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On the geometric construction of symmetric crystals via quivers

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Expansion of Combintorial Representation Theory
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1. Introduction

1.1.

Recently in [5] and [6] with M. Kashiwara, the author presented an analogue of the LLTA conjecture for the affine Hecke algebra of *type* B. In [6], we considered $U_v(\mathfrak{g})$ and its Dynkin diagram involution θ and constructed an analogue $B_\theta(\mathfrak{g})$ of the reduced v-analogue $B_v(\mathfrak{g})$ (for the definition, see Definition 2.9 below). We gave a $B_\theta(\mathfrak{g})$-module $V_\theta(\lambda)$ for a dominant integral weight λ such that $\theta(\lambda)=\lambda$, which is an analogue of the $B_v(\mathfrak{g})$-module $U_v^-(\mathfrak{g})$ (for the definition, see Definition 2.10 below). We defined the notion of symmetric crystals and conjectured the existence of the global basis. In the case $\mathfrak{g}=\mathfrak{gl}_\infty$, $I=\mathbb{Z}_{\mathrm{odd}}$, $\theta(i)=-i$ and $\lambda=0$, we constructed the PBW type basis and the lower (and upper) global basis parametrized by the θ-restricted multi-segments. We conjectured that irreducible modules of the affine Hecke algebras of type B are described by the global basis associated to the symmetric crystals.

In the paper [4], we construct the lower global basis for the symmetric crystals by using a geometry of quivers (with a Dynkin diagram involution). Hence for any symmetric quantized Kac-Moody algebra $U_v(\mathfrak{g})$, we establish the existence of a crystal basis and a global basis for $V_\theta(0)$. This is analogous to Lusztig's geometric construction of $U_v^-(\mathfrak{g})$ and its lower global basis.

1.2.

Lusztig's theory is summarized as follows.

Let $\mathfrak{g}$ be a symmetric Kac-Moody algebra and I an index set of simple roots of $\mathfrak{g}$. For a fixed set of arrows Ω, we consider (I,Ω) as a (finite) oriented graph. We call (I,Ω) a quiver. For an I-graded vector space $\mathbf{V}$, we define the moduli space of representations of quiver (I,Ω) by

$$\mathbf{E}_{\mathbf{V},\Omega}=\bigoplus_{i\xrightarrow{\Omega}j}\mathrm{Hom}(\mathbf{V}_i,\mathbf{V}_j).$$

The algebraic group $G_\mathbf{V}=\prod_{i\in I}GL(\mathbf{V}_i)$ acts on $\mathbf{E}_{\mathbf{V},\Omega}$. Lusztig introduced a certain full subcategory $\mathscr{Q}_{\mathbf{V},\Omega}$ of $\mathscr{D}(\mathbf{E}_{\mathbf{V},\Omega})$ where $\mathscr{D}(\mathbf{E}_{\mathbf{V},\Omega})$ is the bounded derived category of constructible complexes of sheaves on $\mathbf{E}_{\mathbf{V},\Omega}$ (for the definition, see section 3). Let $K(\mathscr{Q}_{\mathbf{V},\Omega})$ be the Grothendieck group of $\mathscr{Q}_{\mathbf{V},\Omega}$. He constructed the induction operators f_i and the restriction operators e_i' on the Grothendieck group $K_\Omega:=\oplus_\mathbf{V}K(\mathscr{Q}_{\mathbf{V},\Omega})$, where $\mathbf{V}$ runs over the isomorphism classes of I-graded vector spaces. He proved the following theorem.

Theorem 1.1 (Lusztig).

(i) The operators e_i' and f_i define the action of the reduced v-analogue $B_v(\mathfrak{g})$ of $\mathfrak{g}$ on $K_\Omega\otimes_{\mathbb{Z}[v,v^{-1}]}\mathbb{Q}(v)$, and $K_\Omega\otimes_{\mathbb{Z}[v,v^{-1}]}\mathbb{Q}(v)$ is isomorphic to $U_v^-(\mathfrak{g})$ as a $B_v(\mathfrak{g})$-module. The involution induced by the Verdier duality functor coincides with the bar involution on $U_v^-(\mathfrak{g})$.

(ii) The simple perverse sheaves in $\bigoplus_\mathbf{V}\mathscr{Q}_{\mathbf{V},\Omega}$ give the lower global basis of $U_v^-(\mathfrak{g})$.

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1.3.

This paper is a summarized version of [4].

We introduce the notion of θ-quivers. This is a quiver (I,Ω) with an involution $\theta : I \to I$ (and $\theta : \Omega \to \Omega$) satisfing some conditions (see Definition 4.1). This notion is partially motivated by Syu Kato's construction [11] of the irreducible representations of the affine Hecke algebras of type B.

We also introduce the θ-symmetric I-graded vector spaces. This is an I-graded vector space $\mathbf{V} = (\mathbf{V}_i)_{i\in I}$ endowed with a non-degenerate symmetric bilinear form such that $\mathbf{V}_i$ and $\mathbf{V}_j$ are orthogonal if $j \neq \theta(i)$. For a θ-quiver (I,Ω) and a θ-symmetric I-graded vector space $\mathbf{V}$, we define the moduli space ${}^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$ of representations of (I,Ω) adding a skew-symmetric condition on $\mathbf{E}_{\mathbf{V},\Omega}$ with respect to the involution θ.

Similarly to Lusztig's arguments, we consider a certain full subcategory ${}^{\theta}\mathscr{Q}_{\mathbf{V},\Omega}$ of $\mathscr{D}({}^{\theta}\mathbf{E}_{\mathbf{V},\Omega})$ and its Grothendieck group ${}^{\theta}K_{\mathbf{V},\Omega}$. We define the induction operators F_i and the restriction operators E_i on ${}^{\theta}K_{\Omega} := \oplus_{\mathbf{V}}{}^{\theta}K_{\mathbf{V},\Omega}$ where $\mathbf{V}$ runs over the isomorphism classes of the θ-symmetric I-graded vector spaces. We prove the following main theorem which is an analogous result of Lusztig's geometric construction.

Theorem 1.2 (Theorem 5.12). *${}^{\theta}K_{\Omega} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(v) \cong V_{\theta}(0)$ as $B_{\theta}(\mathfrak{g})$-modules. The simple perverse sheaves in ${}^{\theta}K_{\Omega}$ give a lower global basis of $V_{\theta}(0)$.*

Though Lusztig proved Theorem 1.1 using some inner product on K_{Ω}, we prove Theorem 1.2 using a criterion of crystals (Theorem 2.14) and certain estimates for the actions of E_i and F_i on simple perverse sheaves (Theorem 5.3).

Theorem 5.3 and Lemma 5.5 are the most essential points of our proof of Theorem 1.2. But we omit the proof of them. But in the latter of section 5, we can know how to use them for our proof.

Remark 1.3. We give two remarks on a difference from the "folding" procedure and an overlap with perverse sheaves arising from graded Lie algebras by Lusztig.

(i) Our construction is different from Lusztig's construction, "Quiver with automorphisms", in his book [15, Chapter.12-14].

He considered actions $a : I \to I$ and $a : H \to H$ induced from a finite cyclic group C generated by a. Put an orientation Ω such that $\mathrm{out}(a(h)) = a(\mathrm{out}(h))$ and $\mathrm{in}(a(h)) = a(\mathrm{in}(h))$. He said this orientation "compatible". Let $\mathcal{V}^a$ be the category of I-graded vector spaces $\mathbf{V}$ such that $\dim \mathbf{V}_i = \dim \mathbf{V}_{a(i)}$ for any $i \in I$. For $\mathbf{V} \in \mathcal{V}^a$, a induces a natural automorphism on $\mathbf{E}_{\mathbf{V},\Omega}$ and a functor $a^* : \mathscr{D}(\mathbf{E}_{\mathbf{V},\Omega}) \to \mathscr{D}(\mathbf{E}_{\mathbf{V},\Omega})$. He introduced "$C$-equivariant" simple perverse sheaves (B,ϕ), where B is a perverse sheaf and $\phi : a^*B \cong B$. Then he proved that the set $\sqcup_{\mathbf{V}\in\mathcal{V}^a}\mathbf{B}_{\mathbf{V},\Omega}$ of C-equivariant perverse sheaves gives a lower global basis of $\mathbf{U}_v^-(\mathfrak{g})$. Here $\mathfrak{g}$ has a non-symmetric Cartan matrix which is obtained by the "folding" procedure with respect to the C-action on I.

But in our construction, a θ-orientation is not a compatible orientation. Moreover the most essential difference is that his construction has no skew-symmetric condition in our sence. Hence the set of simple perverse sheaves ${}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$ and the space ${}^{\theta}K_{\Omega} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(v) \cong V_{\theta}(0)$ are different from $\mathbf{B}_{\mathbf{V},\Omega}$ and $\mathbf{U}_v^-(\mathfrak{g})$, respectively. The detailed crystal structure of $V_{\theta}(0)$ is unknown except for the case $\mathfrak{g} = \mathfrak{gl}_{\infty}$, $I = \mathbb{Z}_{\mathrm{odd}}$ and $\theta(i) = -i$ in [6].

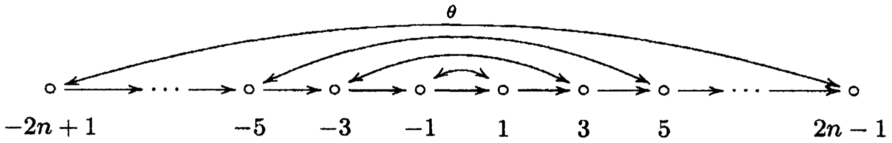
(ii) In some special case, the lower global basis which constructs in this paper is obtained by Lusztig ([16] and [17]). Let us consider the case $G = SO(2n,\mathbb{C})$. Let $\mathfrak{g}$ be the Lie algebra of G and T a fixed maximal torus of G. Set ε_{2i-1} $(1 \le i \le n)$ the fundamental characters of T. Asuume $q \in \mathbb{C}^*$ is not a root of unity. We choose a semisimple element $s \in T$ such that $\varepsilon_{2i-1}(s) \in q^{\mathbb{Z}_{\mathrm{odd},\ge 0}}$ for any i and put $d_{2i-1} = \{j | \varepsilon_{2j-1}(s) = q^{2i-1}\}$. Then the centralizer $G(s)$ of s acts on

$$\mathfrak{g}_2 := \{X \in \mathfrak{g} \mid sXs^{-1} = q^2 X\}$$

which has finitely many $G(s)$-oribits. Lusztig considered the category $\mathscr{Q}(\mathfrak{g}_2)$ of semisimple $G(s)$-equivariant complex on $\mathfrak{g}_2$ and constructed the canonical basis $\mathbf{B}(\mathfrak{g}_2)$ of $K(\mathfrak{g}_2)$ which is the Grothendieck group of $\mathscr{Q}(\mathfrak{g}_2)$.

On the other hand, let us consider the θ-symmetric vector space $\mathbf{V}$ such that $\mathrm{wt}(\mathbf{V}) = \sum_{i=1}^{n} d_{2i-1}(\alpha_{2i-1}+$

α_{-2i+1}) and the following θ-quiver of type A_{2n} and the θ-orientation Ω:



In this case, we have $G(s) = \prod_{i=1}^{n} GL(d_{2i-1}) = {}^{\theta}\mathbf{G}_{\mathbf{V}}$ and $\mathfrak{g}_2 = {}^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$. Thus the set ${}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$ of simple perverse sheaves conincide with $\mathbf{B}(\mathfrak{g}_2)$.

Remark 1.4. After writing the paper [4], the author found the notion of θ-quivers has been already introduced by Derksen-Weyman in [3].

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2. Preliminaries

2.1. Quantum enveloping algebras

2.1.1 Quantum enveloping algebras and reduced v-analogue

We shall recall the quantized universal enveloping algebra $U_v(\mathfrak{g})$. In this paper, we treat only the symmetric Cartan matrix case. Let I be an index set (for simple roots), and Q the free $\mathbb{Z}$-module with a basis $\{\alpha_i\}_{i\in I}$. Let $(\bullet,\bullet) : Q \times Q \to \mathbb{Z}$ be a symmetric bilinear form such that $(\alpha_i, \alpha_i) = 2$ and $(\alpha_i, \alpha_j) \in \mathbb{Z}_{\leq 0}$ for $i \neq j$. Let v be an indeterminate and set $\mathbf{K} := \mathbb{Q}(v)$. We define its subrings $\mathbf{A}_0$, $\mathbf{A}_\infty$ and $\mathbf{A}$ as follows.

$$\begin{aligned}
\mathbf{A}_0 &= \{f \in \mathbf{K} \mid f \text{ is regular at } v = 0\}, \\
\mathbf{A}_\infty &= \{f \in \mathbf{K} \mid f \text{ is regular at } v = \infty\}, \\
\mathbf{A} &= \mathbb{Q}[v, v^{-1}].
\end{aligned}$$

Definition 2.1. *The quantized universal enveloping algebra $U_v(\mathfrak{g})$ is the $\mathbf{K}$-algebra generated by elements e_i, f_i and invertible elements t_i $(i \in I)$ with the following defining relations.*

(1) *The t_i's commute with each other.*

(2) $t_j e_i t_j^{-1} = v^{(\alpha_j,\alpha_i)} e_i$ *and* $t_j f_i t_j^{-1} = v^{-(\alpha_j,\alpha_i)} f_i$ *for any $i, j \in I$.*

(3) $[e_i, f_j] = \delta_{ij}\dfrac{t_i - t_i^{-1}}{v - v^{-1}}$ *for i, $j \in I$.*

(4) (v-Serre relation) *For $i \neq j$,*

$$\sum_{k=0}^{b}(-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0, \ \sum_{k=0}^{b}(-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.$$

Here $b = 1 - (\alpha_i, \alpha_j)$ and

$$e_i^{(k)} = e_i^k/[k]_v!\,,\ f_i^{(k)} = f_i^k/[k]_v!\,,\ [k]_v = (v^k - v^{-k})/(v - v^{-1})\,,\ [k]_v! = [1]_v \cdots [k]_v\,.$$

Let us denote by $U_v^-(\mathfrak{g})$ the subalgebra of $U_v(\mathfrak{g})$ generated by the f_i's.

Let e_i' and e_i^* be the operators on $U_v^-(\mathfrak{g})$ defined by

$$[e_i, a] = \frac{(e_i^* a)t_i - t_i^{-1} e_i' a}{v - v^{-1}} \quad (a \in U_v^-(\mathfrak{g})).$$

These operators satisfy the following formulas similar to derivations:

$$e'_i(ab) = (e'_i a)b + (\mathrm{Ad}(t_i)a)e'_i b.$$

The algebra $U_v^-(\mathfrak{g})$ has a unique symmetric bilinear form $(\bullet, \bullet)$ such that $(1,1) = 1$ and

$$(e'_i a, b) = (a, f_i b) \quad \text{for any } a, b \in U_v^-(\mathfrak{g}).$$

It is non-degenerate. The left multiplication operator f_j and e'_i satisfy the commutation relations

$$e'_i f_j = v^{-(\alpha_i, \alpha_j)} f_j e'_i + \delta_{ij}, \tag{1}$$

and the e'_i's satisfy the v-Serre relations (Definition 2.1(4)).

Definition 2.2. *The reduced v-analogue $B_v(\mathfrak{g})$ of $\mathfrak{g}$ is the $\mathbb{Q}(v)$-algebra generated by e'_i and f_i which satisfy (1) and the v-Serre relations for e'_i and f_i $(i, j \in I)$ as the defining relations.*

2.1.2 Review on crystal bases and global bases of U_v^-

Since e'_i and f_i satisfy the v-boson relation, any element $a \in U_v^-(\mathfrak{g})$ can be uniquely written as

$$a = \sum_{n \geq 0} f_i^{(n)} a_n \quad \text{with } e'_i a_n = 0.$$

Here $f_i^{(n)} = \dfrac{f_i^n}{[n]_v!}$.

Definition 2.3. *We define the modified root operators $\tilde{e}_i$ and $\tilde{f}_i$ on $U_v^-(\mathfrak{g})$ by*

$$\tilde{e}_i a = \sum_{n \geq 1} f_i^{(n-1)} a_n, \quad \tilde{f}_i a = \sum_{n \geq 0} f_i^{(n+1)} a_n.$$

Theorem 2.4 ([8]). *We define*

$$\begin{aligned} L(\infty) &= \sum_{\ell \geq 0,\, i_1, \ldots, i_\ell \in I} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \subset U_v^-(\mathfrak{g}), \\ B(\infty) &= \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \mod vL(\infty) \mid \ell \geq 0, i_1, \cdots, i_\ell \in I \right\} \subset L(\infty)/vL(\infty). \end{aligned}$$

Then we have

(1) $\tilde{e}_i L(\infty) \subset L(\infty)$ and $\tilde{f}_i L(\infty) \subset L(\infty)$,

(2) $B(\infty)$ is a basis of $L(\infty)/vL(\infty)$,

(3) $\tilde{f}_i B(\infty) \subset B(\infty)$ and $\tilde{e}_i B(\infty) \subset B(\infty) \cup \{0\}$.

We call $(L(\infty), B(\infty))$ the crystal basis of $U_v^-(\mathfrak{g})$.

Definition 2.5. *We define $\varepsilon_i(b) := \max\{m \in \mathbb{Z}_{\geq 0} | \tilde{e}_i^m b \neq 0\}$ for $i \in I$ and $b \in B(\infty)$.*

Let $-$ be the automorphism of $\mathbf{K}$ sending v to v^{-1}. Then $\overline{\mathbf{A}_0}$ coincides with $\mathbf{A}_\infty$.

Let V be a vector space over $\mathbf{K}$, L_0 an $\mathbf{A}$-submodule of V, L_∞ an $\mathbf{A}_\infty$- submodule, and $V_{\mathbf{A}}$ an $\mathbf{A}$-submodule. Set $E := L_0 \cap L_\infty \cap V_{\mathbf{A}}$.

Definition 2.6 ([8]). *We say that $(L_0, L_\infty, V_{\mathbf{A}})$ is* balanced *if each of L_0, L_∞ and $V_{\mathbf{A}}$ generates V as a $\mathbf{K}$-vector space, and if one of the following equivalent conditions is satisfied.*

(1) $E \to L_0/vL_0$ is an isomorphism,

(2) $E \to L_\infty/v^{-1}L_\infty$ is an isomorphism,

(3) $(L_0 \cap V_{\mathbf{A}}) \oplus (v^{-1}L_\infty \cap V_{\mathbf{A}}) \to V_{\mathbf{A}}$ is an isomorphism.

(4) $\mathbf{A}_0 \otimes_{\mathbf{Q}} E \to L_0$, $\mathbf{A}_\infty \otimes_{\mathbf{Q}} E \to L_\infty$, $\mathbf{A} \otimes_{\mathbf{Q}} E \to V_{\mathbf{A}}$ and $\mathbf{K} \otimes_{\mathbf{Q}} E \to V$ are isomorphisms.

Let $-$ be the ring automorphism of $U_v(\mathfrak{g})$ sending v, t_i, e_i, f_i to v^{-1}, t_i^{-1}, e_i, f_i.

Let $U_v(\mathfrak{g})_{\mathbf{A}}$ be the $\mathbf{A}$-subalgebra of $U_v(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$ and t_i. Similarly we define $U_v^-(\mathfrak{g})_{\mathbf{A}}$.

Theorem 2.7. *$(L(\infty), L(\infty)^-, U_v^-(\mathfrak{g})_{\mathbf{A}})$ is balanced.*

Let

$$G^{\text{low}}\colon L(\infty)/vL(\infty) \xrightarrow{\sim} E := L(\infty) \cap L(\infty)^- \cap U_v^-(\mathfrak{g})_{\mathbf{A}}$$

be the inverse of $E \xrightarrow{\sim} L(\infty)/vL(\infty)$. Then $\{G^{\text{low}}(b) \mid b \in B(\infty)\}$ forms a basis of $U_v^-(\mathfrak{g})$. We call it a (lower) *global basis*. It is first introduced by G. Lusztig ([13]) under the name of "canonical basis" for the A, D, E cases.

Definition 2.8. *Let $\{G^{\text{up}}(b) \mid b \in B(\infty)\}$ be the dual basis of $\{G^{\text{low}}(b) \mid b \in B(\infty)\}$ with respect to the inner product $(\bullet, \bullet)$. We call it the upper global basis of $U_v^-(\mathfrak{g})$.*

2.2. Symmetric Crystals

Let θ be an automorphism of I such that $\theta^2 = \text{id}$ and $(\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j)$. Hence it extends to an automorphism of the root lattice Q by $\theta(\alpha_i) = \alpha_{\theta(i)}$, and induces an automorphism of $U_v(\mathfrak{g})$.

Definition 2.9. *Let $B_\theta(\mathfrak{g})$ be the $\mathbf{K}$-algebra generated by E_i, F_i, and invertible elements $T_i (i \in I)$ satisfying the following defining relations:*

(i) *the T_i's commute with each other,*

(ii) *$T_{\theta(i)} = T_i$ for any i,*

(iii) *$T_i E_j T_i^{-1} = v^{(\alpha_i + \alpha_{\theta(i)}, \alpha_j)} E_j$ and $T_i F_j T_i^{-1} = v^{(\alpha_i + \alpha_{\theta(i)}, -\alpha_j)} F_j$ for $i, j \in I$,*

(iv) *$E_i F_j = v^{-(\alpha_i, \alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j} T_i)$ for $i, j \in I$,*

(v) *the E_i's and the F_i's satisfy the v-Serre relations.*

We set $F_i^{(n)} = F_i^n / [n]_v!$.

Proposition 2.10 ([6, Proposition2.11.]). *Let*

$$\lambda \in P_+ := \{\lambda \in \operatorname{Hom}(Q, \mathbb{Q}) \mid \lambda(\alpha_i) \in \mathbb{Z}_{\geq 0} \text{ for any } i \in I\}$$

be a dominant integral weight such that $\theta(\lambda) = \lambda$.

(i) *There exists a $B_\theta(\mathfrak{g})$-module $V_\theta(\lambda)$ generated by a non-zero vector ϕ_λ such that*

  (a) *$E_i \phi_\lambda = 0$ for any $i \in I$,*

  (b) *$T_i \phi_\lambda = v^{(\alpha_i, \lambda)} \phi_\lambda$ for any $i \in I$,*

  (c) *$\{u \in V_\theta(\lambda) \mid E_i u = 0 \text{ for any } i \in I\} = \mathbf{K}\phi_\lambda$.*

  Moreover such a $V_\theta(\lambda)$ is irreducible and unique up to an isomorphism.

(ii) *There exists a unique non-degenerate symmetric bilinear form $(\bullet, \bullet)$ on $V_\theta(\lambda)$ such that $(\phi_\lambda, \phi_\lambda) = 1$ and $(E_i u, v) = (u, F_i v)$ for any $i \in I$ and $u, v \in V_\theta(\lambda)$.*

(iii) *There exists an endomorphism $-$ of $V_\theta(\lambda)$ such that $\overline{\phi_\lambda} = \phi_\lambda$ and $\overline{av} = \bar{a}\bar{v}$, $\overline{F_i v} = F_i \bar{v}$ for any $a \in \mathbf{K}$ and $v \in V_\theta(\lambda)$.*

Hereafter we assume further that

$$\textit{there is no } i \in I \textit{ such that } \theta(i) = i.$$

In [6], we conjectured that $V_\theta(\lambda)$ has a crystal basis. This means the following. Since E_i and F_i satisfy the v-boson relation $E_i F_i = v^{-(\alpha_i, \alpha_i)} F_i E_i + 1$, we define the modified root operators:

$$\tilde{E}_i(u) = \sum_{n \geq 1} F_i^{(n-1)} u_n \text{ and } \tilde{F}_i(u) = \sum_{n \geq 0} F_i^{(n+1)} u_n,$$

when writing $u = \sum_{n\geq 0} F_i^{(n)} u_n$ with $E_i u_n = 0$. Let $L_\theta(\lambda)$ be the $\mathbf{A}_0$-submodule of $V_\theta(\lambda)$ generated by $\widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell} \phi_\lambda$ ($\ell \geq 0$ and $i_1, \ldots, i_\ell \in I$), and let $B_\theta(\lambda)$ be the subset

$$\left\{ \widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell} \phi_\lambda \mod vL_\theta(\lambda) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I \right\}$$

of $L_\theta(\lambda)/vL_\theta(\lambda)$.

Conjecture 2.11. *Let λ be a dominant integral weight such that $\theta(\lambda) = \lambda$.*

(1) $\tilde{F}_i L_\theta(\lambda) \subset L_\theta(\lambda)$ and $\tilde{E}_i L_\theta(\lambda) \subset L_\theta(\lambda)$,

(2) $B_\theta(\lambda)$ is a basis of $L_\theta(\lambda)/vL_\theta(\lambda)$,

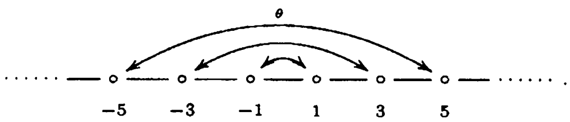
(3) $\tilde{F}_i B_\theta(\lambda) \subset B_\theta(\lambda)$, and $\tilde{E}_i B_\theta(\lambda) \subset B_\theta(\lambda) \sqcup \{0\}$,

(4) $\tilde{F}_i \tilde{E}_i(b) = b$ for any $b \in B_\theta(\lambda)$ such that $\tilde{E}_i b \neq 0$, and $\tilde{E}_i \tilde{F}_i(b) = b$ for any $b \in B_\theta(\lambda)$.

Moreover we conjectured that $V_\theta(\lambda)$ has a global crystal basis. Namely we have

Conjecture 2.12. *$(L_\theta(\lambda), \overline{L_\theta(\lambda)}, V_\theta(\lambda)^{low}_{\mathbf{A}})$ is balanced. Here $V_\theta(\lambda)^{low}_{\mathbf{A}} := U_v^-(\mathfrak{g})_{\mathbf{A}} \phi_\lambda$.*

Example 2.13. Suppose $\mathfrak{g} = \mathfrak{gl}_\infty$, the Dynkin diagram involution θ of I defined by $\theta(i) = -i$ for $i \in I = \mathbb{Z}_{\mathrm{odd}}$.



And assume $\lambda = 0$. In this case, we can prove

$$V_\theta(0) \cong U_v^- / \sum_{i \in I} U_v^- (f_i - f_{\theta(i)}).$$

Moreover we can construct a PBW type basis, a crystal basis and an upper and lower global basis on $V_\theta(0)$ parametrized by "the θ-restricted multisegments". For more details, see [6].

2.3. Criterion for crystals

Let $\mathbf{K}[e, f]$ be the ring generated by e and f with the defining relation $ef = v^{-2} fe + 1$. We call this algebra the v-boson algebra. Let P be a free $\mathbb{Z}$-module, and let α be a non-zero element of P. Let M be a $\mathbf{K}[e, f]$-module. Assume that M has a weight decomposition $M = \oplus_{\xi \in P} M_\xi$ and $eM_\lambda \subset M_{\lambda+\alpha}$ and $fM_\lambda \subset M_{\lambda-\alpha}$. Asuume the following finiteness conditions:

$$\text{for any } \lambda \in P, \dim M_\lambda < \infty \text{ and } M_{\lambda+n\alpha} = 0 \text{ for } n \gg 0.$$

Hence for $u \in M$, we can write $u = \sum_{n\geq 0} f^{(n)} u_n$ with $eu_n = 0$. We define endmorphisms $\widetilde{e}$ and $\widetilde{f}$ of M by

$$\widetilde{e}u = \sum_{n\geq 1} f^{(n-1)} u_n, \quad \widetilde{f}u = \sum_{n\geq 0} f^{(n+1)} u_n.$$

Let B be a crystal with weight decomposition by P in the following sense. We have $\mathrm{wt}\colon B \to P$, $\widetilde{f}\colon B \to B$, $\widetilde{e}\colon B \to B \sqcup \{0\}$ and $\varepsilon\colon B \to \mathbb{Z}_{\geq 0}$ satisfying the following properties, where $B_\lambda = \mathrm{wt}^{-1}(\lambda)$:

(i) $\widetilde{f} B_\lambda \subset B_{\lambda-\alpha}$ and $\widetilde{e} B_\lambda \subset B_{\lambda+\alpha} \sqcup \{0\}$ for any $\lambda \in P$,

(ii) $\widetilde{f}\widetilde{e}b = b$ if $\widetilde{e}b \neq 0$, and $\widetilde{e} \circ \widetilde{f} = \mathrm{id}_B$,

(iii) for any $\lambda \in P$, B_λ is a finite set and $B_{\lambda+n\alpha} = \phi$ for $n \gg 0$,

(iv) $\varepsilon(b) = \max\{n \geq 0 \mid \widetilde{e}^n b \neq 0\}$ for any $b \in B$.

Set $\operatorname{ord}(a) = \sup\{n \in \mathbb{Z} \mid a \in v^n \mathbf{A}_0\}$ for $a \in \mathbf{K}$. We understand $\operatorname{ord}(0) = \infty$.

Let $\{G(b)\}_{b\in B}$ be a system of generators of M with $G(b) \in M_{\mathrm{wt}(b)}$. Asuume that we have expressions:

$$eG(b) = \sum_{b'\in B} E_{b,b'}G(b), \quad fG(b) = \sum_{b'\in B} F_{b,b'}G(b).$$

Now consider the following conditions for these data, where $\ell = \varepsilon(b)$ and $\ell' = \varepsilon(b')$:

$$\operatorname{ord}(F_{b,b'}) \geq 1 - \ell', \tag{2}$$

$$\operatorname{ord}(E_{b,b'}) \geq -\ell', \tag{3}$$

$$F_{b,\tilde{f}b} \in v^{-\ell}(1 + v\mathbb{A}_0), \tag{4}$$

$$E_{b,\tilde{f}b} \in v^{1-\ell}(1 + v\mathbb{A}_0), \tag{5}$$

$$\operatorname{ord}(F_{b,b'}) > 1 - \ell' \text{ if } \ell < \ell' \text{ and } b' \neq \tilde{f}b, \tag{6}$$

$$\operatorname{ord}(E_{b,b'}) > -\ell' \text{ if } \ell < \ell' + 1 \text{ and } b' \neq \tilde{e}b. \tag{7}$$

Theorem 2.14 ([6, Theorem 4.1, Corollary 4.4]). *Assume the conditions (2)–(7). Let L be the $\mathbf{A}_0$-submodule $\sum_{b\in B} \mathbf{A}_0 G(b)$ of M. Then we have $\tilde{e}L \subset L$ and $\tilde{f}L \subset L$. Moreover we have*

$$\tilde{e}G(b) \equiv G(\tilde{e}b) \mod vL, \quad \tilde{f}G(b) \equiv G(\tilde{f}b) \mod vL$$

for any $b \in B$. Here we understand $G(0) = 0$.

In [6], this theorem is proved under weaker assumptions.

2.4. Perverse Sheaves

2.4.1 Perverse Sheaves

In this paper, we consider algebraic varieties over $\mathbb{C}$. Let $\mathscr{D}(X)$ be the bounded derived category of constructible complexes of sheaves on an algebraic variety X. For a morphism $f\colon X \to Y$ of algebraic varieties X and Y, let f^* be the inverse image, $f_!$ the direct image with proper support and $D\colon \mathscr{D}(X) \to \mathscr{D}(X)$ the Verdier duality functor. Let $({}^p\mathscr{D}^{\leq 0}(X), {}^p\mathscr{D}^{\geq 0}(X))$ be the perverse t-structure and $\operatorname{Perv}(X) := {}^p\mathscr{D}^{\leq 0}(X) \cap {}^p\mathscr{D}^{\geq 0}(X)$. Let ${}^pH^k(\bullet)$ be the k-th perverse cohomology sheaf. We say that an object L in $\mathscr{D}(X)$ is semisimple if L is isomorphic to the direct sum $\oplus_k {}^pH^k(L)[-k]$ and if each ${}^pH^k(L)$ is a semisimple perverse sheaf. Assume that we are given an action of a connected algebraic group G on X. A semisimple object L in $\mathscr{D}(X)$ is said to be G-equivariant if each ${}^pH^i(L)$ is a G-equivariant perverse sheaf. We denote by $\mathbf{1}_X$ the constant sheaf on X.

2.4.2 Fourier-Sato-Deligne transforms

Let $E \to S$ be a vector bundle and $E^* \to S$ the dual vector bundle. Hence $\mathbb{C}^\times$ acts on E and E^*. We say that $L \in \mathscr{D}(E)$ is monodromic if $H^j(L)$ is locally constant on every $\mathbb{C}^*$-orbit of E. Let $\mathscr{D}_{\mathrm{mono}}(E)$ be the full subcategory of $\mathscr{D}(E)$ consisting of monodromic objects. Then we can define the Fourier transform

$$\Phi_{E/S}\colon \mathscr{D}_{\mathrm{mono}}(E) \to \mathscr{D}_{\mathrm{mono}}(E^*).$$

2.5. Quivers

Let I and α_i's be as in 2.1.

Definition 2.15. *A double quiver (I, H) associated with the symmetric Cartan matrix is a following data:*

(i) *a set H,*

(ii) *two maps* $\operatorname{out}, \operatorname{in}\colon H \to I$ *such that* $\operatorname{out}(h) \neq \operatorname{in}(h)$ *for any $h \in H$,*

(iii) *an involution $h \mapsto \bar{h}$ on H satisfying* $\operatorname{out}(\bar{h}) = \operatorname{in}(h)$ *and* $\operatorname{in}(\bar{h}) = \operatorname{out} h$,

(iv) $\sharp\{h \in H \mid \mathrm{out}(h) = i, \mathrm{in}(h) = j\} = -(\alpha_i, \alpha_j)$ *for* $i \neq j$.

An orientation of a double quiver (I, H) *is a subset* Ω *of* H *such that* $\Omega \cap \overline{\Omega} = \phi$ *and* $\Omega \cup \overline{\Omega} = H$. *For an orientation* Ω, *we call* (I, Ω) *a quiver.*

For a fixed orientation Ω, *we call a vertex* $i \in I$ *a sink if* $\mathrm{out}(h) \neq i$ *for any* $h \in \Omega$.

Definition 2.16. *Let* $\mathcal{V}$ *be the category of* I*-graded vector spaces* $\mathbf{V} = (\mathbf{V}_i)_i$ *with morphisms being linear maps respecting the grading. Put* $\mathrm{wt}(\mathbf{V}) = \sum_{i\in I}(\dim \mathbf{V}_i)\alpha_i$.

Let $\mathbf{S}_i$ be an I-graded vector space such that $\mathrm{wt}(\mathbf{S}_i) = \alpha_i$.

Definition 2.17. *For* $\mathbf{V} \in \mathcal{V}$ *and a subset* Ω *of* H, *we define*

$$\mathbf{E}_{\mathbf{V},\Omega} := \bigoplus_{h\in\Omega} \mathrm{Hom}(\mathbf{V}_{\mathrm{out}(h)}, \mathbf{V}_{\mathrm{in}(h)}).$$

The algebraic group $\mathbf{G}_{\mathbf{V}} = \prod_{i\in I} GL(\mathbf{V}_i)$ *acts on* $\mathbf{E}_{\mathbf{V},\Omega}$ *by* $(g, x) \mapsto gx$ *where* $(gx)_h = g_{\mathrm{in}(h)} x_h g^{-1}_{\mathrm{out}(h)}$.
The group $(\mathbb{C}^\times)^\Omega$ *also acts on* $\mathbf{E}_{\mathbf{V},\Omega}$ *by* $x_h \mapsto c_h x_h$ $(h \in \Omega, c_h \in \mathbb{C}^\times)$.
For $x \in \mathbf{E}_{\mathbf{V},\Omega}$, *an* I*-graded subspace* $\mathbf{W} \subset \mathbf{V}$ *is* x*-stable if* $x_h(\mathbf{W}_{\mathrm{out}(h)}) \subset \mathbf{W}_{\mathrm{in}(h)}$ *for any* $h \in \Omega$.

Note that $E_{\mathbf{S}_i,\Omega} \cong \{\mathrm{pt}\}$.

3. A Review of Lusztig's geometric construction

We give a quick review on Lusztig's theory in [13] and [14] (cf. [15]). For a sequence $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$ and a sequence $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m_{\geq 0}$, a flag of type $(\mathbf{i}, \mathbf{a})$ is by definition a finite decreasing sequence $F = (\mathbf{V} = \mathbf{F}^0 \supset \mathbf{F}^1 \supset \cdots \supset \mathbf{F}^m = \{0\})$ of I-graded subspaces of $\mathbf{V}$ such that the I-graded vector space $\mathbf{F}^{\ell-1}/\mathbf{F}^\ell$ vanishes in degrees $\neq i_\ell$ and has dimension a_ℓ in degree i_ℓ. We denote by $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega}$ the set of pairs (x, F) such that $x \in \mathbf{E}_{\mathbf{V},\Omega}$ and F is an x-stable flag of type $(\mathbf{i}, \mathbf{a})$. The group $G_{\mathbf{V}}$ acts on $\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega}$. The first projection $\pi_{\mathbf{i},\mathbf{a}}\colon \widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega} \to \mathbf{E}_{\mathbf{V},\Omega}$ is a $G_{\mathbf{V}}$-equivariant projective morphism.

By the decomposition theorem [2], $L_{\mathbf{i},\mathbf{a};\Omega} := (\pi_{\mathbf{i},\mathbf{a}})_!(1_{\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega}}) \in \mathscr{D}(\mathbf{E}_{\mathbf{V},\Omega})$ is a semisimple complex. We define $\mathscr{P}_{\mathbf{V},\Omega}$ as the set of the isomorphism classes of simple perverse sheaves $L \in \mathscr{D}(\mathbf{E}_{\mathbf{V},\Omega})$ satisfying the following property: L appears as a direct summand of $L_{\mathbf{i},\mathbf{a};\Omega}[d]$ for some d and $(\mathbf{i}, \mathbf{a})$. We denote by $\mathscr{Q}_{\mathbf{V},\Omega}$ the full subcategory of $\mathscr{D}(\mathbf{E}_{\mathbf{V},\Omega})$ consisting of all objects which are isomorphic to finite direct sums of complexes of the form $L[d]$ for various $L \in \mathscr{P}_{\mathbf{V},\Omega}$ and various integers d. Any complex in $\mathscr{P}_{\mathbf{V},\Omega}$ is $G_{\mathbf{V}} \times (\mathbb{C}^\times)^\Omega$-equivariant.

Let $\mathbf{T}, \mathbf{W}, \mathbf{V}$ be I-graded vector spaces such that $\mathrm{wt}(\mathbf{V}) = \mathrm{wt}(\mathbf{W}) + \mathrm{wt}(\mathbf{T})$. We consider the following diagram

$$\mathbf{E}_{\mathbf{T},\Omega} \times \mathbf{E}_{\mathbf{W},\Omega} \xleftarrow{p_1} \mathbf{E}'_\Omega \xrightarrow{p_2} \mathbf{E}''_\Omega \xrightarrow{p_3} \mathbf{E}_{\mathbf{V},\Omega}.$$

Here $\mathbf{E}''_\Omega$ is the variety of (x, W) where $x \in \mathbf{E}_{\mathbf{V},\Omega}$ and W is an x-stable I-graded subspace of $\mathbf{V}$ such that $\mathrm{wt}\, W = \mathrm{wt}\, \mathbf{W}$. The variety $\mathbf{E}'_\Omega$ consists of $(x, W, \varphi^{\mathbf{W}}, \varphi^{\mathbf{T}})$ where $(x, W) \in \mathbf{E}''_\Omega$, $\varphi^{\mathbf{W}}\colon \mathbf{W} \cong W$, and $\varphi^{\mathbf{T}}\colon \mathbf{T} \cong \mathbf{V}/W$. The morphisms p_1, p_2 and p_3 are given by $p_1(x, W, \varphi^{\mathbf{W}}, \varphi^{\mathbf{T}}) = (x|_{\mathbf{T}}, x|_{\mathbf{W}})$, $p_2(x, W, \varphi^{\mathbf{W}}, \varphi^{\mathbf{T}}) = (x, W)$ and $p_3(x, W) = x$. Then p_1 is smooth with connected fibers, p_2 is a principal $G_{\mathbf{T}} \times G_{\mathbf{W}}$-bundle, and p_3 is projective. For a $G_{\mathbf{T}}$-equivariant semisimple complex $K_{\mathbf{T}}$ and a $G_{\mathbf{W}}$-equivariant semisimple complex $K_{\mathbf{W}}$, there exists a unique semisimple complex K'' satisfying $p_1^*(K_{\mathbf{T}} \boxtimes K_{\mathbf{W}}) = p_2^* K''$. We define $K_{\mathbf{T}} * K_{\mathbf{W}} := (p_3)_!(K'') \in \mathscr{D}(\mathbf{E}_{\mathbf{V},\Omega})$.

For an I-graded subspace $\mathbf{U}$ of $\mathbf{V}$ such that $\mathbf{V}/\mathbf{U} \cong \mathbf{T}$, we also consider the following diagram

$$\mathbf{E}_{\mathbf{T},\Omega} \times \mathbf{E}_{\mathbf{U},\Omega} \xleftarrow{p} \mathbf{E}(\mathbf{U}, \mathbf{V})_\Omega \xhookrightarrow{\iota} \mathbf{E}_{\mathbf{V},\Omega}.$$

Here $\mathbf{E}(\mathbf{U}, \mathbf{V})_\Omega$ is the variety of $x \in \mathbf{E}_{\mathbf{V},\Omega}$ such that $\mathbf{U}$ is x-stable. For $K \in \mathscr{D}(\mathbf{E}_{\mathbf{V},\Omega})$, we define $\mathrm{Res}_{\mathbf{T},\mathbf{U}}(K) := p_! \iota^*(K)$.

We define $K_{\mathbf{V},\Omega}$ as the Grothendieck group of $\mathscr{Q}_{\mathbf{V},\Omega}$. It is the additive group generated by the isomorphism classes (L) of objects $L \in \mathscr{Q}_{\mathbf{V},\Omega}$ with the relation $(L) = (L') + (L'')$ when $L \cong L' \oplus L''$. The group $K_{\mathbf{V},\Omega}$ has a $\mathbb{Z}[v, v^{-1}]$-module structure by $v(L) = (L[1])$ and $v^{-1}(L) = (L[-1])$ for $L \in \mathscr{Q}_{\mathbf{V},\Omega}$.

Hence, $K_{\mathbf{V},\Omega}$ is a free $\mathbb{Z}[v,v^{-1}]$-module with a basis $\{(L) | L \in \mathscr{P}_{\mathbf{V},\Omega}\}$. We define $K_\Omega \colon = \bigoplus_{\mathbf{V}} K_{\mathbf{V},\Omega}$ where $\mathbf{V}$ runs over the isomorphism classes of I-graded vector spaces. Recall that $\mathbf{S}_i$ is an I-graded vector space such that $\mathrm{wt}(\mathbf{S}_i) = \alpha_i$. Then we can define the induction $f_i \colon K_{\mathbf{W},\Omega} \to K_{\mathbf{V},\Omega}$ and the restriction $e'_i \colon K_{\mathbf{V},\Omega} \to K_{\mathbf{W},\Omega}$ by

$$f_i(K) \colon = v^{\dim \mathbf{W}_i + \sum_{i \xrightarrow{\Omega} j} \dim \mathbf{W}_j}(1_{\mathbf{S}_i} * K), \quad e'_i(K) \colon = v^{-\dim \mathbf{W}_i + \sum_{i \xrightarrow{\Omega} j} \dim \mathbf{W}_j} \operatorname{Res}_{\mathbf{S}_i,\mathbf{V}}(K).$$

Then Lusztig's main theorem is stated as follows.

Theorem 3.1 (Lusztig).

(i) The operators e'_i and f_i define the action of the reduced v-analogue $B_v(\mathfrak{g})$ of $\mathfrak{g}$ on $K_\Omega \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(v)$. The $B_v(\mathfrak{G})$-module $K_\Omega \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(v)$ is isomorphic to $U_v^-(\mathfrak{g})$. The involution induced by the Verdier duality functor coincides with the bar involution on $U_v^-(\mathfrak{g})$.

(ii) The simple perverse sheaves in $\sqcup_{\mathbf{V}} \mathscr{P}_{\mathbf{V},\Omega}$ give a lower global basis of $U_v^-(\mathfrak{g})$.

4. Quivers with an Involution θ

4.1. Quivers with an involution θ

Definition 4.1. *A double θ-quiver is a data:*

(1) a double quiver (I, H),

(2) involutions $\theta \colon I \to I$ and $\theta \colon H \to H$,

satisfying

(a) $\mathrm{out}(\theta(h)) = \theta(\mathrm{in}(h))$ *and* $\mathrm{in}(\theta(h)) = \theta(\mathrm{out}(h))$,

(b) If $\theta(\mathrm{out}(h)) = \mathrm{in}(h)$, *then* $\theta(h) = h$,

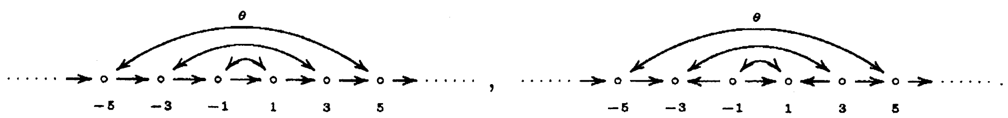
(c) $\theta(\overline{h}) = \overline{\theta(h)}$,

(d) There is no $i \in I$ such that $\theta(i) = i$

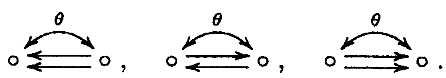
A θ-orientation is an orientation of (I, H) such that Ω is stable by θ. For a θ-orientation Ω, we call (I, Ω) a θ-quiver.

From the assumption (d), any vertex i is a sink with respect to some θ-orientation Ω.

Example 4.2. We give two θ-orientations for the case of Example 2.13. The vertex 1 is a sink in the right example.



Example 4.3. Our definition of a θ-quiver contains the case of type $A_1^{(1)}$. The following three figures are three θ-orientations in this case.



Definition 4.4. *A θ-symmetric I-graded vector space $\mathbf{V}$ is an I-graded vector space endowed with a non-degenerate symmetric bilinear form $(\bullet, \bullet) \colon \mathbf{V} \times \mathbf{V} \to \mathbb{C}$ such that $\mathbf{V}_i$ and $\mathbf{V}_j$ are orthogonal if $j \neq \theta(i)$. For an I-graded subspace $\mathbf{W}$ of $\mathbf{V}$, we set*

$$\mathbf{W}^\perp \colon = \{v \in \mathbf{V} \mid (v, w) = 0 \text{ for any } w \in \mathbf{W}\}.$$

Hence $(\mathbf{W}^\perp)_{\theta(i)} \cong (\mathbf{V}_i/\mathbf{W}_i)^*$.

Note that if $\mathbf{W} \supset \mathbf{W}^{\perp}$, then $\mathbf{W}/\mathbf{W}^{\perp}$ has a structure of θ-symmetric I-graded vector space. Note that two θ-symmetric I-graded vector spaces with the same dimension are isomorphic.

Definition 4.5. *Let (I, H) be a θ-quiver. For a θ-symmetric I-graded vector space $\mathbf{V}$ and a θ-stable subset Ω of H, we define*

$$^{\theta}\mathbf{E}_{\mathbf{V},\Omega} \colon = \{x \in \mathbf{E}_{\mathbf{V},\Omega} \mid x_{\theta(h)} = -{}^t x_h \in \mathrm{Hom}(\mathbf{V}_{\theta(\mathrm{in}(h))}, \mathbf{V}_{\theta(\mathrm{out}(h))}) \text{ for any } h \in \Omega\}.$$

The algebraic group $^{\theta}\mathbf{G}_{\mathbf{V}} \colon = \{g \in \mathbf{G}_{\mathbf{V}} \mid {}^t g_i^{-1} = g_{\theta(i)}$ for any $i\}$ naturally acts on $^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$.

Set $(\mathbb{C}^{\times})^{\Omega,\theta} \colon = \{(c_h)_{h\in\Omega} \mid c_h \in \mathbb{C}^{\times}$ and $c_{\theta(h)} = c_h\}$. The group $(\mathbb{C}^{\times})^{\Omega,\theta}$ also acts on $^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$ by $x_h \mapsto c_h x_h$ $(h \in \Omega)$. These two actions commute with each other.

Definition 4.6. *For a θ-symmetric I-graded vector space $\mathbf{V}$, a sequence $\mathbf{i} = (i_1, \dots, i_{2m}) \in I^{2m}$ such that $\theta(i_\ell) = i_{2m-\ell+1}$ and a sequence $\mathbf{a} = (a_1, \dots, a_{2m}) \in \mathbb{Z}^m_{\geq 0}$ such that $a_{2m-\ell+1} = a_\ell$, we say that a flag of I-graded subspace of $\mathbf{V}$*

$$F = (\mathbf{V} = \mathbf{F}^0 \supset \mathbf{F}^1 \supset \cdots \supset \mathbf{F}^m \supset \mathbf{F}^{m+1} \supset \cdots \supset \mathbf{F}^{2m} = \{0\})$$

is of type $(\mathbf{i}, \mathbf{a})$ if

(i) $\dim(\mathbf{F}^{\ell-1}/\mathbf{F}^{\ell})_i = \begin{cases} a_\ell & (i = i_\ell) \\ 0 & (i \neq i_\ell) \end{cases}$,

(ii) $\mathbf{F}^{2m-\ell} = (\mathbf{F}^{\ell})^{\perp}$.

Then we have $\mathrm{wt}\,\mathbf{V} = \sum_{1\leq\ell\leq 2m} a_\ell \alpha_{i_\ell}$. We denote by $^{\theta}\mathcal{F}_{\mathbf{i},\mathbf{a}}$ the set of flags of type $(\mathbf{i}, \mathbf{a})$.

For $x \in {}^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$, a flag F of type $(\mathbf{i}, \mathbf{a})$ is x-stable if $\mathbf{F}^{\ell}$ $(\ell = 1, \dots, 2m)$ are x-stable. We define

$$^{\theta}\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega} \colon = \{(x, F) \in {}^{\theta}\mathbf{E}_{\mathbf{V},\Omega} \times {}^{\theta}\mathcal{F}_{\mathbf{i},\mathbf{a}} \mid F \text{ is } x\text{-stable}\}.$$

The group $^{\theta}\mathbf{G}_{\mathbf{V}}$ naturally acts on $^{\theta}\mathcal{F}_{\mathbf{i},\mathbf{a}}$ and $^{\theta}\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega}$.

Note that $x \colon \mathbf{V} \to \mathbf{V} \cong \mathbf{V}^*$ in $^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$ may be regarded as a skew-symmetric form on $\mathbf{V}$, and the condition that F is x-stable is equivalent to the one $x(\mathbf{F}^{\ell}, \mathbf{F}^{2m-\ell}) = 0$ for any ℓ.

The following lemma is obvious.

Lemma 4.7. *The variety $^{\theta}\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega}$ is smooth and irreducible. The first projection $^{\theta}\pi_{\mathbf{i},\mathbf{a}} \colon {}^{\theta}\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega} \to {}^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$ is $^{\theta}\mathbf{G}_{\mathbf{V}} \times (\mathbb{C}^{\times})^{\Omega,\theta}$-equivariant and projective.*

4.2. Perverse sheaves on $^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$

Let Ω be a θ-orientation. By Lemma 4.7 and the decomposition theorem [2],

$$^{\theta}L_{\mathbf{i},\mathbf{a};\Omega} \colon = ({}^{\theta}\pi_{\mathbf{i},\mathbf{a}})_!(\mathbf{1}_{{}^{\theta}\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega}})$$

is a semisimple complex in $\mathscr{D}({}^{\theta}\mathbf{E}_{\mathbf{V},\Omega})$.

Definition 4.8. *We define $^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$ as the set of the isomorphism classes of simple perverse sheaves L in $\mathscr{D}({}^{\theta}\mathbf{E}_{\mathbf{V},\Omega})$ satisfying the property: L appears in $^{\theta}L_{\mathbf{i},\mathbf{a};\Omega}[d]$ as a direct summand for some integer d and $(\mathbf{i}, \mathbf{a})$. We denote by $^{\theta}\mathscr{Q}_{\mathbf{V},\Omega}$ the full subcategory of $\mathscr{D}({}^{\theta}\mathbf{E}_{\mathbf{V},\Omega})$ consisting of objects which are isomorphic to finite direct sums of $L[d]$ with $L \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$ and $d \in \mathbb{Z}$.*

Note that any object in $^{\theta}\mathscr{Q}_{\mathbf{V},\Omega}$ is $^{\theta}\mathbf{G}_{\mathbf{V}} \times (\mathbb{C}^{\times})^{\Omega,\theta}$-equivariant.

4.3. Multiplications and Restrictions

Fix θ-symmetric and I-graded vector spaces $\mathbf{V}$ and $\mathbf{W}$, and an I-graded vector space $\mathbf{T}$ such that $\mathrm{wt}(\mathbf{V}) = \mathrm{wt}(\mathbf{W}) + \mathrm{wt}(\mathbf{T}) + \theta(\mathrm{wt}(\mathbf{T}))$.

We consider the following diagram

$$\mathbf{E}_{\mathbf{T},\Omega} \times {}^{\theta}\mathbf{E}_{\mathbf{W},\Omega} \xleftarrow{p_1} {}^{\theta}\mathbf{E}'_{\Omega} \xrightarrow{p_2} {}^{\theta}\mathbf{E}''_{\Omega} \xrightarrow{p_3} {}^{\theta}\mathbf{E}_{\mathbf{V},\Omega}\,.$$

Here ${}^\theta\mathbf{E}''_\Omega$ is the variety of (x, V) where $x \in {}^\theta\mathbf{E}_{\mathbf{V},\Omega}$ and V is an x-stable I-graded subspace of $\mathbf{V}$ such that $V \supset V^\perp$ and $\mathrm{wt}(\mathbf{V}/V) = \mathrm{wt}(\mathbf{T})$, and we denote by ${}^\theta\mathbf{E}'_\Omega$ the variety of $(x, V, \varphi^{\mathbf{W}}, \varphi^{\mathbf{T}})$ where $(x, V) \in {}^\theta\mathbf{E}''_\Omega$, $\varphi^{\mathbf{W}}\colon \mathbf{W} \xrightarrow{\sim} V/V^\perp$ is an isomorphism of θ-symmetric I-graded vector spaces and $\varphi^{\mathbf{T}}\colon \mathbf{T} \xrightarrow{\sim} \mathbf{V}/V$ is an isormorphism of I-graded vector spaces. We define p_1, p_2 and p_3 by $p_1(x, V, \varphi^{\mathbf{W}}, \varphi^{\mathbf{T}}) = (x^{\mathbf{T}}, x^{\mathbf{W}})$, $p_2(x, V, \varphi^{\mathbf{W}}, \varphi^{\mathbf{T}}) = (x, V)$ and $p_3(x, V) = x$. Here the morphism $x^{\mathbf{W}}, x^{\mathbf{T}}$ are defined by

$$x^{\mathbf{W}}_h = {\varphi^{\mathbf{W}}_{\mathrm{in}(h)}}^{-1} \circ (x|_{V/V^\perp})_h \circ \varphi^{\mathbf{W}}_{\mathrm{out}(h)}, \quad x^{\mathbf{T}}_h = {\varphi^{\mathbf{T}}_{\mathrm{in}(h)}}^{-1} \circ (x|_{\mathbf{V}/V})_h \circ \varphi^{\mathbf{T}}_{\mathrm{out}(h)}.$$

Then p_1 is smooth with connected fibers, p_2 is a principal $\mathbf{G_T} \times {}^\theta\mathbf{G_W}$-bundle and p_3 is projective.

For a $\mathbf{G_T}$-equivariant semisimple object $K_{\mathbf{T}} \in \mathscr{Q}_{\mathbf{T},\Omega}$ and a ${}^\theta\mathbf{G_W}$-equivariant semisimple object $K_{\mathbf{W}} \in {}^\theta\mathscr{Q}_{\mathbf{W},\Omega}$, there exists a unique semisimple object $K'' \in \mathscr{D}({}^\theta\mathbf{E}''_\Omega)$ satisfying $p_1^*(K_{\mathbf{T}} \boxtimes K_{\mathbf{W}}) = p_2^* K''$.

Definition 4.9. *We define* $K_{\mathbf{T}} * K_{\mathbf{W}} := (p_3)_!(K'') \in \mathscr{D}({}^\theta\mathbf{E}_{\mathbf{V},\Omega})$.

Next, we fix an I-graded vector space U such that

$$\mathbf{V} \supset U \supset U^\perp \supset \{0\}.$$

We also fix an isomorphism $\mathbf{W} \cong U/U^\perp$ as θ-symmetric I-graded vector spaces and an isomorphism $\mathbf{T} \cong \mathbf{V}/U$ as I-graded vector spaces. We consider the following diagram

$$\mathbf{E}_{\mathbf{T},\Omega} \times {}^\theta\mathbf{E}_{\mathbf{W},\Omega} \xleftarrow{\ p\ } {}^\theta\mathbf{E}(\mathbf{W}, \mathbf{V})_\Omega \xhookrightarrow{\ \iota\ } {}^\theta\mathbf{E}_{\mathbf{V},\Omega}$$

where

$${}^\theta\mathbf{E}(\mathbf{W}, \mathbf{V})_\Omega = \{x \in {}^\theta\mathbf{E}_{\mathbf{V},\Omega} \mid U \text{ is } x\text{-stable}\}$$

and $p(x) = (x^{\mathbf{T}}, x^{\mathbf{W}})$, $\iota(x) = x$.

Definition 4.10. *For* $K \in \mathscr{D}({}^\theta\mathbf{E}_{\mathbf{V},\Omega})$*, we define* $\mathrm{Res}_{\mathbf{T},\mathbf{W}}(K) := p_! \iota^*(K)$.

Proposition 4.11. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be θ-symmetric I-graded vector spaces such that* $\mathrm{wt}\,\mathbf{V} = \mathrm{wt}\,\mathbf{W} + \alpha_i + \alpha_{\theta(i)}$. *For $a \in \mathbb{Z}_{\geq 0}$, let $\mathbf{S}_i^a$ be an I-graded vector space such that* $\mathrm{wt}(\mathbf{S}_i^a) = a\alpha_i$.

(i) *Suppose* ${}^\theta L_{\mathbf{i},\mathbf{a};\Omega} \in \mathscr{D}({}^\theta\mathbf{E}_{\mathbf{W},\Omega})$. *We have*

$$\mathbf{1}_{\mathbf{S}_i^a} * {}^\theta L_{\mathbf{i},\mathbf{a};\Omega} = L_{(i,\mathbf{i},\theta(i)),(a,\mathbf{a},a)}.$$

for $a \in \mathbb{Z}_{\geq 0}$.

(ii) *Suppose* ${}^\theta L_{\mathbf{i},\mathbf{a};\Omega} \in \mathscr{D}({}^\theta\mathbf{E}_{\mathbf{V},\Omega})$ *and $a_\ell > 0$ for all ℓ such that $i_\ell = i$. For $1 \leq k \leq 2m$ such that $i_k = i$, we define* $\mathbf{a}^{(k)} = (a_1^{(k)}, \cdots, a_{2m}^{(k)})$ *by* $a_\ell^{(k)} = a_\ell - \delta_{\ell,k} - \delta_{\ell,2m-k+1}$ *and we set*

$$M_{i,k}(\mathbf{i}, \mathbf{a}^{(k)}) = \sum_{i_\ell = i, \ell < k} a_\ell^{(k)} + \sum_{k < \ell, h \in \Omega; \mathrm{out}(h) = i, \mathrm{in}(h) = i_\ell} a_\ell^{(k)}.$$

Then we have

$$\mathrm{Res}_{\mathbf{S}_i,\mathbf{W}}({}^\theta L_{\mathbf{i},\mathbf{a};\Omega}) = \bigoplus_{i_k = i} {}^\theta L_{\mathbf{i},\mathbf{a}^{(k)};\Omega}[-2M_{i,k}(\mathbf{i}, \mathbf{a}^{(k)})].$$

Lemma 4.12. *Let* $\mathbf{T}^1$ *and* $\mathbf{T}^2$ *be I-graded vector spaces. Let* $\mathbf{W}$ *and* $\mathbf{V}$ *be θ-symmetric I-graded vector spaces such that* $\mathrm{wt}\,\mathbf{V} = \mathrm{wt}\,\mathbf{T}^1 + \theta(\mathrm{wt}\,\mathbf{T}^1) + \mathrm{wt}\,\mathbf{T}^2 + \theta(\mathrm{wt}\,\mathbf{T}^2) + \mathrm{wt}\,\mathbf{W}$.

For $\mathbf{G}_{\mathbf{T}^j}$-equivariant semisimple objects $L_j \in \mathscr{D}(\mathbf{E}_{\mathbf{T}^j,\Omega})$ $(j = 1, 2)$ and a ${}^\theta\mathbf{G_W}$-equivariant semisimple obejct $L \in \mathscr{D}({}^\theta\mathbf{E}_{\mathbf{W},\Omega})$, we have $(L_1 * L_2) * L \cong L_1 * (L_2 * L)$. Here, $L_1 * L_2$ is the Lusztig's multiplication defined in Section 3.

4.4. Restriction functor E_i, Induction functors F_i and $F_i^{(a)}$

We consider the following diagram

$$\mathbf{E}_{\mathbf{T},\Omega} \times {}^\theta\mathbf{E}_{\mathbf{W},\Omega} \xleftarrow{p_1} {}^\theta\mathbf{E}'_\Omega \xrightarrow{p_2} {}^\theta\mathbf{E}''_\Omega \xrightarrow{p_3} {}^\theta\mathbf{E}_{\mathbf{V},\Omega}\,.$$

Lemma 4.13. *Suppose $\mathbf{T} = \mathbf{S}_i$. Let d_{p_1} and d_{p_2} be the dimension of the fibers of p_1 and p_2, respectively. The we have*

$$d_{p_1} - d_{p_2} = \dim {}^\theta\mathbf{E}''_\Omega - \dim {}^\theta\mathbf{E}_{\mathbf{W},\Omega} = \dim \mathbf{W}_i + \sum_{h\in\Omega:\ \mathrm{out}(h)=i} \dim \mathbf{W}_{\mathrm{in}(h)}.$$

Definition 4.14.

(i) For $\mathbf{T} = \mathbf{S}_i$ and a ${}^\theta\mathrm{G}_\mathbf{W}$-equivariant semisimple object K in ${}^\theta\mathscr{Q}_{\mathbf{W},\Omega}$, we define the operator F_i by

$$F_i(K):\ = (\mathbf{1}_{\mathbf{S}_i} * K)\,[d_{F_i}]$$

where

$$d_{F_i} = d_{p_1} - d_{p_2} = \dim \mathbf{W}_i + \sum_{h\in\Omega:\ \mathrm{out}(h)=i} \dim \mathbf{W}_{\mathrm{in}(h)}.$$

(ii) For $\mathbf{T} = \mathbf{S}_i$, we define the functor $E_i\colon \mathscr{D}({}^\theta\mathbf{E}_{\mathbf{V},\Omega}) \to \mathscr{D}({}^\theta\mathbf{E}_{\mathbf{W},\Omega})$ by

$$E_i(K):\ = \mathrm{Res}_{\mathbf{S}_i,\mathbf{W}}(K)\,[d_{E_i}]$$

where

$$d_{E_i} = d_{F_i} - 2\dim \mathbf{W}_i = -\dim \mathbf{W}_i + \sum_{h\in\Omega:\ \mathrm{out}(h)=i} \dim \mathbf{W}_{\mathrm{in}(h)}.$$

By Prposition 4.11, E_i and F_i induce the restriction functor ${}^\theta\mathscr{Q}_{\mathbf{V},\Omega} \to {}^\theta\mathscr{Q}_{\mathbf{W},\Omega}$, induction functor ${}^\theta\mathscr{Q}_{\mathbf{W},\Omega} \to {}^\theta\mathscr{Q}_{\mathbf{V},\Omega}$, respectively.

Definition 4.15. *For $a \in \mathbb{Z}_{>0}$, let $\mathbf{W}$ and $\mathbf{V}$ be θ-symmetric I-graded vector spaces such that* $\mathrm{wt}(\mathbf{V}) = \mathrm{wt}(\mathbf{W}) + a(\alpha_i + \alpha_{\theta(i)})$. *For a ${}^\theta\mathrm{G}_\mathbf{W}$-equivariant semisimple object $L \in {}^\theta\mathscr{P}_{\mathbf{W},\Omega}$, we define $F_i^{(a)}(L):\ = \mathbf{1}_{\mathbf{S}_i^a} * L[d_a]$ where*

$$d_a = a\left(\dim \mathbf{W}_i + \sum_{h\in\Omega:\ \mathrm{out}(h)=i} \dim \mathbf{W}_{\mathrm{in}(h)}\right) + \frac{a(a-1)}{2}\sharp\{h \in \Omega \mid \mathrm{out}(h) = i, \mathrm{in}(h) = \theta(i)\}.$$

We call $F_i^{(a)}$ the a-th divided power of F_i.

By Proposition 4.11(1), we have the following lemma.

Lemma 4.16. *The object ${}^\theta L_{\mathbf{i},\mathbf{a};\Omega}$ is isomorphic to $F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_m}^{(a_m)}\mathbf{1}_{\mathrm{pt}}$ up to shift.*

Lemma 4.17. *The operator $F_i^{(a)}$ gives a functor ${}^\theta\mathscr{Q}_{\mathbf{W},\Omega} \to {}^\theta\mathscr{Q}_{\mathbf{V},\Omega}$ and satisfy $F_iF_i^{(a)} = F_i^{(a)}F_i = [a+1]_vF_i^{(a+1)}$.*

4.5. Commutativity with Fourier transforms

For two θ-orientations Ω and Ω', we have $\overline{\Omega\backslash\Omega'} = \Omega'\backslash\Omega$. Then we can regard ${}^\theta\mathbf{E}_{\mathbf{V},\Omega} \to {}^\theta\mathbf{E}_{\mathbf{V},\Omega\cap\Omega'}$ and ${}^\theta\mathbf{E}_{\mathbf{V},\Omega'} \to {}^\theta\mathbf{E}_{\mathbf{V},\Omega\cap\Omega'}$ as vector bundles and they are the dual vector bundle to each other by the form $\sum_{h\in\Omega\backslash\Omega'} \mathrm{tr}(x_h x_{\bar{h}})$ on ${}^\theta\mathbf{E}_{\mathbf{V},\Omega} \times {}^\theta\mathbf{E}_{\mathbf{V},\Omega'}$. We say that $L \in \mathscr{D}({}^\theta\mathbf{E}_{\mathbf{V},\Omega})$ is $(\mathbb{C}^\times)^{\Omega,\theta}$-monodromic if $H^j(L)$ is locally constant on every $(\mathbb{C}^\times)^{\Omega,\theta}$-orbit on ${}^\theta\mathbf{E}_{\mathbf{V},\Omega}$. Let $\mathscr{D}_{(\mathbb{C}^\times)^{\Omega,\theta}-\mathrm{mono}}({}^\theta\mathbf{E}_{\mathbf{V},\Omega})$ be the full subcategory of $\mathscr{D}({}^\theta\mathbf{E}_{\mathbf{V},\Omega})$ consisting of $(\mathbb{C}^\times)^{\Omega,\theta}$-monodromic objects. Hence we have the Fourier transform

$$\Phi_\mathbf{V}^{\Omega\Omega'} : \mathscr{D}_{(\mathbb{C}^\times)^{\Omega,\theta}-\mathrm{mono}}({}^\theta\mathbf{E}_{\mathbf{V},\Omega}) \to \mathscr{D}_{(\mathbb{C}^\times)^{\Omega,\theta}-\mathrm{mono}}({}^\theta\mathbf{E}_{\mathbf{V},\Omega'}).$$

The following lemma is obvious.

Lemma 4.18. *For three θ-orientations Ω, Ω' and Ω'', we have*

$$\Phi_{\mathbf{V}}^{\Omega'\Omega''} \circ \Phi_{\mathbf{V}}^{\Omega\Omega'} \cong a^* \circ \Phi_{\mathbf{V}}^{\Omega\Omega''} : \mathscr{D}_{(\mathbb{C}^\times)^{\Omega,\theta}-\mathrm{mono}}({}^\theta\mathbf{E}_{\mathbf{V},\Omega}) \to \mathscr{D}_{(\mathbb{C}^\times)^{\Omega,\theta}-\mathrm{mono}}({}^\theta\mathbf{E}_{\mathbf{V},\Omega''})$$

where $a : {}^\theta\mathbf{E}_{\mathbf{V},\Omega''} \to {}^\theta\mathbf{E}_{\mathbf{V},\Omega''}$ is defined by $x_h \mapsto -x_h$ or x_h according that $h \in \Omega'' \cap \overline{\Omega'} \cap \Omega$ or not. In particular, $\mathscr{D}_{(\mathbb{C}^\times)^{\Omega,\theta}-\mathrm{mono}}({}^\theta\mathbf{E}_{\mathbf{V},\Omega})$ does not depend on Ω.

Since any object in ${}^\theta\mathscr{Q}_{\mathbf{V},\Omega}$ is ${}^\theta\mathbf{G}_{\mathbf{V}} \times (\mathbb{C}^\times)^{\Omega,\theta}$-equivariant, it is a monodromic object. By the commutativity between E_i, F_i and $(\mathbb{C}^\times)^{\Omega,\theta}$-action, the functors E_i and F_i preserve the category $(\mathbb{C}^\times)^{\Omega,\theta}$-monodromic objects.

Theorem 4.19. *Let $\mathbf{V}$ and $\mathbf{W}$ be θ-symmetric I-graded vector spaces such that* $\mathrm{wt}\,\mathbf{V} = \mathrm{wt}\,\mathbf{W} + \alpha_i + \alpha_{\theta(i)}$, *and Ω and Ω' be two θ-symmetric orientations.*

(1) *Let F_i^Ω and $F_i^{\Omega'}$ be the induction functors with respect to Ω and Ω', respectively. For a ${}^\theta\mathbf{G}_{\mathbf{W}}$-equivariant semisimple obejct $L \in {}^\theta\mathscr{Q}_{\mathbf{W},\Omega}$, we have $\Phi_{\mathbf{V}}^{\Omega\Omega'} \circ F_i^\Omega(L) \cong F_i^{\Omega'} \circ \Phi_{\mathbf{W}}^{\Omega\Omega'}(L)$.*

(2) *Let ${E_i}^\Omega$ and ${E_i}^{\Omega'}$ be the restriction functors with respect to Ω and Ω', respectively. For a ${}^\theta\mathbf{G}_{\mathbf{V}}$-equivariant semisimple obejct $K \in {}^\theta\mathscr{Q}_{\mathbf{W},\Omega}$, we have $\Phi_{\mathbf{W}}^{\Omega\Omega'} \circ {E_i}^\Omega(K) \cong {E_i}^{\Omega'} \circ \Phi_{\mathbf{V}}^{\Omega\Omega'}(K)$.*

(3) *The Fourier transform $\Phi_{\mathbf{V}}^{\Omega\Omega'}$ gives an isomorphism between ${}^\theta\mathscr{P}_{\mathbf{V},\Omega}$ and ${}^\theta\mathscr{P}_{\mathbf{V},\Omega'}$ and an equivalence between ${}^\theta\mathscr{Q}_{\mathbf{V},\Omega}$ and ${}^\theta\mathscr{Q}_{\mathbf{V},\Omega'}$.*

Similarly, we can prove the commutativity of $F_i^{(a)}$'s and the Fourier transforms.

Proposition 4.20. *Let $\mathbf{W}$ and $\mathbf{V}$ be θ-symmetric I-graded vector spaces such that* $\mathrm{wt}(\mathbf{V}) = \mathrm{wt}(\mathbf{W}) + a(\alpha_i + \alpha_{\theta(i)})$. *Let ${F_i^{(a)}}^\Omega$ and ${F_i^{(a)}}^{\Omega'}$ be the a-th divided powers with respect to θ-orientations Ω and Ω', respectively. For a ${}^\theta\mathbf{G}_{\mathbf{W}}$-equivariant semisimple obejct $L \in {}^\theta\mathscr{Q}_{\mathbf{W},\Omega}$, we have $\Phi_{\mathbf{V}}^{\Omega\Omega'} \circ {F_i^{(a)}}^\Omega(L) \cong {F_i^{(a)}}^{\Omega'} \circ \Phi_{\mathbf{W}}^{\Omega\Omega'}(L)$.*

5. A quiver construction of symmetric crystals

5.1. Grothendieck group

For a θ-orientation Ω and a θ-symmetric and I-graded vector space $\mathbf{V}$, we define ${}^\theta K_{\mathbf{V},\Omega}$ as the Grothendieck group of ${}^\theta\mathscr{Q}_{\mathbf{V},\Omega}$. Namely ${}^\theta K_{\mathbf{V},\Omega}$ is generated by (L) for $L \in {}^\theta\mathscr{Q}_{\mathbf{V},\Omega}$ with the relation $(L) = (L') + (L'')$ when $L \cong L' \oplus L''$. This is a $\mathbb{Z}[v, v^{-1}]$-module by $v(L) = (L[1])$ and $v^{-1}(L) = (L[-1])$ for $L \in {}^\theta\mathscr{Q}_{\mathbf{V},\Omega}$. Hence, ${}^\theta K_{\mathbf{V},\Omega}$ is a free $\mathbb{Z}[v, v^{-1}]$-module with a basis $\{(L) \mid L \in {}^\theta\mathscr{P}_{\mathbf{V},\Omega}\}$. For another θ-symmetric and I-graded vector space $\mathbf{V}'$ such that $\mathrm{wt}\,\mathbf{V} = \mathrm{wt}\,\mathbf{V}'$, we have ${}^\theta K_{\mathbf{V},\Omega} \cong {}^\theta K_{\mathbf{V}',\Omega}$. We define

$${}^\theta K_\Omega :\, = \bigoplus_{\mathbf{V}} {}^\theta K_{\mathbf{V},\Omega}$$

where $\mathbf{V}$ runs over the isomorphism classes of θ-symmetric I-graded vector spaces. For two θ-orientations Ω and Ω', the Fourier transform induces an equivalence ${}^\theta\mathscr{Q}_{\mathbf{V},\Omega} \to {}^\theta\mathscr{Q}_{\mathbf{V},\Omega'}$ and the isomorphism ${}^\theta K_{\mathbf{V},\Omega} \xrightarrow{\sim} {}^\theta K_{\mathbf{V},\Omega'}$. Therefore ${}^\theta K_\Omega \cong {}^\theta K_{\Omega'}$.

We set ${}^\theta K = {}^\theta K_\Omega, {}^\theta\mathscr{P}_{\mathbf{V}} = {}^\theta\mathscr{P}_{\mathbf{V},\Omega}$. By Lemma 4.18, they are well-defined.

5.2. Actions of E_i and F_i

The functors E_i and $F_i^{(a)}$ induce the action on ${}^\theta K_\Omega$.Since E_i and F_i commute with the Fourier transforms, they also act on ${}^\theta K$. The submodule ${}^\theta K' :\, = \sum_{(\mathrm{i},\mathrm{a})} \mathbb{Z}[v, v^{-1}]({}^\theta L_{\mathrm{i},\mathrm{a};\Omega}) \subset {}^\theta K$ is stable by E_i and F_i by Proposition 4.11. We define

$$T_i|_{{}^\theta K_{\mathbf{V},\Omega}} = v^{-(\alpha_i, \mathrm{wt}\,\mathbf{V})}\,\mathrm{id}_{{}^\theta K_{\mathbf{V},\Omega}}.$$

Proposition 5.1. *The operators E_i, F_i and T_i $(i \in I)$ regarded as operators on ${}^\theta K'$ satisfy*

$$E_i F_j - v^{-(\alpha_i,\alpha_j)} F_j E_i = \delta_{ij} + \delta_{\theta(i),j} T_i$$

and

$$T_i E_j T_i^{-1} = v^{(\alpha_i+\alpha_{\theta(i)},\alpha_j)} E_j, \quad T_i F_j T_i^{-1} = v^{(\alpha_i+\alpha_{\theta(i)},-\alpha_j)} F_j.$$

5.3. Key estimates of coefficients

Let Ω be a θ-orientation and suppose that a vertex i is a sink. For a θ-symmetric I-graded vector space $\mathbf{V}$ and $r \in \mathbb{Z}_{\geq 0}$, we define

$$^{\theta}\mathbf{E}_{\mathbf{V},\Omega,r} := \left\{ x \in {}^{\theta}\mathbf{E}_{\mathbf{V},\Omega} \,\middle|\, \dim \operatorname{Coker}\left(\bigoplus_{h\in\Omega;\mathrm{in}(h)=i} \mathbf{V}_{\mathrm{out}(h)} \to \mathbf{V}_i \right) = r \right\}.$$

Then we have ${}^{\theta}\mathbf{E}_{\mathbf{V},\Omega} = \sqcup_{r\geq 0}{}^{\theta}\mathbf{E}_{\mathbf{V},\Omega,r}$, and ${}^{\theta}\mathbf{E}_{\mathbf{V},\Omega,\geq r} := \sqcup_{r'\geq r}{}^{\theta}\mathbf{E}_{\mathbf{V},\Omega,r'}$ is a closed subset of ${}^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$.

Definition 5.2. *For $L \in {}^{\theta}\mathscr{P}_{\mathbf{V}}$ and $i \in I$, choose a θ-orientation Ω such that i is a sink with respect to Ω, and regard L as an element of ${}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$. We define $\varepsilon_i(L)$ as the largest integer r satisfying* $\operatorname{Supp}(L) \subset {}^{\theta}\mathbf{E}_{\mathbf{V},\Omega,\geq r}$. *This does not depend on the choice of Ω.*

Note that $0 \leq \varepsilon_i(L) \leq \dim V_i$.

We shall prove the following key estimates with respect to $F_i(L)$ and $E_i(L)$.

Theorem 5.3. *Assume that θ-symmetric and I-graded vector spaces $\mathbf{V}$ and $\mathbf{W}$ satisfy* $\operatorname{wt}\mathbf{V} = \operatorname{wt}\mathbf{W} + \alpha_i + \alpha_{\theta(i)}$. *Fix a θ-orientation Ω such that the vertex i is a sink.*

(1) *For $L \in {}^{\theta}\mathscr{P}_{\mathbf{W},\Omega}$, there exists a unique simple perverse sheaf $L_0 \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$ such that $\varepsilon_i(L_0) = \varepsilon_i(L) + 1$ and*

$$F_i(L) = [\varepsilon_i(L)+1]_v(L_0) + \sum_{L' \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}\,:\, \varepsilon_i(L') > \varepsilon_i(L)+1} a_{L'}(L')$$

for $a_{L'} \in v^{2-\varepsilon_i(L')}\mathbb{Z}[v]$.

We define the map $\tilde{F}_i \colon {}^{\theta}\mathscr{P}_{\mathbf{W}} \cong {}^{\theta}\mathscr{P}_{\mathbf{W},\Omega} \to {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega} \cong {}^{\theta}\mathscr{P}_{\mathbf{V}}$ by $\tilde{F}_i(L) = L_0$. It does not depend on the choice of Ω.

(2) *Let $K \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$. If $\varepsilon_i(K) > 0$, there exists a unique simple perverse sheaf $K_0 \in {}^{\theta}\mathscr{P}_{\mathbf{W},\Omega}$ such that $\varepsilon_i(K_0) = \varepsilon_i(K) - 1$ and*

$$E_i(K) = v^{1-\varepsilon_i(K)}(K_0) + \sum_{K' \in {}^{\theta}\mathscr{P}_{\mathbf{W},\Omega}\,:\, \varepsilon_i(K') > \varepsilon_i(K)-1} b_{K'}(K')$$

for $b_{K'} \in v^{-\varepsilon_i(K')+1}\mathbb{Z}[v]$. Here we regard $K_0 = 0$ if $\varepsilon_i(K) = 0$.

We define the map $\tilde{E}_i \colon {}^{\theta}\mathscr{P}_{\mathbf{V}} \cong {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega} \to {}^{\theta}\mathscr{P}_{\mathbf{W},\Omega} \sqcup \{0\} \cong {}^{\theta}\mathscr{P}_{\mathbf{W}} \sqcup \{0\}$ by $\tilde{E}_i(K) = K_0$ if $\varepsilon_i(K) > 0$ and $\tilde{E}_i(K) = 0$ if $\varepsilon_i(K) = 0$. It does not depend on the choise of Ω.

Lemma 5.4. *Suppose* $\operatorname{wt}\mathbf{V} \neq 0$. *For any $L \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$, there exists $i \in I$ such that $\varepsilon_i(L) > 0$.*

Proof. If $\mathbf{V} \neq \{0\}$, there exists an integer d, $\mathbf{i} = (i_1, \ldots, i_{2m})$ and $\mathbf{a}$ such that $L[d]$ appears in a direct summand of ${}^{\theta}L_{\mathbf{i},\mathbf{a};\Omega}$. We may assume $a_1 > 0$. Then, taking Ω such that i_1 is a sink, we have $\operatorname{Supp}(L) \subset \operatorname{Supp}({}^{\theta}L_{\mathbf{i},\mathbf{a};\Omega}) \subset {}^{\theta}\mathbf{E}_{\mathbf{V},\Omega,\geq 1}$. By the definition of ε_i, we have $\varepsilon_{i_1}(L) \neq 0$. □

Lemma 5.5. *For $L \in \mathscr{P}_{\mathbf{V}}$, we have $\tilde{E}_i\tilde{F}_i(L) = (L)$, and if $\tilde{E}_i(L) \neq 0$, we have $\tilde{F}_i\tilde{E}_i(L) = L$.*

5.4. Verdier duality functor

The Verdier duality functor $D \colon \mathscr{D}({}^{\theta}\mathbf{E}_{\mathbf{V},\Omega}) \to \mathscr{D}({}^{\theta}\mathbf{E}_{\mathbf{V},\Omega})$ satisfies $D(L[d]) = D(L)[-d]$ for $L \in \mathscr{D}({}^{\theta}\mathbf{E}_{\mathbf{V},\Omega})$, $d \in \mathbb{Z}$. Then D induces the involution $v^{\pm 1} \mapsto v^{\mp 1}$.

Proposition 5.6.

(i) $D({}^{\theta}L_{\mathbf{i},\mathbf{a};\Omega}) = {}^{\theta}L_{\mathbf{i},\mathbf{a};\Omega}[2\dim {}^{\theta}\widetilde{\mathcal{F}}_{\mathbf{i},\mathbf{a};\Omega}]$.

(ii) For any $L \in {}^{\theta}\mathscr{Q}_{\mathbf{V},\Omega}$, we have $D(F_iL) = F_iD(L)$.

(iii) For any $L \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$, we have $D(L) \cong L$.

Proof. (i) and (ii) follow from the general property of the Verdier duality functor. To prove (iii), we use the induction on wt $\mathbf{V}$.

When wt $\mathbf{V} = 0$, the claim is clear by ${}^{\theta}\mathscr{P}_{\mathbf{V},\Omega} = \{\mathbf{1}_{\mathrm{pt}}\}$ and $D(\mathbf{1}_{\mathrm{pt}}) = \mathbf{1}_{\mathrm{pt}}$.

Suppose wt $\mathbf{V} \neq 0$. By Lemma 5.4, there exists i such that $\varepsilon_i(L) > 0$. We shall prove $D(L) = L$ by the descending induction on $\varepsilon_i(L)$. By Theorem 5.3 and Lemma 5.5, we have

$$F_i(\tilde{E}_i L) = [\varepsilon_i(L)]_v(L) + \sum_{L' \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}:\, \varepsilon_i(L') > \varepsilon_i(L)} a_{L'}(L').$$

By the induction hypothesis on wt $\mathbf{V}$, $D(\tilde{E}_i L) = \tilde{E}_i L$. Hence the lefthand side is D-invariant by (ii). We restrict $F_i(\tilde{E}_i L)$ on the open subset ${}^{\theta}\mathbf{E}_{\mathbf{V},\Omega,\leq\varepsilon_i(L)}$. Then it is isomorphic to $[\varepsilon_i L]_v(L)|_{{}^{\theta}\mathbf{E}_{\mathbf{V},\Omega,\leq\varepsilon_i(L)}}$ and D-invariant. Since L is the minimal extension of $L|_{{}^{\theta}\mathbf{E}_{\mathbf{V},\Omega,\leq\varepsilon_i(L)}}$, L is D-invariant. □

Remark 5.7. By the result of (iii), we have $a_{L'}(v) = a_{L'}(v^{-1})$ in Theorem 5.3 (1).

Lemma 5.8. *For* $L \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$, *we have*

$$F_i^{(a)}(L) = \begin{bmatrix} \varepsilon_i(L) + a \\ a \end{bmatrix}_v (\tilde{F}_i^a L) + \sum_{L':\, \varepsilon_i(L') > \varepsilon_i(L)+a} c_{L'}(L')$$

with $c_{L'} \in \mathbb{Z}[v, v^{-1}]$.

Proof. We shall prove the claim by the induction on a. If $a = 1$, the claim follows from Theorem 5.3. If $a > 1$, by the induction hypothesis and Theorem 5.3, we have

$$\begin{aligned} F_i F_i^{(a)}(L) &= \begin{bmatrix} \varepsilon_i(L) + a \\ a \end{bmatrix}_v F_i(\tilde{F}_i^a L) + \sum_{L':\, \varepsilon_i(L') > \varepsilon_i(L)+a} c_{L'} F_i(L') \\ &= [a+1]_v \left(\begin{bmatrix} \varepsilon_i(L) + a + 1 \\ a+1 \end{bmatrix}_v (\tilde{F}_i^{a+1} L) + \sum_{L'':\, \varepsilon_i(L'') > \varepsilon_i(L)+a+1} d_{L''}(L'') \right), \end{aligned}$$

where $d_{L''} \in \mathbb{Q}(v)$. Hence

$$F_i^{(a+1)} L = \begin{bmatrix} \varepsilon_i(L) + a + 1 \\ a+1 \end{bmatrix}_v (\tilde{F}_i^{a+1} L) + \sum_{L'':\, \varepsilon_i(L'') > \varepsilon_i(L)+a+1} d_{L''}(L'').$$

On the other hand, since $F_i^{(a+1)} L = \mathbf{1}_{\mathbf{S}_i^{a+1}} * L[d_{a+1}]$ is semisimple, we conclude $d_{L''} \in \mathbb{Z}[v, v^{-1}]$. □

Proposition 5.9. *We have* ${}^{\theta}K = \sum \mathbb{Z}[v, v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} \mathbf{1}_{\{\mathrm{pt}\}}$.

Proof. For $L \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}$ such that wt $\mathbf{V} \neq 0$, there exists i such that $\varepsilon_i(L) > 0$. We shall prove that (L) is contained in $\sum \mathbb{Z}[v, v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} \mathbf{1}_{\{\mathrm{pt}\}}$ by the induction on wt $\mathbf{V}$ and the descending induction on $\varepsilon_i(L)$. We have

$$F_i^{(\varepsilon_i(L))}(\tilde{E}_i^{\varepsilon_i(L)} L) = (L) + \sum_{L' \in {}^{\theta}\mathscr{P}_{\mathbf{V},\Omega}:\, \varepsilon_i(L') > \varepsilon_i(L)} c_{L'}(L')$$

by Lemma 5.8 and Lemma 5.5. By the induction hypothesis, we have $c_{L'}(L')$ and $\tilde{E}_i^{\varepsilon_i(L)} L$ are contained in $\sum \mathbb{Z}[v, v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} \mathbf{1}_{\{\mathrm{pt}\}}$.

Thus $(L) \in \sum \mathbb{Z}[v, v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} \mathbf{1}_{\{\mathrm{pt}\}}$. □

5.5. Main Theorem

Let us recall

$${}^{\theta}K' := \sum_{(\mathbf{i},\mathbf{a})} \mathbb{Z}[v, v^{-1}]({}^{\theta}L_{\mathbf{i},\mathbf{a};\Omega}) = \sum \mathbb{Z}[v, v^{-1}] F_{i_1}^{(a_1)} \cdots F_{i_k}^{(a_k)} \mathbf{1}_{\{\mathrm{pt}\}} \subset {}^{\theta}K.$$

Theorem 5.10.

(i) ${}^\theta K = {}^\theta K'$.

(ii) *For* $L \in {}^\theta\mathscr{P}_{\mathbf{V}}$, *we define* $\mathrm{wt}(L) = -\mathrm{wt}\,\mathbf{V}$. *Then* $(\mathrm{wt}, \tilde{E}_i, \tilde{F}_i, \varepsilon_i)$ *gives a crystal structure on* ${}^\theta\mathscr{P} := \sqcup_{\mathbf{V}} {}^\theta\mathscr{P}_{\mathbf{V}}$ *in the sence of section 2.3. Here* $\mathbf{V}$ *runs over all isomorphism classes of* θ*-symmetric* I*-graded vector spaces.*

(iii) *Let* $\mathcal{L}$ *be the* $\mathbf{A}_0$*-submodule* $\sum_{(L)\in{}^\theta\mathscr{P}} \mathbf{A}_0(L)$ *of* ${}^\theta K$. *Then* $\{(L) \mod v\mathcal{L} | L \in {}^\theta\mathscr{P}\}$ *gives a crystal basis of* ${}^\theta K$. *Especially, the actions of modified root operators* $\tilde{E}_i$ *and* $\tilde{F}_i$ *on* $\mathcal{L}/v\mathcal{L}$ *are compatible with the actions of* $\tilde{E}_i$ *and* $\tilde{F}_i$ *on* ${}^\theta\mathscr{P}$ *introduced in Theorem 5.3.*

Proof. (i) is nothing but Proposition 5.9.
(ii) By the definition of $\varepsilon_i(L), \tilde{F}_i$ and $\tilde{E}_i$, and Lemma 5.5, we conculde that $(\mathrm{wt}, \tilde{E}_i, \tilde{F}_i, \varepsilon_i)$ gives a crystal structure on ${}^\theta\mathscr{P} := \sqcup_{\mathbf{V}} {}^\theta\mathscr{P}_{\mathbf{V}}$ in the sence of section 2.3(i)-(iv). By the estimates in Theorem 5.3, the actions of E_i and F_i on (L) $(L \in {}^\theta\mathscr{P})$ satisfy the conditions (2)-(7) in section 2.3. Thus we obtain the claim.
(iii) follows from Theorem 2.14. □

Lemma 5.11. *We have* $\{v \in {}^\theta K \mid E_i v = 0 \text{ for any } i \in I\} = \mathbb{Z}[v, v^{-1}]\mathbf{1}_{\{\mathrm{pt}\}}$.

Proof. Suppose that $E_i\left(\sum a_L(L)\right) = 0$ for any L. Then $a_L \in v^c\mathbb{Z}[v]$ for some c. Put $\widetilde{a_L} = v^{-c}a_L \in \mathbb{Z}[v]$. By the definition of the modified root operators and Theorem 5.10(iii), we have $\tilde{E}_i\left(\sum \widetilde{a_L}(L)\right) = 0$. Specializing v to 0, we have $\widetilde{a_L}(0) = 0$ if $\tilde{E}_i L \neq 0$. But for any L such that $\mathrm{wt}(L) \neq 0$, there exists $i \in I$ such that $\varepsilon_i(L) > 0$. Hence we obtain $\widetilde{a_L} \in v\mathbb{Z}[v]$ and hence $a_L \in v^{c+1}\mathbb{Z}[v]$. By the induction on c, we have $a_L \in v^c\mathbb{Z}[v]$ for any c. Thus we conclude $a_L = 0$ for $\mathrm{wt}(L) \neq 0$. □

Theorem 5.12.

(i) ${}^\theta K \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(v) \cong V_\theta(0)$ *as a* $B_\theta(\mathfrak{g})$*-module. The involution induced by the Verdier duality functor coincides with the bar involution on* $V_\theta(0)$.

(ii) $\{(L) \mid L \in {}^\theta\mathscr{P}\}$ *gives the lower global basis on* $V_\theta(0)$.

Proof. (i) By Proposition 5.1, to check the defining relations of $B_\theta(\mathfrak{g})$, we only need to prove the v-Serre relations. Put

$$S_e = \sum_{k=0}^{b}(-1)^k E_i^{(k)} E_j E_i^{(b-k)}, \quad S_f = \sum_{k=0}^{b}(-1)^k F_i^{(k)} F_j F_i^{(b-k)}$$

and note that $F_k S_e = S_e F_k$ and $E_k S_f = S_f E_k$ for any $k \in I$.

Since ${}^\theta K_\Omega$ is generated by $F_k^{(n)}$'s from $\phi := \mathbf{1}_{\{\mathrm{pt}\}}$ and $S_e\phi = 0$, we have $S_e v = 0$ for any $v \in {}^\theta K_\Omega$. We show $S_f(L) = 0$ for any $L \in {}^\theta\mathscr{P}_{\mathbf{V},\Omega}$ by the induction on $\mathrm{wt}\,\mathbf{V}$. If $\mathrm{wt}(S_f(L)) \neq 0$, we have we have $E_k S_f(L) = S_f E_k(L) = 0$ for any $k \in I$ by applying the induction hypothesis to $E_k(L)$. Since $\mathrm{wt}(S_f(L)) \neq 0$, we have $S_f(L) = 0$ by Lemma 5.11. Hence ${}^\theta K$ is a $B_\theta(\mathfrak{g})$-module. Note that $T_i\mathbf{1}_{\{\mathrm{pt}\}} = \mathbf{1}_{\{\mathrm{pt}\}}$ for any $i \in I$. We conclude ${}^\theta K \cong V_\theta(0)$ by Lemma 5.11 and the characterization of $V_\theta(0)$ in Proposition 2.10.
(ii) We already know that $\mathcal{L} = \sum_{L\in{}^\theta\mathscr{P}} \mathbf{A}_0(L)$ is a crystal lattice and $\{(L) \mod v\mathcal{L}\}$ is a basis of $\mathcal{L}/v\mathcal{L}$. Note that $\sum_{L\in{}^\theta\mathscr{P}} \mathbb{Z}[v, v^{-1}](L)$ is stable under the actions of E_i's and $F_i^{(a)}$'s by Lemma 5.8 and L is D-invariant, namely bar-invariant. Moreover $\{(L) \mid L \in {}^\theta\mathscr{P}\}$ is a basis of the $\mathbf{A}_0$-module $\mathcal{L}$ and also a basis of the $\mathbb{Z}[v, v^{-1}]$-module ${}^\theta K$. Hence we conclude that $\{(L) \mid L \in {}^\theta\mathscr{P}\}$ gives the lower global basis on $V_\theta(0)$. □

Corollary 5.13. *For any Kac-Moody algebra* $\mathfrak{g}$ *with a symmetric Cartan matrix, the* $B_\theta(\mathfrak{g})$*-module* $V_\theta(0)$ *has a crystal basis and a lower global basis, namely Conjecture 2.11 and Conjecture 2.12 is true if* $\lambda = 0$.

Example 5.14. Let us consider the case $\mathfrak{g} = \mathfrak{sl}_3$, $I = \{\pm 1\}$ and $\theta(i) = -i$. Fix a θ-symmetric orientation $-1 \xrightarrow{\Omega} 1$. For a θ-symmetric I-graded vector space $\mathbf{V}$ such that $\mathrm{wt}(\mathbf{V}) = n(\alpha_{-1} + \alpha_1)$, ${}^\theta\mathbf{E}_{\mathbf{V},\Omega}$ is the set of skew symmetric matrix x of size n. Its ${}^\theta\mathbf{G}_{\mathbf{V}}$-orbits are parametrized by the rank $2r$ $(0 \leq r \leq \lfloor \frac{n}{2} \rfloor)$ of x. We denote $\mathcal{O}_r^n$ by the orbit consisting of $n \times n$ skew symmetric matrices x of rank $2r$. Note that any ${}^\theta G_{\mathbf{V}}$-equivariant simple local system on each ${}^\theta\mathbf{G}_{\mathbf{V}}$-orbit is trivial. Let us

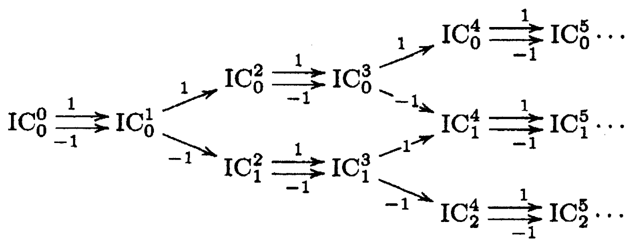
denote IC_r^n by the simple perverse sheaves corresponding to the orbit $\mathcal{O}_r^n$. Note that $\varepsilon_1(IC_r^n) = n - 2r$.
Let $\mathbf{W}$ be a θ-symmetric I-graded vector space such that $\mathrm{wt}(\mathbf{W}) = (n-1)(\alpha_{-1} + \alpha_1)$. We consider the diagram:

$$^{\theta}\mathbf{E}_{\mathbf{W},\Omega} \xleftarrow[p_1]{} {}^{\theta}\mathbf{E}'_{\Omega} \xrightarrow[p_2]{} {}^{\theta}\mathbf{E}''_{\Omega} \xrightarrow[p_3]{} {}^{\theta}\mathbf{E}_{\mathbf{V},\Omega}.$$

Note that the fibers of p_3 on $\mathcal{O}_r^n$ is isomorphic to $\mathbf{P}^{n-1-2r}$. Then

$$F_1(IC_r^{n-1}) = [n-2r]_v(IC_r^n) + \sum_{k=0}^{r-1} a_k(IC_k^n)$$

where $a_k \in v^{2-n+2k}\mathbb{Z}[v]$. We obtain the crystal graph:



In this case, all indecomposable representations are described by

$$\mathbb{C} \xrightarrow{0} \mathbb{C} \quad \text{and} \quad \mathbb{C}^2 \xrightarrow{J} \mathbb{C}^2$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We denote $\langle 1 \rangle$ and $\langle -1, 1 \rangle$ by above indecomposable representations, respectively. Thus we can parametrized $^{\theta}G_{\mathbf{V}}$-orbit in $^{\theta}\mathbf{E}_{\mathbf{V},\Omega}$ and associated simple perverse sheaves by $a\langle 1 \rangle + b\langle -1, 1 \rangle$ $(a, b \in \mathbb{Z}_{\geq 0})$, especially $\mathcal{O}_r^n$ (and IC_r^n) corresponds to $(n-2r)\langle 1 \rangle + r\langle -1, 1 \rangle$. Therefore we recover the crystal graph parametrized by "θ-restricted multi-segments" in [6, Example 4.7 (1)].

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