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SPRINGER THEORY FOR COMPLEX REFLECTION GROUPS

PRAMOD N. ACHAR

ABSTRACT. Many complex reflection groups behave as though they were the Weyl groups of "nonexistent algebraic groups": one can associate to them various representation-theoretic structures and carry out calculations that appear to describe the geometry and representation theory of an unknown object. This paper is a survey of a project to understand the geometry of the "unipotent variety" of a complex reflection group (enumeration of unipotent classes, Springer correspondence, Green functions), based on the author's joint work with A.-M. Aubert.

A *complex reflection group* is a finite group of automorphisms of a finite-dimensional complex vector space V that is generated by *reflections*, i.e., linear transformations that fix some hyperplane pointwise. Some complex reflection groups can actually be realized on a real vector space, and a famous theorem of Coxeter states that these are precisely the finite Coxeter groups. Among those, the reflection groups that can be realized on a $\mathbb{Q}$-vector space are particularly important: these are the groups that occur as Weyl groups of reductive algebraic groups.

Since the early 1990's, there has been a growing awareness that many complex reflection groups that cannot be realized over $\mathbb{Q}$ nevertheless behave as though they were the Weyl groups of certain "nonexistent" algebraic groups. The first important step was the discovery [4, 5, 13] that their group algebras admit deformations resembling Iwahori–Hecke algebras of Coxeter groups. Those deformations are now known as *cylcotomic Hecke algebras.* Subsequent work by a number of authors showed that complex reflection groups admit analogues of Coxeter presentations [13], root systems [17, 33] and root lattices [33], length functions [8, 9], generic degrees [28, 30], and Green functions [36, 37].

A theme in these developments is that statements that are regarded as theorems in the setting of Weyl groups are often adopted as definitions in the setting of complex reflection groups. For instance, families of representations for Weyl groups are defined in terms of the Kazhdan–Lusztig basis for the Hecke algebra, but a theorem of Rouquier [34] gives an alternate description of families in terms of blocks over a suitable coefficient ring. For cyclotomic Hecke algebras, Kazhdan–Lusztig bases are unavailable, but "Rouquier blocks" still make sense, and have been adopted as a definition [10, 23, 32].

The present paper is an exposition of how this philosophy may be applied to the theory of unipotent classes and the Springer correspondence. Many features of the geometry of the unipotent variety of an algebraic group—including the number of conjugacy classes, their dimensions and closure relations, and their local intersection cohomology—can be computed from elementary knowledge of the Weyl group. Remarkably, analogous calculations for complex reflection groups often yield sensible results, with surprisingly "geometric" integrality and positivity properties, even though there is not (yet?) an actual "unipotent variety" attached to a general complex reflection group.

The ideas and results described here come from a series of joint papers by the author and A.-M. Aubert [1, 2, 3]. There are no new theorems in this paper. However, the last two sections give the results of various calculations in the exceptional groups that have not previously been published.

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1. Overview of Complex Reflection Groups

1.1. Examples and classification. The easiest example of a complex reflection group is the cyclic group $\mathfrak{C}_d$ of order d, acting on $\mathbb{C}$ by multiplication by d-th roots of unity. Simiarly, the n-fold product $(\mathfrak{C}_d)^n$ acts on $\mathbb{C}^n$ as a (reducible) complex reflection group. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{C}^n$ by permuting coordinate axes, and this action is generated by reflections and normalizes the action of $(\mathfrak{C}_d)^n$. The semidirect product

$$G(d, 1, n) = (\mathfrak{C}_d)^n \rtimes \mathfrak{S}_n$$

is an irreducible complex reflection group. If $d = 2$, this is the Weyl group of type B_n or C_n.

Next, replace d by a product of positive integers de, and consider the group $G(de, 1, n)$. Define a map $\phi_e : G(de, 1, n) \to \mathfrak{C}_e$ by $\phi_e = p \circ f$, where $f : G(de, 1, n) \to \mathfrak{C}_{de}$ assigns to an element of $(\mathfrak{C}_{de})^n \rtimes \mathfrak{S}_n$ the product of the components in the $(\mathfrak{C}_{de})^n$ part, and $p : \mathfrak{C}_{de} \to \mathfrak{C}_e$ is the obvious quotient map. Then let

$$G(de, e, n) = \ker \phi_e.$$

This is also an irreducible complex reflection group, and it is obviously a normal subgroup of index e in $G(de, 1, n)$. The Weyl groups of type D_n are of this form; they are the groups $G(2, 2, n)$. The dihedral groups also occur in this series: the group $I_2(m)$ of order $2m$ is $G(m, m, 2)$.

According to the classification theorem due to Shephard and Todd [35], there are, in addition to the infinite family $G(de, e, n)$ (called *imprimitive* groups), exactly thirty-four other irreducible complex reflection groups (called *primitive* or *exceptional* groups). In their notation, which has now become standard, these are denoted

$$G_4, \ldots, G_{37}.$$

The exceptional finite Coxeter groups occur in this list as follows:

$$H_3 \simeq G_{23}, \quad F_4 \simeq G_{28}, \quad H_4 \simeq G_{30}, \quad E_6 \simeq G_{35}, \quad E_7 \simeq G_{36}, \quad E_8 \simeq G_{37}.$$

1.2. **Cyclotomic Hecke algebras.** All complex reflection groups are known to have "Coxeter-like" presentations [13]. Such presentations involve two kinds of relations: (i) "braid-like relations," which are homogeneous relations of certain form involving two or more generators, and (ii) "order relations," which simply specify the order of each generator. (In the Coxeter case, all generators have order 2.) Suppose W has such a presentation with generators $t_1, \ldots, t_r$ and order relations $t_1^{e_1} = \cdots = t_r^{e_r} = 1$.

Let $\mathcal{H}(W)$ be the algebra over the Laurent polynomial ring $\mathbb{Z}[u, u^{-1}]$ with generators $T_1, \ldots, T_r$, subject to the same braid-like relations as the generators of W, and to the following additional relations:

$$(T_i - u)(T_i^{e_i - 1} + T^{e_i - 1} + \cdots + 1) = 0 \qquad \text{for each } i \in \{1, \ldots, r\}.$$

Clearly, under the specialization $u \mapsto 1$, the latter relations become the order relations for W, and $\mathcal{H}(W)$ becomes the group ring $\mathbb{Z}W$. $\mathcal{H}(W)$ is known as the *spetsial cyclotomic Hecke algebra* for W. (For the generic cyclotomic Hecke algebra and its other specializations, see [13].) In case W is a Coxeter group, $\mathcal{H}(W)$ is simply its usual single-parameter Iwahori–Hecke algebra.

Assumption 1.1. *The spetsial cyclotomic Hecke algebra $\mathcal{H}(W)$ is a free $\mathbb{Z}[u, u^{-1}]$-module of rank $|W|$.*

This assumption is known to hold for all imprimitive complex reflection groups [4, 5] and many exceptional ones [11]. The importance of this assumption comes from the fact that it allows us to invoke the machinery of Tits' deformation theorem (see [21, Chap. 7]). In particular, over a sufficiently large field $\mathcal{K}$ containing $\mathbb{Z}[u, u^{-1}]$, the algebra

$$\mathcal{K}\mathcal{H}(W) = \mathcal{K} \otimes_{Z[u,u^{-1}]} \mathcal{H}(W)$$

is isomorphic to the group algebra $\mathcal{K}W$, so the set of irreducible representations of $\mathcal{K}\mathcal{H}(W)$ may be identified with those of W. The field $\mathcal{K}$ will be called a *splitting field* for $\mathcal{H}(W)$. According to [29], $\mathcal{K}$ may be taken to be the field of rational functions in some root of the indeterminate u.

The group algebra $\mathbb{Z}W$ admits a canonical *symmetrizing trace*, i.e., a linear function $t : \mathbb{Z}W \to \mathbb{Z}$ such that the bilinear form $(h_1, h_2) \mapsto t(h_1 h_2)$ is symmetric and nondegenerate. It is given by the formula $t(w) = \delta_{w,1}$ for $w \in W$. Similarly, $\mathcal{K}\mathcal{H}(W) \simeq \mathcal{K}W$ admits a symmetrizing trace $\tau_{\mathcal{K}} : \mathcal{K}\mathcal{H}(W) \to \mathcal{K}$. Such a trace is necessarily a class function on W, so we can write it as a linear combination of irreducible characters:

$$\tau_{\mathcal{K}} = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{c_\chi} \chi.$$

The elements $c_\chi \in \mathcal{K}$ are called *Schur elements*. In the imprimitive groups, it is known [31] that the trace $\tau_{\mathcal{K}}$ arises by extension of scalars from a trace $\tau : \mathcal{H}(W) \to \mathbb{Z}[u, u^{-1}]$, and that is conjectured (see [12]) to hold in general.

Related to Schur elements are the *generic degrees*, given by the formula

$$D_\chi(u) = P(W)/c_\chi.$$

Here, $P(W)$ denotes the *Poincaré polynomial* of W, given by

$$P(W) = (u - 1)^{-r} \prod_{i=1}^{r} (u^{d_i} - 1),$$

where $d_1, \ldots, d_r$ are the exponents of W. (The terminology comes from the fact that if W is a Weyl group, then when u is specialized to a prime power q, the generic degrees $D_\chi(q)$ give the dimensions of a certain irreducible representations of a reductive group over the field of q elements.)

1.3. **Spetsial complex reflection groups.** In general, generic degrees are elements of the splitting field $\mathcal{K}$, but for certain class of groups, including all Coxeter groups, they turn out to be polynomials in u. These groups, known as *spetsial* groups, have a number of equivalent characterizations [30, Proposition 8.1], and are in certain ways better-behaved than general complex reflection groups. The irreducible imprimitive spetsial complex reflection groups are those of the form

$$G(d,1,n) \qquad \text{or} \qquad G(d,d,n).$$

There are eighteen irreducible primitive spetsial complex reflection groups:

$$G_4, G_6, G_8, G_{14}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27}, G_{28}, G_{29}.G_{30}, G_{32}, G_{33}, G_{34}, G_{35}, G_{36}, G_{37}.$$

For a spetsial complex reflection group W, we attach an integer to each irreducible representation $\chi \in \mathrm{Irr}(W)$ by the formula

$$a(\chi) = \min\{i \mid u^i \text{ appears with nonzero coefficient in } D_\chi\}.$$

This quantity is known simply as the *a-invariant* of χ.

1.4. **The coinvariant algebra and j-induction.** For any character χ of W, we define the *fake degree* of χ to be the polynomial

$$R(\chi) = \frac{(u-1)^r}{|W|} P(W) \prod_{w \in W} \frac{\det_V(w)\chi(w)}{\det(u \cdot \mathrm{id}_V - w)}.$$

If χ is an irreducible character of W, this polynomial may be interpreted as the graded multiplicity of χ in the *coinvariant algebra* $C(W)$, which is the quotient of the symmetric algebra $S(V)$ by the ideal generated by the ring $S(V)^W_+$ of W-invariants of strictly positive degree. The latter is a homogeneous ideal, so $C(W)$ inherits a grading $C(W) = \bigoplus_i C^i(W)$ from $S(V)$. We then have

$$R(\chi) = \sum_i [C^i(W) : \chi] u^i.$$

The *b-invariant* of an irreducible character χ is given by

$$b(\chi) = \min\{i \mid [C^i(W) : \chi] \neq 0\} = \min\{i \mid u^i \text{ appears with nonzero coefficient in } R(\chi)\}.$$

Since $C^i(W)$ is a quotient of $S^i(W)$, any character χ appears with nonzero multiplicity in $S^{b(\chi)}(W)$.

Theorem 1.2 (see [21, Section 5.2]). *Let $W' \subset W$ be a reflection subgroup, and let $\chi' \in \mathrm{Irr}(W')$ be an irreducible representation. Assume that χ' occurs with multiplicity 1 in $S^{b(\chi')}(V)$. Then the smallest W-stable subspace of $S^{b(\chi)}(V)$ containing χ' affords an irreducible W-representation χ with the property that $b(\chi) = b(\chi')$. Moreover, χ occurs with multiplicity 1 in $S^{b(\chi)}(V)$.*

The representation χ is said to be obtained by *MacDonald–Lusztig–Spaltenstein induction* or *j-induction* from χ'; we write

$$\chi = j_{W'}^W \chi'.$$

The last statement in the theorem above enables us to repeat the j-induction operation when we have a chain of reflection subgroups $W'' \subset W' \subset W$. It is clear from the definition of this operation that

$$j_{W''}^W \simeq j_{W'}^W \circ j_{W''}^{W'}.$$

1.5. **Rouquier blocks and special representations.** Let $\mathcal{O}$ be the ring obtained from $\mathbb{Z}[u, u^{-1}]$ by inverting all elements of $1 + u\mathbb{Z}[u]$, i.e., all polynomials with constant term 1. The ring $\mathcal{O}\mathcal{H}(W) = \mathcal{O} \otimes_{\mathbb{Z}[u,u^{-1}]} \mathcal{H}(W)$ is not semisimple in general. Given a block of $\mathcal{O}\mathcal{H}(W)$, one can further extend scalars to $\mathcal{K}$ and then ask which irreducible representations of $\mathcal{K}\mathcal{H}(W)$ occur in that block. For Weyl groups, this question has been answered by Rouquier as follows.

Theorem 1.3 ([34]). *Assume that W is Weyl group. Then two representations in $\mathrm{Irr}(W)$ belong to the same two-sided cell if and only if the corresponding $\mathcal{K}\mathcal{H}(W)$-representations belong to the same block of $\mathcal{O}\mathcal{H}(W)$.*

For general complex reflection groups, there is currently no analogue of Kazhdan–Lusztig theory, and hence no way to define two-sided cells. Instead, we adopt the above theorem as a definition of a certain way of partitioning $\mathrm{Irr}(W)$.

Definition 1.4. The *families* in $\mathrm{Irr}(W)$ are the subsets characterized by the following property: two representations in $\mathrm{Irr}(W)$ belong to the same family if and only if the corresponding $\mathcal{KH}(W)$-representations belong to the same block of $\mathcal{OH}(W)$.

Families of characters have been studied by Broué–Kim [10], Kim [23], and Malle–Rouquier [32], and most recently by Chlouveraki [15, 16].

Definition 1.5. A representation $\chi \in \mathrm{Irr}(W)$ is *special* if $a(\chi) = b(\chi)$.

Special representations of Weyl groups play an important role in many aspects of geometric representation theory, appearing, for instance, in Lusztig's parametrization of unipotent characters of finite reductive groups [25], and in connection with special conjugacy classes in the unipotent variety [39]. A well-known property of the set of all special representations of a Weyl group is that exactly one occurs in each two-sided cell. An analogous statement holds for spetsial complex reflection groups in general:

Theorem 1.6 ([32, Théorème 5.3]). *Assume that W is spetsial. Then each family of* $\mathrm{Irr}(W)$ *contains a unique special representation.*

This statement can fail in nonspetsial groups: see [32, Exemple 5.5] for a family in the nonspetsial group G_5 containing no special representation.

2. Springer Theory for Algebraic Groups

2.1. **Overview.** Let G be a semisimple algebraic group over $\mathbb{C}$ (or any algebraically closed field of good characteristic), and let $T \subset G$ be a maximal torus. Let W denote the Weyl group of G, and L the root lattice, both with respect to T. Then W is naturally a reflection group acting on the complex vector space $V = \mathbb{C} \otimes_{\mathbb{Z}} L$.

Let $\mathcal{U}$ denote the variety of unipotent elements in G, and let Υ denote the set of pairs (C, E), where $C \subset \mathcal{U}$ is a conjugacy class, and E is a G-equivariant local system on C. Recall that the *Springer correspondence* is a certain injective map

$$\nu : \mathrm{Irr}(W) \to \Upsilon.$$

One way of defining the Springer correspondence is as follows: let $\mathcal{B}$ denote the variety of Borel subgroups of G, and let $\tilde{\mathcal{U}} = \{(x, B) \in \mathcal{U} \times \mathcal{B} \mid x \in B\}$. This is a smooth variety, and the obvious map $\pi : \tilde{\mathcal{U}} \to \mathcal{U}$ is a resolution of singularities, known as the *Springer resolution*. The derived push-forward of the constant sheaf $R\pi_*\underline{\mathbb{C}}$ is a semisimple perverse sheaf on $\mathcal{U}$, so it has the general form

$$R\pi_*\underline{\mathbb{C}} = \bigoplus_{(C,E)\in\Upsilon} \mathrm{IC}(C, E) \otimes V_{C,E}, \tag{2.1}$$

where the $V_{C,E}$ are various finite-dimensional vector spaces. Following Borho–MacPherson [6], the $V_{C,E}$ carry actions of W. Moreover, the nonzero $V_{C,E}$ carry irreducible representations of W, and every irreducible representation occurs as exactly one $V_{C,E}$. The Springer correspondence ν is defined by matching each element of $\mathrm{Irr}(W)$ with the vector space $V_{C,E}$ on which it is realized. (Because some $V_{C,E}$ may vanish, the Springer correspondence is not, in general, surjective.)

2.2. **Enumerating unipotent classes.** Let $\Upsilon_0 \subset \Upsilon$ be the set of pairs (C, E) with E trivial. Of course, Υ_0 may be thought of simply as the set of unipotent classes. It turns out that Υ_0 is always contained in the image of the Springer correspondence. Moreover, the representations appearing in $\nu^{-1}(\Upsilon_0)$ admit an elementary description in terms of certain subgroups of W.

A subgroup $W' \subset W$ is called *pseudoparabolic* if it is generated by reflections corresponding to a proper subset of the extended Dynkin diagram of type dual to G. Equivalently, one may consider the Langlands dual group $G^\vee$, together with the dual maximal torus $T^\vee \subset G^\vee$. A subgroup $H^\vee \subset G^\vee$ is called an *endoscopic group* (for G) if it is the identity component of the centralizer of some element of $T^\vee$. Endoscopic subgroups are automatically reductive. A subgroup $W' \subset W$ is pseudoparabolic if and only if it is the Weyl group of some endoscopic group.

The following theorem describes the relationship between pseudoparabolic subgroups and unipotent classes.

Theorem 2.1 (see [14]). *The following two conditions on a representation $\chi \in \mathrm{Irr}(W)$ are equivalent:*

1. $\nu(\chi) \in \Upsilon_0$.
2. $\chi \simeq j_{W'}^{W}\chi'$, *where $W' \subset W$ is some pseudoparabolic subgroup, and $\chi' \in \mathrm{Irr}(W')$ is a special representation of W'.*

Definition 2.2. A representation $\chi \in \mathrm{Irr}(W)$ satisfying the equivalent conditions of Theorem 2.1 is called a *Springer representation.*

By taking $W' = W$ in the second part of the theorem above, we obtain the following.

Corollary 2.3. *Every special representation of W is a Springer representation.*

2.3. **Green functions and the Lusztig–Shoji algorithm.** In considering the perverse sheaf (2.1), a natural problem is the determination of the stalks of the various simple perverse sheaves $\mathrm{IC}(C, E)$. We will describe these stalks in the following way. Given another pair $(C', E') \in \Upsilon$, consider the object $\mathrm{IC}(C, E)|_{C'}$. This is a certain complex of sheaves whose cohomology sheaves are local systems on C'. We may then ask what the multiplicity of the irreducible local system E' in the local system $\mathcal{H}^i(\mathrm{IC}(C, E)|_{C'})$ is. Finding all such multiplicities is equivalent to determining the polynomials

$$\Pi_{(C,E),(C',E')}(u) = \sum_i [\mathcal{H}^i(\mathrm{IC}(C, E)|_{C'}) : E']u^{i/2}. \tag{2.2}$$

(It is known that $\mathcal{H}^i(\mathrm{IC}(C, E)) = 0$ for odd i, so these are indeed polynomials in u.) These polynomials are called *Green functions.* Most of them can be computed using only elementary linear algebra, by a method which we now describe. Let

$$\Upsilon' = \text{the image of } \nu \subset \Upsilon.$$

According to [26, Theorem 24.8(c)],

$$\Pi_{(C,E),(C',E')} = 0 \qquad \text{if } (C, E) \in \Upsilon' \text{ but } (C', E') \notin \Upsilon'.$$

We henceforth restrict our attention to those $\Pi_{(C,E),(C',E')}$ with $(C, E), (C', E') \in \Upsilon'$. The next two theorems together enable us to effectively compute these polynomials.

Theorem 2.4 (see [20, Section 2]). *Let W be a complex reflection group acting on a vector space V, and suppose $\mathrm{Irr}(W)$ is equipped with a fixed partition into an ordered collection of disjoint subsets:*

$$\mathrm{Irr}(W) = \mathcal{C}_1 \sqcup \cdots \mathcal{C}_n.$$

Let $b(\mathcal{C}_i) = \min\{b(\chi) \mid \chi \in \mathcal{C}_i\}$. Define a matrix $\Omega = (\omega_{\chi,\chi})_{\chi,\chi' \in \mathrm{Irr}(W)}$ by

$$\omega_{\chi,\chi'} = u^{N^*} R(\chi \otimes \chi' \otimes \overline{\det}),$$

where N^* is the number of reflections in W, and $\overline{\det}$ denotes the complex conjugate of the determinant character of W. Then there is a unique pair of matrices $P = (P_{\chi,\chi'})_{\chi,\chi' \in \mathrm{Irr}(W)}$, $\Lambda = (\Lambda_{\chi,\chi'})_{\chi,\chi' \in \mathrm{Irr}(W)}$ with entries in $\mathbb{Q}(u)$ satisfying the matrix equation

$$P \Lambda P^t = \Omega$$

and subject to following additional conditions:

$$P_{\chi,\chi'} = \begin{cases} 0 & \text{if } \chi \in \mathcal{C}_i,\ \chi' \in \mathcal{C}_j \text{ with } i < j, \\ u^{b(\mathcal{C}_i)} & \text{if } \chi, \chi' \in \mathcal{C}_i, \end{cases} \tag{2.3}$$
$$\Lambda_{\chi,\chi'} = 0 \qquad \text{if } \chi \in \mathcal{C}_i,\ \chi' \in \mathcal{C}_j \text{ with } i \neq j.$$

The proof of this theorem is elementary and consists mostly of a description of a procedure for producing the matrices P and Λ. That procedure is known as the *Lusztig–Shoji algorithm.*

When W is the Weyl group of a reductive algebraic group, a particular class of ordered partitions of $\mathrm{Irr}(W)$ arises naturally in connection with the Springer correspondence. Let us say that a partition

$$\mathrm{Irr}(W) = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_n$$

is *of Springer type* if the following two conditions hold:

1. Two representations χ and χ' belong to the same $\mathcal{C}_i$ if and only if the Springer correspondence attaches them both to local systems on the same unipotent class. (Thus, there is a bijection between the collection of subsets $\{\mathcal{C}_i\}$ and the set of unipotent classes.)
2. Suppose that $\mathcal{C}_i$ corresponds to $C \subset \mathcal{U}$ and $\mathcal{C}_j$ to $C' \subset \mathcal{U}$. If $C' \subset \overline{C}$, then $j \leq i$.

The second condition simply says that the total order on the $\mathcal{C}_i$ refines the closure partial order on unipotent classes.

Theorem 2.5 ([26, Theorem 24.8]). *Let W be the Weyl group of a reductive algebraic group, and assume that $\mathrm{Irr}(W)$ is equipped with an ordered partition into disjoint subsets of Springer type. Let P and Λ be the matrices resulting from the Lusztig–Shoji algorithm.*

(1) *We have* $P_{\chi,\chi'}(u) = \Pi_{\nu(\chi),\nu(\chi')}(u)$. *In particular, the entries of* P *lie in* $\mathbb{Z}[u]$ *and have nonnegative coefficients.*
(2) *The entries of* Λ *also lie in* $\mathbb{Z}[u]$.

Note that the conditions (2.3) in Theorem 2.4 say that the matrix P is upper-triangular (no condition is imposed if $\chi \in \mathcal{C}_i$, $\chi' \in \mathcal{C}_j$ with $i > j$) and that Λ is block-diagonal. The former corresponds to the fact that the stalks of $\mathrm{IC}(C, E)$ vanish outside $\overline{C}$, and the latter is related to an interpretation in [26, Theorem 24.8] of the entries of Λ in terms of inner products of certain characteristic functions supported on a single unipotent class.

3. Pseudoparabolic Subgroups of Complex Reflection Groups

In the preceding section, we saw how to reduce the determination of unipotent classes and the calculation of Green functions into elementary calculations in terms of the Weyl group, via Theorems 2.1 and 2.5. Our aim is to carry out analogous calculations for complex reflection groups, in the hope that the results describe some as-yet unknown "unipotent variety" in that case as well. However, those calculations require some auxiliary data

(1) A suitable notion of "pseudoparabolic subgroup," allowing us to adopt part of Theorem 2.1 to define Springer representations (cf. Definition 2.2) of complex reflection groups.
(2) A way of partitioning $\mathrm{Irr}(W)$ into an ordered collection of disjoint subsets satisfying appropriate axioms, enabling us to carry out the algorithm of Theorem 2.4.

The former was studied in [3]; this is the subject of the present section. The latter, which is much less well understood, will be treated in the next section.

3.1. **Root lattices and stabilizers.** Many ideas in this section and the following one depend not only on a complex reflection group W, but also on the choice of a *root lattice* L in the sense of Nebe [33]. This phenomenon is to be expected, as it already occurs in the realm of algebraic groups: groups of types B_n and C_n have isomorphic Weyl groups but inequivalent root lattices and different Springer correspondences.

Nebe's definition depends on the fact that every reflection group over $\mathbb{C}$ can actually be realized over a much smaller field $K \subset \mathbb{C}$; in fact, K can be taken to be a finite abelian extension of $\mathbb{Q}$. (For a table of the minimal fields over which various complex reflection groups can be realized, see [13].) Inside K, we have the ring of integers $\mathbb{Z}_K$, and we may consider $\mathbb{Z}_K$-lattices inside K-vector spaces.

Definition 3.1. Let W be a complex reflection group, acting on the vector space V. Assume that W can be realized over the abelian number field K. A *root lattice* is a W-stable $\mathbb{Z}_K$-submodule $L \subset V$ such that $V \simeq C \otimes_{\mathbb{Z}_K} L$, and such that L is spanned by the W-orbit of one element.

(In Nebe's terminology, these were called *primitive* root lattices; general root lattices were not required to be spanned by the W-orbit of a single element. Here, however, all root lattices will be assumed to be primitive.) The root lattices for all irreducible complex reflection groups have been classified by Nebe. Each irreducible primitive spetsial complex reflection group other than G_6, G_{26}, and G_{28} admits a unique root lattice. By an abuse of notation and language, we will usually write, for instance, "G_{14}" instead of "the pair (G_{14}, L) where L is the unique root lattice." The group $G_{28} = F_4$ admits two root lattices; they are exchanged by the automorphism of F_4 which swaps long and short roots.

The groups G_6 and G_{26} also admit two isomorphism classes of root lattices each. In each case, Nebe [33] has denoted one of the root lattices L_1 and the other L_2. Continuing the abuse of notation, we will simply write "G_6" and "G_{26}" to refer to the pairs (G_6, L_1) and (G_{26}, L_1). The pairs (G_6, L_2) and (G_{26}, L_2) will be abbreviated G_6' and G_{26}', respectively.

Thus, from the viewpoint of Springer theory, there are 21 cases to study among the irreducible primitive spetsial complex reflection groups. In the sequel, we will omit the Weyl groups E_6, E_7, E_8, and F_4 (with its two root lattices) from the discussion, since there is nothing new to say about their Springer theory, and we will focus on the remaining 16 cases:

$$G_4, G_6, G_6', G_8, G_{14}, G_{23}, G_{24}, G_{25}, G_{26}, G_{26}', G_{27}, G_{29}.G_{30}, G_{32}, G_{33}, G_{34}.$$

3.2. **Stabilizers of torus points.** Recall that if W is the Weyl group of an algebraic group G, then a subgroup $W' \subset W$ is pseudoparabolic if and only if it is the Weyl group of the centralizer in the dual group $G^\vee$ of a point of the dual torus $T^\vee$. Of course, W itself acts on $T^\vee$. We can bypass the notions of Weyl groups and centralizers in $G^\vee$ and observe simply that $W' \subset W$ is pseudoparabolic if and only if it is the stabilizer in W of a point of $T^\vee$. Finally, we note that over $\mathbb{C}$, the torus $T^\vee$ can be identified in a

W-equivariant way with V/L, where L is the root lattice of W and $V = \mathbb{C} \otimes_{\mathbb{Z}} L$. That last observation is a statement that makes sense for general complex reflection groups, and it seems reasonable to adopt it as a definition.

Approximate Definition 3.2. Roughly, a subgroup W' of a complex reflection group W should be called *pseudoparabolic* with respect to the root lattice L if it is the stabilizer of some point of V/L.

Unfortunately, many things can go wrong if this is taken as a literal definition: pseudoparabolic subgroups may fail to be reflection subgroups (so the Hecke algebra and special representations may be undefined), and even when they are reflection groups, they may fail to be spetsial (so special representations may not behave as expected). However, these problems turn out not to be very serious: instead of taking the full stabilizer of a point of V/L, one constructs from it a certain large spetsial reflection subgroup, and calls that group "pseudoparabolic." For the full definition, the reader is referred to [3, Section 8].

3.3. **Finding pseudoparabolic subgroups.** The determination of all pseudoparabolic subgroups (with respect to any root lattice) in all imprimitive spetsial complex reflection groups was carried out in [3], and the results appear there as Théorèmes 8.11 and 8.15. A typical pseudoparabolic subgroup of $G(d,1,n)$ is a product of various subgroups of the form $G(d,1,m)$, $G(d,d,m)$, and $G(1,1,m)$ with $m \le n$, subject to various constraints depending on d and on the choice of root lattice. Pseudoparabolic subgroups of $G(d,d,n)$ with $n \ge 3$ are similar, although no factors of type $G(d,1,n)$ may appear in that case.

The dihedral groups $G(d,d,2)$ behave somewhat differently from the other imprimitive complex reflection groups, mainly because they can be defined over fields of the form $K = \mathbb{Q}(\zeta_d + \zeta_d^{-1})$ (where ζ_d is a primitive d-th root of unity) rather than over $\mathbb{Q}(\zeta_d)$. The rank-2 pseudoparabolic subgroups of $G(d,d,2)$ are precisely the smaller dihedral groups $G(p^k,p^k,2)$ where p is a prime and p^k divides d.

Finally, for the primitive spetsial complex reflection groups, the determination of pseudoparabolic subgroups has been done by computer, using the CHEVIE package for the GAP computer algebra system [18]. Given a point in V/L, it is straightforward to identify the associated pseudoparabolic subgroup of W. However, to find all pseudoparabolic subgroups in this way, we must show how to reduce the problem of checking all points of V/L to that of checking a finite number of points. The next two subsections describe this reduction. The results of the calculations will be given in Section 3.6.

3.4. **Maximal-rank pseudoparabolic subgroups.** We begin by observing that any pseudoparabolic subgroup is contained in some parabolic subgroup of the same rank. In other words, the list of all pseudoparabolic subgroups of W is simply the union of the lists of maximal-rank pseudoparabolic subgroups of all parabolic subgroups of W. The parabolic subgroups of all complex reflection groups are known (see [13], for instance), so we have reduced the problem of finding all pseudoparabolic subgroups to that of finding all those of maximal rank. In the remainder of this subsection, we show that there exists a finite set of points $P \subset V/L$ such that any maximal-rank pseudoparabolic subgroup arises from some point of P.

Following Nebe [33], we can associate to W a *root system* $R \subset L$. Such a root system consists of a W-stable finite set of $\mathbb{Z}_K^*$-orbits of vectors (called *roots*), with one such orbit for each cylic reflection subgroup in W, subject to a certain integrality condition. (Since $\mathbb{Z}_K^*$ may be infinite, R may be infinite as well.) Specifically, if α is a root for the reflection s, let $\alpha^\vee \in V^*$ be the element such that

$$s(x) = x - \langle \alpha^\vee, x\rangle \alpha.$$

The integrality condition states that $\langle \alpha^\vee, \beta\rangle \in \mathbb{Z}_K$ for all $\beta \in R$, and hence for all $\beta \in L$.

Now, let $x \in V$ be a point whose image in V/L gives rise to a pseudoparabolic subgroup W' of maximal rank in W. For any reflection $s \in W'$, we must have

$$x - s(x) = \langle \alpha^\vee, x\rangle \alpha \in L,$$

and hence

$$\langle \beta^\vee, x - s(x)\rangle = \langle \alpha^\vee, x\rangle \langle \beta^\vee, \alpha\rangle \in \mathbb{Z}_K.$$

Let $\mathfrak{p}_\alpha \subset \mathbb{Z}_K$ be the ideal generated by the elements $\langle \beta^\vee, \alpha\rangle$ as β ranges over all roots. Then $\langle \alpha^\vee, x\rangle$ belongs to the fractional ideal $\mathfrak{p}_\alpha^{-1} \subset K$. Next, let $\mathfrak{q}_\alpha \subset \mathbb{Z}_K$ be the ideal that is the image of $\langle \alpha^\vee, \cdot\rangle : L \to \mathbb{Z}_K$. We clearly have $\mathfrak{q}_\alpha \subset \mathfrak{p}_\alpha^{-1}$. Choose a set of coset representatives $a_1, \dots, a_t$ in $\mathfrak{p}_\alpha^{-1}$ for the finite group $\mathfrak{p}_\alpha^{-1}/\mathfrak{q}_\alpha$. By replacing x by a suitable element of $x + L$, we may assume that x lies on one of the finitely many affine hyperplanes defined by equations of the form

$$\langle \alpha^\vee, \cdot\rangle = a_i \qquad (i \in \{1, \dots, t\}). \tag{3.1}$$

We could repeat this process beginning with any other reflection preserving $x + L$, and thereby achieve that x simultaneously lies in on various hyperplanes of the form (3.1) corresponding to different roots. Because W' is assumed to have maximal rank, there exists a set of reflections in W' whose associated roots span V, so we may insist that x belong to the following set:

$$\tilde{P} = \left\{ x \;\middle|\; \begin{array}{c} x \text{ satisfies equations of the form (3.1)} \\ \text{for all } \alpha \text{ in some set of roots spanning } V \end{array} \right\}. \tag{3.2}$$

It is clear that $\tilde{P}$ is finite. Taking P to be the image of $\tilde{P}$ in V/L, we have established the following.

Proposition 3.3. *There is a finite set $P \subset V/L$ such that every maximal-rank pseudoparabolic subgroup of W is associated to some point of P.*

3.5. **Cartan integers for complex reflection groups.** As noted earlier, it is straightforward to identify explicitly the pseudoparabolic subgroup $W' \subset W$ associated to a point of V/L. The preceding proposition tells us that it suffices to check points in a finite set P, but to do the calculation by computer, we first need an algorithmic means of listing the points of P. One sees from the definition of P that it suffices to know all possible values of $\langle \alpha^\vee, \beta\rangle$, and up to multiplication by a unit in $\mathbb{Z}_K$, there are only finitely many such values for a fixed reflection group and root system. By analogy with the Weyl group case, we call the quantities $\langle \alpha^\vee, \beta\rangle$ *Cartan integers* (of course, they are now algebraic integers in general, and not necessarily elements of $\mathbb{Z}$).

Let s and t be reflections, with corresponding roots α and β in some root system. To determine the possible Cartan integers, we first consider the following related elements of $\mathbb{Z}_K$:

$$N_{s,t} = \langle \alpha^\vee, \beta\rangle\langle \beta^\vee, \alpha\rangle.$$

As the notation suggests, $N_{s,t}$ depends only on the reflections s and t, and not on the choice of roots α and β, or even on the choice of root system. (To see this, note that if we replace, say, α by another root $c\alpha$ with $c \in K^*$, then we must also replace $\alpha^\vee$ by $(c\alpha)^\vee = \bar{c}^{-1}\alpha^\vee$.)

Of course, $N_{s,t}$ also remains unchanged if we replace s by another reflection with the same root. We may thus assume that s and t have the eigenvalue property appearing in the following definition.

Definition 3.4. A triple of positive integers (a, b, l) is called *admissible* if there exist reflections s and t of some complex vector space V whose nontrivial eigenvalues are $e^{2\pi i/a}$ and $e^{2\pi i/b}$, respectively, and which satisfy the "braid relation"

$$\underbrace{sts\cdots}_{l \text{ factors}} = \underbrace{tst\cdots}_{l \text{ factors}},$$

but do not satisfy any shorter braid relation.

The usefulness of this notion lies in the fact that [1, Proposition 3.9] gives us a formula for $N_{s,t}$ just in terms of the admissible triple determined by s and t. The paper [1] also gives a classification of all admissible triples (see also [22]), and from [1, Table 1] and [13], it is easy to read off the list of admissible triples occurring in primitive spetsial complex reflection groups. (There are other admissible triples that occur only in imprimitive or nonspetsial groups.)

This list is given Table 1, together with the corresponding values of $N_{s,t}$. The third column records the norm of $N_{s,t}$ over $\mathbb{Q}$: we see that in these cases, $N_{s,t}$ is either a unit or else the generator of a power of a prime ideal over (2) or (3) $\subset \mathbb{Z}$. The last column of the table gives the smallest extension of $\mathbb{Q}$ over which a given admissible triple may be realized.

We now return to the problem of determining the possible Cartan integers. Any Cartan integer $\langle \alpha^\vee, \beta\rangle$ is a divisor (in $\mathbb{Z}_K$) of some $N_{s,t}$, and from the list in Table 1, we see that every such $N_{s,t}$ is a divisor of either 2 or 3. The list of all possible Cartan integers (up to multiplication by a unit) can then be obtained simply by writing down the factorizations of the numbers 2 and 3 in the ring $\mathbb{Z}_K$. (As noted in the proof of [33, Corollary 13], these rings are all unique factorization domains.) These factorizations are given in Table 2.

3.6. **Determination of pseudoparabolic subgroups of primitive spetsial groups.** We are now ready to put everything together into an algorithm. For a fixed spetsial complex reflection group W defined over the number field K, together with a fixed Nebe root lattice L over $\mathbb{Z}_K$, we look up the factorizations of 2 and 3 in $\mathbb{Z}_K$ in Table 2. That list of factors is equivalent to the list of ideals appearing in the union below:

$$\tilde{P} = \bigcup_{\substack{\mathfrak{p} \subset \mathbb{Z}_K \text{ an ideal such} \\ \text{that } (2) \subset \mathfrak{p} \text{ or } (3) \subset \mathfrak{p}}} \mathfrak{p}^{-1}L/L \subset V/L.$$

Admissible triples	$N_{s,t}$	$\text{Norm}(N_{s,t})$	*Field*
$(2,2,3)$	1	1	$\mathbb{Q}$
$(2,2,4)$	2	2	$\mathbb{Q}$
$(2,2,5)$	$\frac{3+\sqrt{5}}{2}$	1	$\mathbb{Q}(\sqrt{5})$
$(2,3,4)$	$1-\omega$	3	$\mathbb{Q}(\omega)$
$(2,3,6)$	$\frac{\omega(1-i)}{1-\xi}, \frac{1-i}{1+\xi^{-1}}$	4	$\mathbb{Q}(\xi)$
$(2,3,8)$	$1-\omega+\xi\sqrt{2}$	1	$\mathbb{Q}(\omega,\sqrt{-2})$
$(3,3,3)$	$-\omega$	1	$\mathbb{Q}(\omega)$
$(3,3,4)$	-2ω	4	$\mathbb{Q}(\omega)$
$(4,4,3)$	$-i$	1	$\mathbb{Q}(i)$

TABLE 1. Admissible triples in primitive spetsial complex reflection groups. *Notation*: $\omega = e^{2\pi i/3}$, $\xi = e^{2\pi i/12}$.

field	groups	2	3
$\mathbb{Q}$	G_{33}, G_{34}	2	3
$\mathbb{Q}(\sqrt{5})$	G_{23}, G_{30}	2	3
$\mathbb{Q}(\sqrt{-7})$	G_{24}	$(\frac{1+\sqrt{-7}}{2})(\frac{1-\sqrt{-7}}{2})$	3
$\mathbb{Q}(\omega)$	$G_4, G_{25}, G_{26}, G_{32}$	2	$-\omega^2(1-\omega)^2$
$\mathbb{Q}(\omega,\sqrt{-2})$	G_{14}	$(\sqrt{-2})^2$	$-\omega^2(\omega+\sqrt{-2})^2(\omega-\sqrt{-2})^2$
$\mathbb{Q}(\omega,\sqrt{5})$	G_{27}	$(\omega+\frac{1+\sqrt{5}}{2})(\omega+\frac{1-\sqrt{5}}{2})$	$-\omega^2(1-\omega)^2$
$\mathbb{Q}(i)$	G_8, G_{29}	$i(1-i)^2$	3
$\mathbb{Q}(\xi)$	G_6	$i(1-i)^2$	$-\omega^2(1-\omega)^2$

TABLE 2. Factorizations of 2 and 3 in various number fields

Group	*Pseudoparabolics*
G_4	none
G_6	$A_1 \times A_1, G_4$
G'_6	B_2
G_{14}	$A_1 \times A_1, \mathfrak{C}_3 \times \mathfrak{C}_3, A_2, B_2, G(3,1,2), G_4$
$G_{23} = H_3$	$A_1 \times A_1 \times A_1$
G_{24}	$A_1 \times B_2, A_3, B_3$
G_{25}	$\mathfrak{C}_3 \times \mathfrak{C}_3 \times \mathfrak{C}_3$
G_{26}	$A_1 \times G_4, G(3,1,3)$
G'_{26}	$A_1 \times G_4, \mathfrak{C}_3 \times G(3,1,2), G_{25}$
G_{27}	$A_3, B_3, H_3, G(3,3,3)$
G_{29}	$A_1 \times A_3, A_1 \times B_3, A_4, B_4, D_4, G(4,4,4)$
$G_{30} = H_4$	$A_1 \times H_3, A_2 \times A_2, I_2(5) \times I_2(5), A_4, D_4$
G_{32}	$\mathfrak{C}_3 \times G_{25}, G_4 \times G_4$
G_{33}	$A_1 \times D_4, A_5$
G_{34}	$A_1 \times G_{33}, A_6, D_6, E_6, G(3,3,6)$

TABLE 3. Maximal-rank pseudoparabolic subgroups in primitive spetsial complex reflection groups

$\bar{P}$ is a finite set containing the set P defined following (3.2). For each point of $\bar{P}$, we can then find the associated pseudoparabolic subgroup by direct computation. We know from Proposition 3.3 that every maximal-rank pseudoparabolic subgroup arises in this way. The results of this calculation are given in Table 3. (Note the converse is not true: some points of $\bar{P}$ may give rise to pseudoparabolic subgroups that are not of maximal rank. Those groups have been omitted from Table 3.) In this table, as in Section 1, $\mathfrak{C}_d$ denotes the cyclic group of order d, and any direct factor that happens to be a Coxeter group is written with its usual Coxeter name for brevity.

	$\phi_{1,4}$	$\phi_{3,2}$	$\phi_{2,1}$	$\phi_{1,0}$
G_4	$\phi_{1,4}$	$\phi_{3,2}$	$\phi_{2,1}$	$\phi_{1,0}$
$\mathfrak{C}_3$			χ_3	(1)

	$\phi_{1,10}$	$\phi''_{2,5}$	$\phi_{3,4}$	$\phi_{2,1}$	$\phi_{3,2}$	$\phi_{1,4}$	$\phi_{1,0}$
G_6	$\phi_{1,10}$	$\phi''_{2,5}$	$\phi_{3,4}$	$\phi_{2,1}$			$\phi_{1,0}$
A_1				(1^2)			(2)
$\mathfrak{C}_3$				χ_3			(1)
$A_1 \times A_1$				$(1^2) \boxtimes (2)$; $(2) \boxtimes (1^2)$	$(1^2) \boxtimes (1^2)$		$(2) \boxtimes (2)$
G_4				$\phi_{2,1}$	$\phi_{3,2}$	$\phi_{1,4}$	$\phi_{1,0}$

	$\phi_{1,10}$	$\phi''_{2,5}$	$\phi_{3,4}$	$\phi_{2,1}$	$\phi_{1,0}$
G'_6	$\phi_{1,10}$	$\phi''_{2,5}$	$\phi_{3,4}$	$\phi_{2,1}$	$\phi_{1,0}$
A_1				(1^2)	(2)
$\mathfrak{C}_3$				χ_3	(1)
B_2			$(\varnothing; 1^2)$	$(1;1)$	$(2;\varnothing)$

TABLE 4. Springer representations in G_4, G_6, and G'_6

4. Springer Correspondences for Complex Reflection Groups

4.1. **Springer representations.** Following the usual philosophy for generalizing concepts from Weyl groups to other complex reflection groups, we adopt the second part of Theorem 2.1 as a definition:

Definition 4.1. Given a spetsial complex reflection group W and a root lattice L, we say that $\chi \in \mathrm{Irr}(W)$ is a *Springer representation* if it is of the form $\chi \simeq j_{W'}^{W}\chi'$ for some pseudoparabolic subgroup $W' \subset W$ and some special representation $\chi' \in \mathrm{Irr}(W')$.

The list of special representations has been determined by Broué–Kim [10] in the imprimitive case and by Malle–Rouquier [32] in the primitive case. Using those results, it is easy to compute the list of all Springer representations. The results in the small examples of G_4, G_6, and G'_6 are given in Table 4. In each table, the list of all pseudoparabolic subgroups (not just the maximal-rank ones) appears on the left-hand side, and the list of all Springer representations appears along the top. The interior of the table encodes j-induction: a special representation χ' appears in the row labelled by the subgroup W' and the column labelled by the representation χ exactly when $\chi \simeq j_{W'}^{W}\chi'$.

The horizontal dividing line in the tables for G_6 and G'_6 separates parabolic subgroups (above the line) from pseudoparabolic subgroups that are *not* parabolic (below). (Recall that G_4 has no pseudoparabolic subgroups of latter kind.) The vertical lines separate the Springer representations by families. In G_4 and G'_6, every Springer representation is special, so they all lie in distinct families. In contrast, in G_6, there are two nonspecial Springer representations, both in the same family as the special representation $\phi_{2,1}$.

The notation for representations of primitive complex reflection groups follows that of [32]. The general principle is that $\phi_{r,s}$ is an irreducible representation of dimension r and b-invariant s. When these two properties fail to uniquely characterize a representation, the various representations with the same dimension and b-invariant may be denoted, for instance, $\phi'_{r,s}$ and $\phi''_{r,s}$. When a classical-type Weyl group occurs, its representations are labelled by partitions or bipartitions as in [14].

Unfortunately, it would be impractical to reproduce such tables of j-induction data here for most larger primitive spetsial complex reflection groups: G_{34}, for instance, has fifty-three Springer representations. However, the list of Springer representations themselves will appear in Section 4.3.

Finally, we remark briefly on what happens in the imprimitive case. Recall that in classical-type Weyl groups, the Springer correspondence can be described with the aid of combinatorial objects called *symbols* and *u-symbols*. (These are certain arrays of nonnegative integers related to partitions.) The set of all symbols and the set of all u-symbols are both in bijection with $\mathrm{Irr}(W)$, and a representation $\chi \in \mathrm{Irr}(W)$ is a Springer representation if and only if its corresponding u-symbol is *distinguished* (an elementary combinatorial property).

There is a generalization of the notion of u-symbols, due to Malle [27], that is adapted to discussing the irreducible representations of $G(d,1,n)$ or $G(d,d,n)$. This notion was used by Shoji [36, 37] in his study of Green functions for these groups, and implicit in that work was the idea that the representations corresponding to distinguished generalized symbols should be thought of as "Springer representations."

However, the combinatorial set-up of [36, 37] has no obvious generalization to primitive complex reflection groups. One of the aims of the paper [3] was to give "intrinsic" descriptions of some of the combinatorial notions in [36, 37], with a view to generalizing Shoji's work to all spetsial groups. For Springer representations, this is achieved with the following result.

Theorem 4.2 ([3, Théorème 8.11]). *The Springer representations of $G(d,1,n)$ or of $G(d,d,n)$ are precisely those corresponding to distinguished generalized symbols.*

4.2. **Springer correspondences and Green functions.** In the work of Shoji [36, 37] mentioned above, his main aim was the study of Green functions. The essential idea here is simply to run the algorithm from Theorem 2.4 and see how the output behaves. That algorithm requires some input data: namely, an ordered partitioning of $\mathrm{Irr}(W)$ into disjoint subsets. Recall that in the Weyl group case, this data comes from the Springer correspondence, which for classical groups can be encoded with the combinatorics of u-symbols. Specifically, two Weyl group representations are attached to the same unipotent class by the Springer correspondence if and only if their u-symbols are *similar*. By analogy, in his study of $G(d,1,n)$ and $G(d,d,n)$, Shoji partitioned $\mathrm{Irr}(W)$ by similarity classes of generalized symbols.

The natural question to ask is: can one give an "intrinsic," noncombinatorial description of this partition of $\mathrm{Irr}(W)$ that could then be applied to primitive complex reflection groups as well? In other words, we are seeking to do for similarity classes of symbols what Definition 4.1 and Theorem 4.2 do for distinguished symbols. Such a way of partitioning $\mathrm{Irr}(W)$ should be regarded as a generalization of the Springer correspondence for algebraic groups.

This question remains largely open. In this section, we discuss desiderata for a solution to the question, as well as computational examples among primitive groups.

We begin by making a few observations about the Springer correspondence for an algebraic group. It is known [14] that among the representations attached to a given unipotent class, the one attached to the trivial local system (i.e., the Springer representation) is the unique one with minimal b-invariant. Moreover, the b-invariant of that representation is exactly half the codimension of that unipotent class in the full unipotent variety. We have previously noted that every special representation is a Springer representation. It turns out that every nonspecial representation must be attached to a unipotent class in the closure of the one corresponding to the unique special representation in the same family. As we saw in Theorem 2.5, the entries of the matrices P and Λ produced by the Lusztig–Shoji algorithm, a priori only rational functions of u, are actually polynomials. Furthermore, because the entries of P describe stalks of simple perverse sheaves, they have nonnegative coefficients, and they obey a bound on the degrees of terms that may appear, coming from cohomological degree bounds on perverse sheaves.

We now remove the underlying algebraic group from the preceding paragraph, and adopt the list of observations as a definition.

Definition 4.3 (cf. [2, Definition 3]). Given an ordered partition

$$\mathrm{Irr}(W) = C_1 \sqcup \cdots \sqcup C_n, \tag{4.1}$$

let P and Λ denote the output of the Lusztig–Shoji algorithm. The partition (4.1) is called an *abstract Springer correspondence* for W if the following properties hold:

(1) Each C_i contains a unique Springer representation χ_i. Moreover, for any $\chi \in C_i$, $\chi \neq \chi_i$, we have $b(\chi) > b(\chi_i)$.
(2) If $i < j$, then $b(C_i) \geq b(C_j)$.
(3) Suppose $\chi \in C_i$ is a special representation. If $\chi' \in C_j$ belongs to the same family as χ, then $j \leq i$.
(4) P and Λ have entries in $\mathbb{Z}[u]$, and in addition, all coefficients of entries of P are nonnegative.
(5) If $\chi \in C_i$, then $P_{\chi,\chi'}(u)$ is divisible by $u^{b(C_i)}$ for all $\chi' \in \mathrm{Irr}(W)$.

This definition should perhaps be regarded as preliminary. It is certainly satisfied by the actual Springer correspondences for algebraic groups, but it has somewhat of an ad hoc flavor. It is too weak to imply a general uniqueness statement, but it is also too restrictive: as we will see below, condition (5) may be unreasonable for general complex reflection groups. The best result so far is for dihedral groups.

Theorem 4.4 ([2, Theorem 2]). *Let W be a dihedral group $G(d,d,2)$. If d is odd, W admits a unique abstract Springer correspondence. If d is even, W admits a unique abstract Springer correspondence satisfying the following additional property:*

(5) *Each non-Springer representation belongs to the set C_i with i as large as possible, subject to conditions* (1)–(3).

	$\phi_{1,0}$	$\phi_{2,1}$	$\phi_{2,3}$	$\phi_{3,2}$	$\phi_{1,4}$	$\phi_{2,5}$	$\phi_{1,8}$
$\phi_{1,0}$	1	1		1	1		
$\phi_{2,1}$		u		u	u^3+u		
$\phi_{2,3}$			u	u		u^2	
$\phi_{3,2}$				u^2	u^2	u^3	
$\phi_{1,4}$					u^4		
$\phi_{2,5}$						u^4	
$\phi_{1,8}$							u^4

	$\phi_{1,0}$	$\phi_{2,1}$	$\phi'_{2,3}$	$\phi''_{2,3}$	$\phi'_{2,5}$	$\phi_{1,6}$	$\phi_{3,2}$	$\phi_{3,4}$	$\phi_{1,4}$	$\phi_{1,8}$	$\phi''_{2,5}$	$\phi_{2,7}$	$\phi_{1,10}$	$\phi_{1,14}$
$\phi_{1,0}$	1	1					1	1	1		1		1	
$\phi_{2,1}$		u					u	u	u		u		u^9+u	
$\phi'_{2,3}$			u					u^3	u^3		u^3		u^7+u^3	
$\phi''_{2,3}$				u				u^3			u^3		u^3	u^7
$\phi'_{2,5}$					u		u	u				u^3	u^5	u^5
$\phi_{1,6}$						u		u^2					u^6	
$\phi_{3,2}$							u^2	u^2	u^2		u^2	u^4	u^6+u^2	u^6
$\phi_{3,4}$								u^4			u^4		u^8+u^4	u^8
$\phi_{1,4}$									u^4		u^4		u^4	
$\phi_{1,8}$										u^4		u^2		u^4
$\phi''_{2,5}$											u^5		u^5	u^9
$\phi_{2,7}$												u^5	u^7	u^7
$\phi_{1,10}$													u^{10}	
$\phi_{1,14}$														u^{10}

TABLE 5. Local intersection cohomology of the unipotent variety in G_4 and G_6.

To understand the last condition, note that even before choosing a partition of the form (4.1), the number of subsets is determined (they are in bijection with the Springer representations), and they are already endowed with at least a partial order, by condition (2).

Given an abstract Springer correspondence, we can try to develop the analogy with unipotent classes of an algebraic group further, by extracting "geometric" information from the matrix P.

Definition 4.5. Suppose W is equipped with an abstract Springer correspondence. The *closure partial order* on Springer representations is defined by declaring $\chi_i \leq \chi_j$ if $P_{\chi_j,\chi_i} \neq 0$.

It follows from Theorem 2.5 and basic properties of perverse sheaves that in the case of an algebraic group, the partial order defined above coincides with the usual closure partial order on unipotent classes.

Definition 4.6. In an algebraic group, a *special piece* is the union of a special unipotent class C and all nonspecial classes in $\overline{C}$ that are not contained in the closure of any smaller special unipotent class $C' \subset \overline{C}$.

In an abstract Springer correspondence, a *special piece* is a set consisting of one special representation χ and all nonspecial Springer representations $\chi' \leq \chi$ such that there is no other special representation χ_1 with $\chi' \leq \chi_1 < \chi$.

It is clear that these two definitions are compatible in the setting of algebraic groups.

Next, recall that a variety X is *rationally smooth* if the simple perverse sheaf $\mathrm{IC}(X, \underline{\mathbb{C}})$ is simply the constant sheaf $\underline{\mathbb{C}}$. (Another way of saying this is that X is rationally smooth if it obeys Poincaré duality.) A number of important varieties in representation theory turn out to be rationally smooth, including the full unipotent variety of an algebraic group [7] and all its special pieces [24].

Translating rational smoothness into the setting of the Lusztig–Shoji algorithm, we obtain the following notion.

Definition 4.7. Let $X \in \mathrm{Irr}(W)$ be a set of Springer representations with a unique maximal element χ with respect to the closure partial order. X is said to be *rationally smooth* if $P_{\chi,\chi'} = u^{b(\chi)}$ for all $\chi' \in X$.

One surprise that emerged from [2] was the following result.

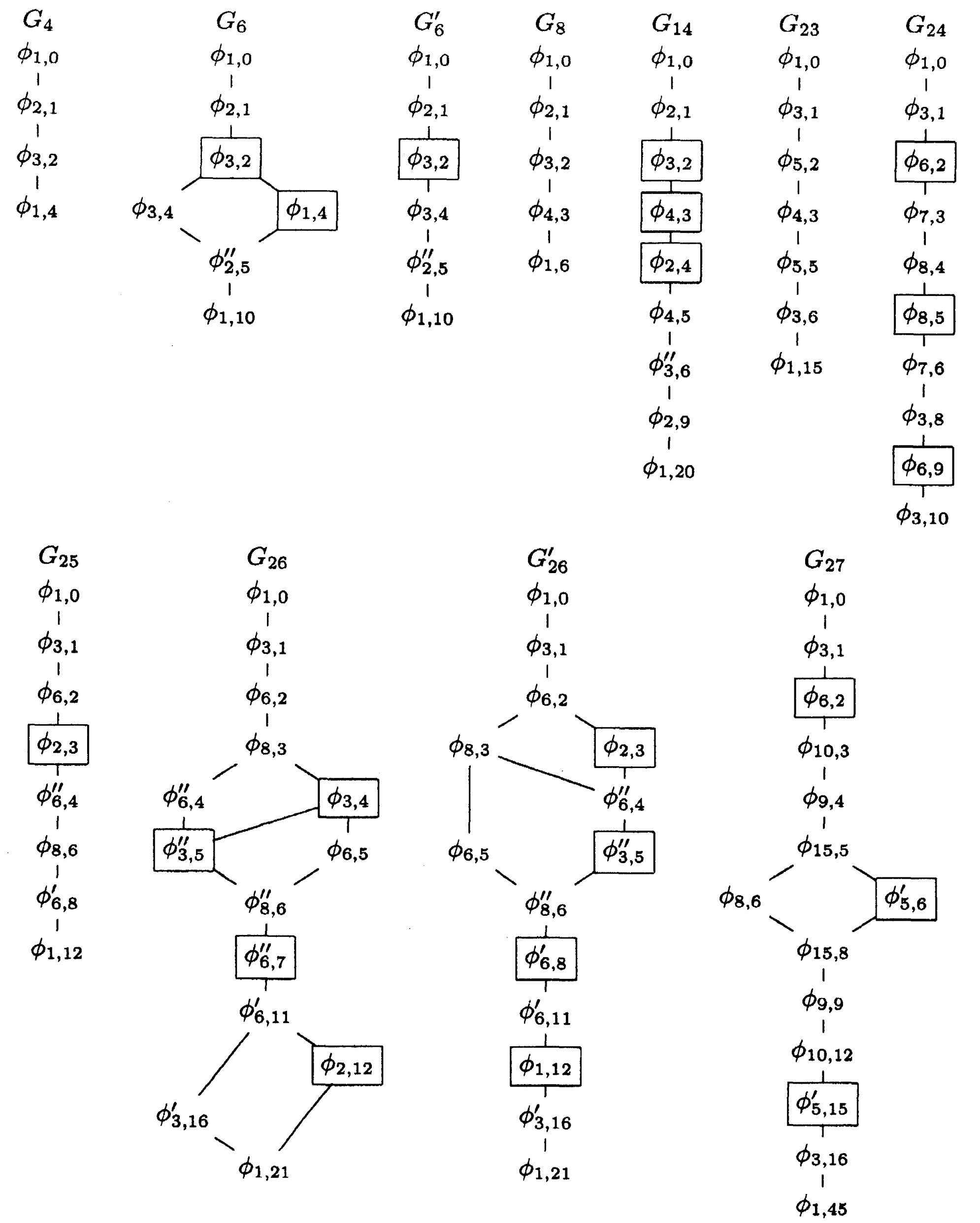

TABLE 6. Partial orders on Springer representations for primitive spetsial complex reflection groups

Theorem 4.8 ([2, Theorem 3]). *In the dihedral groups, each special piece is rationally smooth, as is the whole unipotent variety.*

Here, the "whole unipotent variety" simply means the set of all Springer representations. Of course, there is no known actual variety whose intersection cohomology is obtained by running the Lusztig–Shoji algorithm for a dihedral group, but this kind of result leads one to hope that perhaps one day, such a variety might be found.

4.3. **Calculations in the primitive groups.** We conclude by considering abstract Springer correspondences for the primitive spetsial complex reflection groups. We will treat G_4 and G_6 in detail, and the remaining groups with summary diagrams.

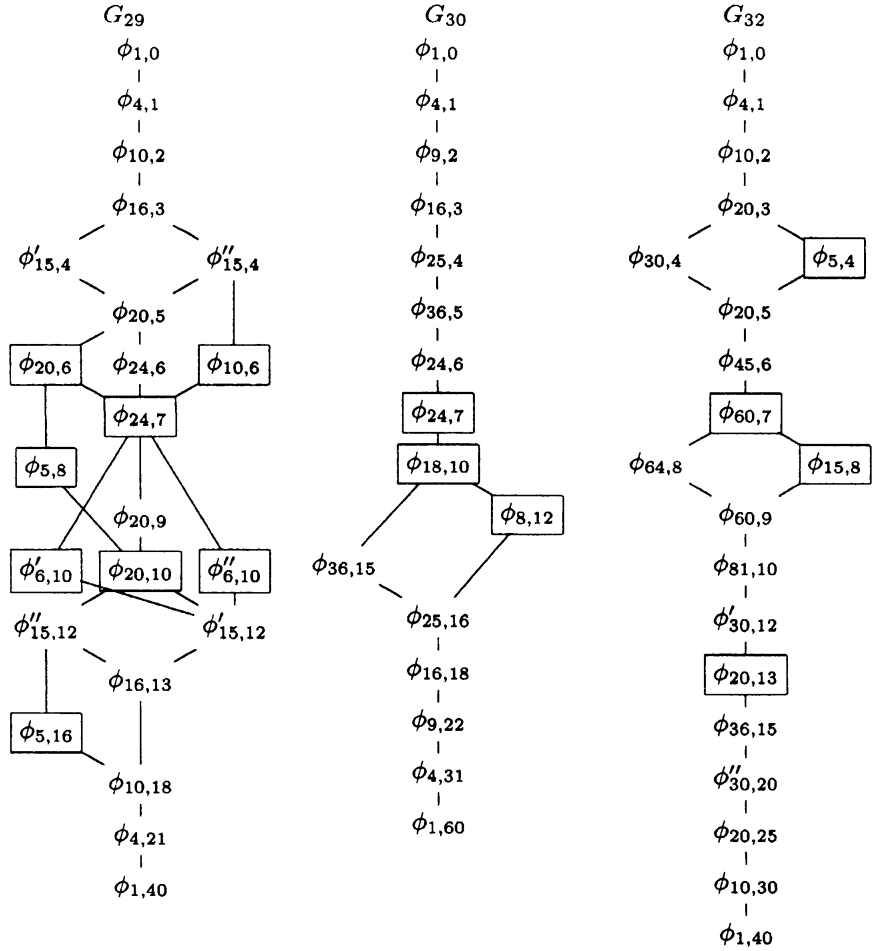

TABLE 6. (continued)

Recall from Section 4.1 that the only Springer representations in G_4 are the special representations. To produce an abstract Springer correspondence, we must decide how to group the three non-Springer representations, denoted (following [32]) $\phi_{2,3}$, $\phi_{2,5}$, and $\phi_{2,8}$. It is readily seen that conditions (1)–(3) of Definition 4.3 imply that $\phi_{2,5}$ and $\phi_{2,8}$ must belong to the same subset as the Springer representation $\phi_{1,4}$. The position of $\phi_{2,3}$ is not determined by these axioms, but when the Lusztig–Shoji algorithm is run, condition (4) fails unless $\phi_{2,3}$ is placed with $\phi_{2,1}$. The resulting matrix P is shown in Table 5. The vertical and horizontal lines show the partition of $\mathrm{Irr}(W)$ into subsets as in (4.1).

In this example, the subsets of that partition turned out to be precisely the families of characters of G_4, as determined by Malle–Rouquier [32]. The idea of carrying out the Lusztig–Shoji algorithm with $\mathrm{Irr}(W)$ partitioned by families, rather than by an actual or abstract Springer correspondence, has been investigated by Geck–Malle [20]. The Lusztig–Shoji algorithm in this case is not well understood, even for Weyl groups. In the Weyl group case, the output is undoubtedly related to the geometry of the unipotent variety, and Geck and Malle formulate some precise conjectures on this topic. Some progress in this direction has been made by Shoji [38], but an analogue of Theorem 2.5 is still lacking.

Next, we turn to G_6. In this case, there are seven Springer representations. There turn out to be six partitions of $\mathrm{Irr}(W)$ satisfying conditions (1)–(4) of Definition 4.3, but unfortunately, all of them violate condition (5). Nevertheless, there is a unique partition satisfying the additional condition (5) appearing in Theorem 4.4. The matrix P obtained by running the Lusztig–Shoji algorithm with this partition is shown in Table 5. Note that the failure of condition (5) is quite mild: it occurs only in the entry $P_{\phi_{1,8},\phi_{2,7}}$. It also has a feature that never occurs in Weyl groups: the representation $\phi_{2,7}$ is not a

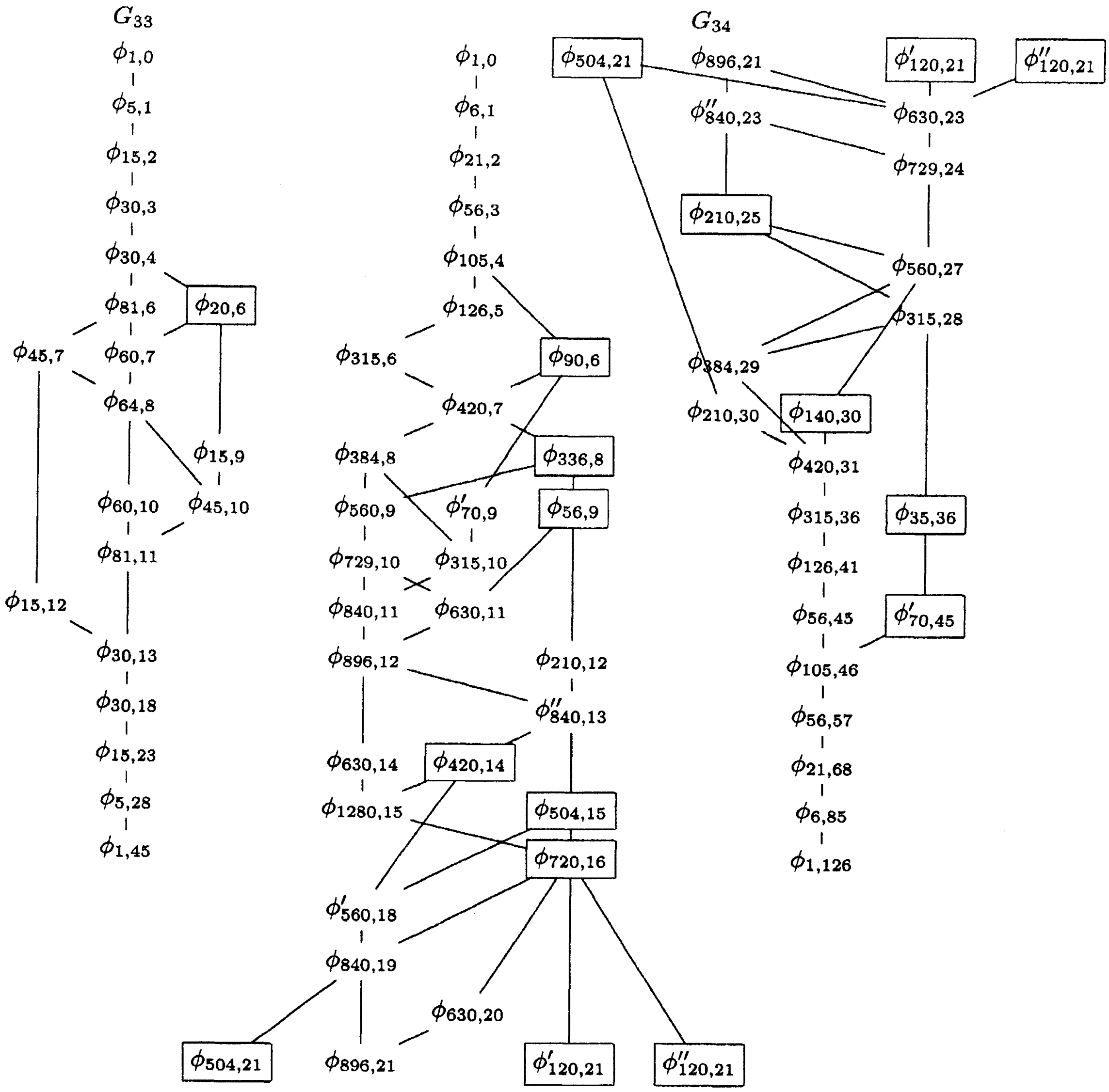

TABLE 6. (continued)

Springer representation, but its complex conjugate $\phi''_{2,5}$ is. (In Weyl groups, all representations are self-conjugate.) Condition (5) should probably be replaced by a slightly different condition to accommodate this kind of occurrence, but it is not known at this time what the correct formulation of such a condition should be.

The Lusztig–Shoji algorithm can similarly be carried out for the remaining primitive groups, using the list of Springer representations from Table 4, and using condition (5) of Theorem 4.4 as a guide for partitioning $\mathrm{Irr}(W)$. Condition (5) often fails in the way seen in G_6, so these partitions are not quite abstract Springer correspondences, but the other conditions hold, and the definitions of the closure partial order and of rational smoothness from Section 4.2 make sense.

The closure partial order on Springer representations in each of the primitive groups (other than the Coxeter groups) is shown in Table 6. By examination of the results of the Lusztig–Shoji algorithm for each of these groups, we obtain the following analogue of Theorem 4.8.

Theorem 4.9. *In the primitive spetsial complex reflection groups, each special piece is rationally smooth, as is the whole unipotent variety.*

References

[1] P. Achar and A.-M. Aubert, *On rank 2 complex reflection groups*, Comm. Algebra **36** (2008), 2092–2132.

[2] P. Achar and A.-M. Aubert, *Springer correspondences for dihedral groups*, Transform. Groups **13** (2008), 1–24.

[3] P. Achar and A.-M. Aubert, *Représentations de Springer pour les groupes de réflexions complexes imprimitifs*, arXiv:0707.0836v1.

[4] S. Ariki and K. Koike, *A Hecke algebra of* $(\mathbb{Z}/r\mathbb{Z})\wr\mathfrak{S}_n$ *and construction of its irreducible representations*, Adv. Math. **106** (1994), 216–243.

[5] S. Ariki, *Representation theory of a Hecke algebra of* $G(r,p,n)$, J. Algebra **177** (1995), 164–185.

[6] W. Borho and R. MacPherson, *Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes*, C.R. Acad. Sci. Paris **292** (1981), 707–710.

[7] W. Borho and R. MacPherson, *Partial resolutions of nilpotent varieties*, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, Astérisque **101–102** (1983), 23–74.

[8] K. Bremke and G. Malle, *Reduced words and a length function for* $G(e,1,n)$, Indag. Math. **8** (1997), 453–469.

[9] K. Bremke and G. Malle, *Root systems and length functions*, Geom. Dedicata **72** (1998), 83–97.

[10] M. Broué and S. Kim, *Familles de caractères des algèbres de Hecke cyclotomiques*, Adv. Math. **172** (2002), 53–136.

[11] M. Broué and G. Malle, *Zyklotomische Heckealgebren*, Astérisque **212** (1993), 119–189.

[12] M. Broué, G. Malle, and J. Michel, *Towards spetses I*, Transform. Groups **4** (1999), 157–218.

[13] M. Broué, G. Malle, and R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127–190.

[14] R.W. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*, John Wiley, New-York, 1985.

[15] M. Chlouveraki, *Rouquier blocks of the cyclotomic Hecke algebras*, C. R. Math. Acad. Sci. Paris **344** (2007), 615–620.

[16] M. Chlouveraki, *Rouquier blocks of the cyclotomic Ariki–Koike algebras*, Algebra Number Theory **2** (2008), 689–720.

[17] A.M. Cohen, *Finite complex reflection groups*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 379–436.

[18] M. Geck, G. Hiss, F. Lübeck, G. Malle, et G. Pfeiffer. CHEVIE—*A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups, and Hecke algebras*, Appl. Algebra Engrg. Comm. Comput. **7** (1996), 175–210.

[19] M. Geck and G. Malle, *On the existence of a unipotent support for the irreducible characters of a finite group of Lie type*, Trans. Amer. Math. Soc. **352** (1999), 429–456.

[20] M. Geck and G. Malle, *On special pieces in the unipotent variety*, J. Experiment. Math **8** (1999), 281–290.

[21] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori–Hecke algebras*, London Mathematical Society Monographs, New Series no. 21, Oxford University Press, New York, 2000.

[22] M. Hughes and A. Morris, *Root systems for two dimensional complex reflection groups*, Sém. Lothar. Combin. **45** (2000/01), Art. B45e, 18 pp.

[23] S. Kim, *Families of the characters of the cyclotomic Hecke algebras of* $G(de,e,r)$, J. Algebra **289** (2005), 346–364.

[24] H. Kraft and C. Procesi, *A special decomposition of the nilpotent cone of a classical Lie algebra*, in *Orbites unipotentes et représentations, III*, Astérisque **173–174** (1989), 271–279.

[25] G. Lusztig, *Characters of reductive groups over a finite field*, Annals of Mathematics Studies, vol. 107, Princeton University Press, Princeton, NJ, 1984.

[26] G. Lusztig, *Character Sheaves, V*, Adv. Math. **61** (1986), 103–155.

[27] G. Malle, *Unipotente Grade imprimitiver komplexer Spiegelungsgruppen*, J. Algebra **177** (1995), 768–826.

[28] G. Malle, *Degrés relatifs des algèbres cyclotomiques associées aux groupes de réflexions complexes de dimension deux*, Finite Reductive Groups: Related Structures and Representations (M. Cabanes, ed.), Progress in Mathematics, vol. 141, Birkhäuser, 1997, pp. 311–332.

[29] G. Malle, *On the rationality and fake degrees of characters of cyclotomic algebras*, J. Math. Sci. Univ. Tokyo **6** (1999), 647–677.

[30] G. Malle, *On the generic degrees of cyclotomic algebras*, Represent. Theory **4** (2000), 342–369.

[31] G. Malle and A. Mathas, *Symmetric cyclotomic Hecke algebras*, J. Algebra **205** (1998), 275–293.

[32] G. Malle and R. Rouquier, *Familles de caractères de groupes de réflexions complexes*, Represent. Theory **7** (2003), 610–640.

[33] G. Nebe, *The root lattices of the complex reflection groups*, J. Group Theory **2** (1999), 15–38.

[34] R. Rouquier, *Familles et blocs d'algèbres de Hecke*, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), 1037–1042.

[35] G.C. Shephard and J.A. Todd, *Finite unitary reflection groups*, Canadian J. Math **6** (1954), 274–304.

[36] T. Shoji, *Green functions associated to complex reflection groups*, J. Algebra **245** (2001), 650–694.

[37] T. Shoji, *Green functions associated to complex reflection groups. II*, J. Algebra **258** (2002), 563–598.

[38] T. Shoji, *Green functions and a conjecture of Geck and Malle*, Beiträge Algebra Geom. **41** (2000), 115–140.

[39] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lecture Notes in Math. **946**, Springer, Berlin Heidelberg New-York, 192.