T. The following fact is easy.

Fact 3.3 *[11] Let A and B be subsets of $\mathcal{M}$. Suppose A and B are free over $A \cap B$. Then for any $A' \le A$ and $B' \le B$ with $A' \cap B' = A \cap B$, $A'B'$ is closed in AB.*

Note that for any finite subset A of $\mathcal{M}$ and any $n < \omega$, the relation $|A| = n$ is definable. So we may assume that the domain of the function $|*|$ is the set of all hyperfinite sets and the range of it is $\mathbb{R}^*$.

Proposition 3.4 *Let A and B be closed subsets of $\mathcal{M}$. Suppose that $A \cap B$ is algebraically closed. Suppose also that A and B are forking independent over $A \cap B$. Then*

1. *A and B are free over $A \cap B$ and*

2. *AB is closed.*

Proof: By the above fact, we can assume that A and B are both algebraically closed.

Claim A *There are sequences $(A_i)_{i<\omega}$ and $(B_i)_{i<\omega}$ satisfying the following properties:*

- $\mathrm{tp}(A_iB_j/A \cap B) = \mathrm{tp}(AB/A \cap B)$ *for any $i, j \in \omega$;*
- $\{A_i : i < \omega\} \cup \{B_i : i < \omega\}$ *is an independent set.*

Proof: Let $(B_i)_{i<\omega}$ be a Morley sequence over $A \cap B$ with $B_0 = B$. By $A \mathop{\smile\!\!\!\!\!\raise0.6ex\hbox{$\scriptstyle |$}}_{A \cap B} B$, we can assume $\mathrm{tp}(AB_i/A \cap B) = \mathrm{tp}(AB/A \cap B)$ for any $i < \omega$. Take A_{n+1} be a realization of a nonforking extension of $\mathrm{tp}(A/\bigcup_{i<\omega} B_i)$ to $\bigcup_{j<n+1} A_j \cup \bigcup_{i<\omega} B_i$ for each n.

(1)Freeness: Suppose not. For simplicity, we assume that there are a ternary relation $R \in L$ and elements $a \in A - (A \cap B)$, $b \in B - (A \cap B)$, and $c \in A \cap B$ such that $R(a, b, c)$ holds. By the above claim, there are $a_i \in A_i$ and $b_i \in B_i$ such that for any $i, j < \omega$, $\mathrm{tp}(a_ib_jc) = \mathrm{tp}(abc)$. In particular, $R(a_i, b_j, c)$ holds for any $i, j < \omega$. Then there are hyperfinite sets A^* with $|A^*| \geq n$ (for each $n < \omega$) and B^* with $|B^*| = |A^*|$ such that $\mathcal{M}^* \models \forall x \in A^* \forall y \in B^* R(x, y, c)$. Then we have

$$\begin{aligned} \delta^*(A^*B^*) &\leq |A^*| + |B^*| - \alpha(|A^*| \times |B^*|) \\ &= |A^*| + |A^*| - \alpha(|A^*| \times |A^*|) \\ &< 0. \end{aligned}$$

A contradiction.

(2)Closedness: Suppose not. For simplicity, we assume that there are elements $d \in \mathrm{acl}(AB) - AB$, $a \in A$, and $b \in B$ such that $\gamma := \delta(d/ab) < 0$. By the claim, for all $i, j \in \omega$, we can find $a_i \in A_i$, $b_i \in B_i$ and d_{ij} such that $\mathrm{tp}(abdAB) = \mathrm{tp}(a_ib_id_{ij}A_iB_j)$.

Claim B $(\bigcup_{(i,j)\in\omega^2} d_{ij}) \cap (\bigcup_{i\in\omega} A_iB_i) = \emptyset$

Proof: Take any $e \in d_{i,j}$. By choice of $d_{i,j}$, we have $e \notin A_iB_j$. By symmetry, it is enough to show that $e \notin A_m$ for any $m \neq i$. Note that $e \in \mathrm{acl}(A_iB_j)$ and $\mathrm{acl}(A_iB_j) \cap A_m = A \cap B$ by $A_iB_j \mathop{\smile\!\!\!\!\!\raise0.6ex\hbox{$\scriptstyle |$}} A_m$. But $e \notin A \cap B$, so $e \notin A_m$.

Claim C *d_{ij}'s are distinct.*

Proof: By way of a contradiction, we assume $d_{i,j} = d_{i',j'}$ for some $(i,j) \neq (i',j')$. Then $d_{i,j} \in \mathrm{acl}(A_iB_j) \cap \mathrm{acl}(A_{i'}B_{j'})$. Note that $\mathrm{acl}(A_iB_j) \cap \mathrm{acl}(A_iB_{j'}) = A_i$ and if $i \neq i'$ and $j \neq j'$, $\mathrm{acl}(A_iB_j) \cap \mathrm{acl}(A_{i'}B_{j'}) = A \cap B$. But $d_{i,j} \notin A_i$. A contradiction.

Note that $\delta(\bigcup_{i<n} a_ib_i) = n\delta(a_0b_0)$ and $\delta(\bigcup_{(i,j)\in n^2} d_{(i,j)}/\bigcup_{i<n} a_ib_i) \leq n^2\gamma$ for each $n < \omega$. So, there are hyperfinite sets A^* with $|A^*| \geq n$ (for each $n < \omega$), B^* with $|B^*| = |A^*|$, and D^* with $|D^*| = |A^*|^2$ such that $\delta^*(A^*B^*) = |A^*|\delta(a_0b_0)$ and $\delta^*(D^*/A^*B^*) \leq |A^*|^2\gamma$.

Then we have

$$\begin{aligned} \delta(D^*A^*B^*) &\leq \delta(D^*/A^*B^*) + \delta(A^*B^*) \\ &\leq |A^*|^2\gamma + |A^*|\delta(a_0b_0) \\ &< 0. \end{aligned}$$

A contradiction.

Definition 3.5 We say that T has amalgamation over closed sets if for any $N_0, N_1 \models T$, $A \in \overline{\mathbb{K}}$, and closed embeddings $f_i : A \to N_i$ $(i = 0, 1)$, there are $N \models T$ and elementary embeddings $g_i : N_i \to N$ $(i = 0, 1)$ such that $g_0 \circ f_0 = g_1 \circ f_1$ on A, equivalently for any $N \models T$ and $A_0, A_1 \subseteq N$, if $A_0 \cong A_1$ and A_0 and A_1 are both closed, then $\mathrm{tp}(A_0) = \mathrm{tp}(A_1)$.

Fact 3.6 *[4, 10] If T has amalgamation over closed sets, then T is stable.*

Corollary 3.7 *If T has amalgamation over closed sets, then for closed sets A and B with $A \cap B$ is algebraically closed, the following are equivalent:*

1. *A and B are forking independent oner $A \cap B$;*

2. *A and B are d-independent over $A \cap B$;*

3. *A and B are free over $A \cap B$ and AB is closed.*

Proof: We have already proved $(1 \Rightarrow 3)$ and $(2 \Leftrightarrow 3)$.

$(3 \Rightarrow 1)$ Suppose 3. Take A' such that A' is independent from B over $A \cap B$ and satisfies $\mathrm{tp}(A/A \cap B)$. By $(1 \Rightarrow 3)$, we have that A' and B are free over $A \cap B$ and $A'B$ is closed. By amalgamation over closed sets, we have $\mathrm{tp}(AB/A \cap B) = \mathrm{tp}(A'B/A \cap B)$. Hence, A is also independent from B over $A \cap B$.

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