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Chern-Simons theory, enumeration and Macdonald functions

名古屋大学・大学院多元数理科学研究科　　菅野　浩明 (Hiroaki Kanno)
Graduate School of Mathematics, Nagoya University

1 Introduction (a conjecture)

Let us begin with the following curious identity

$$\sum_{\lambda} \Lambda^{|\lambda|} \prod_{s\in\lambda} \frac{1 - Qq^{a(s)}t^{\ell(s)+1}}{1 - q^{a(s)}t^{\ell(s)+1}} \frac{1 - Qq^{-a(s)-1}t^{-\ell(s)}}{1 - q^{-a(s)-1}t^{-\ell(s)}} = \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{\Lambda^n}{1-\Lambda^n Q^n} \frac{(1-t^nQ^n)(1-q^{-n}Q^n)}{(1-t^n)(1-q^{-n})} \right\}, \tag{1.1}$$

where the left hand side is a summation over partitions λ, which we will identity with the Young diagrams in this article. If we denote the length of the i-th row of a Young diagram λ by λ_i, we have a non-increasing finite sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i \geq \lambda_{i+1} \geq \cdots \geq \lambda_{\ell(\lambda)} > \lambda_{\ell(\lambda)+1} = 0$ and we define the weight of the diagram by $|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i$. The conjugate (dual) partition which is given by the transpose of the Young diagram is denoted by $\lambda^\vee$. The product in (1.1) is taken over all the "boxes" $s = (i,j)$ in λ and $a(s) := \lambda_i - j$ and $\ell(s) = \lambda_j^\vee - i$ are the arm length and the leg length, respectively. Though we can check this identity by explicit computation for lower orders in Λ, we have no rigorous proof at the moment.

When $Q = 0$ the identity (1.1) gives

$$\sum_{\lambda} \Lambda^{|\lambda|} \prod_{s\in\lambda} \frac{1}{1 - q^{a(s)}t^{\ell(s)+1}} \frac{1}{1 - q^{-a(s)-1}t^{-\ell(s)}} = \exp\left\{ \sum_{n>0} \frac{\Lambda^n}{n} \frac{1}{(1-t^n)(1-q^{-n})} \right\}, \tag{1.2}$$

which has been proved by Nakajima and Yoshioka [20]. Their proof is geometric. Namely we can see that the right hand side is the generating function of the Hilbert series of the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points in $\mathbb{C}^2$;

$$\sum_{n=0}^{\infty} \Lambda^n \mathrm{ch}\, H^0((\mathbb{C}^2)^{[n]}, \mathcal{O}) = \prod_{k_1,k_2\geq 0} \frac{1}{(1 - t^{k_1}q^{-k_2}\Lambda)} = \exp\left\{ \sum_{n>0} \frac{\Lambda^n}{n} \frac{1}{(1-t^n)(1-q^{-n})} \right\}. \tag{1.3}$$

On the other hand the left hand side arises from a computation of the generating function by the localization theorem for toric action, where the fixed points of the toric action are labeled by partitions. We also have a combinatorial proof based on the Cauchy formula of the Macdonald function [17];

$$\sum_{\lambda} P_\lambda(x;q,t) P_{\lambda^\vee}(y;t,q) = \exp\left\{ \sum_{n>0} \frac{(-1)^{n-1}}{n} p_n(x) p_n(y) \right\}. \tag{1.4}$$

The following formula of the principal specialization of the Macdonald function

$$P_\lambda(t^\rho;q,t) = \prod_{s\in\lambda} \frac{(-1)t^{\frac{1}{2}} q^{a(s)}}{1-q^{a(s)}t^{\ell(s)+1}}, \qquad P_{\lambda^\vee}(-q^\rho;t,q) = \prod_{s\in\lambda} \frac{(-1)q^{-\frac{1}{2}} q^{-a(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}}, \tag{1.5}$$

implies the desired identity. When $Q=0$, the conjecture (1.1) is simplified to

$$\sum_{\lambda} \Lambda^{|\lambda|} = \exp\left\{ \sum_{n>0}\sum_{k>0} \frac{1}{n}\Lambda^{n\cdot k} \right\} = \prod_{k>0} \frac{1}{(1-\Lambda^k)} . \tag{1.6}$$

We recognize that this is nothing but the generating function of the number of partitions. Thus our conjecture interpolates the counting of partitions and the enumerative geometry associated with the Hilbert scheme of $\mathbb{C}^2$.

In this article we will explain how we have arrived at the conjecture (1.1). In section 2 we first review the fact that the Chern-Simons theory on a three dimensional manifold M can be regarded as an open topological string theory on the cotangent bundle T^*M. By introducing the Wilson loop operator in the Chern-Simons theory we obtain link invariants on M as topological correlation function. In section 3 we show the relation of the link invariants to the topological vertex, which is a kind of building block of topological string amplitudes on local toric Calabi-Yau threefolds. Finally by the relation of topological string amplitudes to the Nekrasov's partition function which is defined by instanton counting of four dimensional gauge theory, we were led to a refinement of the topological vertex. This is explained in section 4. We will see that the consistency of the computation in terms of the refined topological vertex requires the identity (1.1).

2 Chern-Simons theory and open topological string

The Chern-Simons theory is a three dimensional topological gauge theory with the action

$$S(A) = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3} A\wedge A\wedge A \right) , \tag{2.1}$$

where A is a gauge field (connection one form) on a three dimensional manifold M and Tr means the Killing form (invariant bilinear form) on the Lie algebra of the gauge group G. The parameter k has to be an integer for the gauge invariance and is called level. Since the Chern-Simons theory is topological in the sense that no metric is required to write down the action, we expect that the path integral

$$Z(M) := \int [\mathcal{D}A] e^{-S(A)} , \tag{2.2}$$

if it is appropriately defined, gives invariants of three manifold M. In fact they are known as Witten-Reshetikhin-Turaev invariants [27, 25]. Let us consider a link $\mathcal{L}$ in M and suppose that $\mathcal{L}$ consists of the components (knots) $\mathcal{K}_i$. To each component $\mathcal{K}_i$ we assign an integrable representation R_i of the affine Kac-Moody algebra of the gauge group G. In a seminal paper [27] Witten proposed that the link invariants of $\mathcal{L}$ are identified with the correlation functions of the Chern-Simons theory;

$$\left\langle \prod_{i=1}^{n} W_{R_i}^{\mathcal{K}_i} \right\rangle := \frac{1}{Z(M)} \int [\mathcal{D}A] e^{-S(A)} \prod_{i=1}^{n} W_{R_i}^{\mathcal{K}_i}(A) , \tag{2.3}$$

where

$$W_R^{\mathcal{K}}(A) := \mathrm{Tr}_R \left(P \exp \oint_{\mathcal{K}} A \right) \tag{2.4}$$

is called the Wilson loop operator. The right hand side computes the holonomy of the connection A along the knot $\mathcal{K}$ in the representation R.

Let us consider the Hopf link in S^3. We attach representations μ and ν to each component of the Hopf link. Such link is called colored Hopf link. In the following we consider the case $G = U(N)$. In this case we can associate integrable representations with Young diagrams[1]. The path integral construction of physical states of the Chern-Simons theory tells us that the invariants (2.3) for a colored Hopf link in S^3 are essentially the matrix elements $\mathcal{W}_{\mu\nu}(q, \lambda)$ of the modular S transformation on the space of characters of the integral representations [27]. It is known that they are given by specializations of the Schur function $s_\mu(x)$, which is a basis of the space of symmetric functions labeled by partitions;

$$\mathcal{W}_{\mu\nu}(q, \lambda) = \lambda^{\frac{|\mu|}{2} + \frac{|\nu|}{2}} s_\mu(q^{\nu+\rho}) s_\nu(q^\rho) , \tag{2.5}$$

where

$$q := \exp\left(\frac{2\pi i}{N+k}\right), \quad \lambda := q^N . \tag{2.6}$$

[1] We will eventually consider large N limit.

For a partition μ, the specialization $q^{\mu+\rho}$ means $x_i = q^{\mu_i - i + \frac{1}{2}}$. As a special case of (2.5) we find the quantum dimension of the representation R_μ defined by μ as the invariants of the unknot;

$$\mathcal{W}_\mu(q, \lambda) = \lambda^{\frac{|\mu|}{2}} s_\mu(q^\rho) = \dim_q R_\mu \ , \tag{2.7}$$

where

$$s_\lambda(q^\rho) = \frac{q^{\kappa(\lambda)/4}}{\prod_{(i,j)\in\lambda} q^{h(i,j)/2} - q^{-h(i,j)/2}} \ , \quad h(i,j) : \text{ the hook length} \ . \tag{2.8}$$

For any compact three dimensional manifold M, its cotangent bundle T^*M is a (non-compact) Calabi-Yau manifold and thus a consistent target of the string theory. If we introduce N Lagrangian D branes wrapping on the base space M, then the gauge theory on the branes gives the $U(N)$ Chern-Simons gauge theory. Thus, the Chern-Simons theory on M can be regarded as an open topological string theory on the cotangent bundle T^*M [28]. In this setup a knot $\mathcal{K}$ in M is realized as a Lagrangian submanifold $L_{\mathcal{K}}$ of T^*M which intersects with the base M along the knot $\mathcal{K}$ [22]. The topological invariants which are most naturally related to the string theory are the Gromov-Witten invariants, since they are defined by counting holomorphic maps from a curve to the target space. Here we find that the link invariants, which might look quite different from the Gromov-Witten invariants, are also related to the string theory through the Chern-Simons theory. We can obtain the invariants of the colored Hopf link in S^3 by considering topological string theory on T^*S^3 with an appropriate Lagrangian brane configuration. At this point it is an amusing fact that the cotangent bundle T^*S^3 is obtained by deforming the conifold singularity $x^2 + y^2 + z^2 + w^2 = 0$ in $\mathbb{C}^4$, which is the cone over $S^2 \times S^3$. The conifold is the most fundamental singularity in the Calabi-Yau threefolds and one can show that the deformed conifold defined by $x^2 + y^2 + z^2 + w^2 = \epsilon > 0$ is diffeomorphic to T^*S^3 [4]. On the other hand by blowing up the conifold singularity we have what is called resolved conifold, which is a rank two vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$ over the rational curve.

Now we have to introduce the idea of gauge/gravity correspondence, or open/closed string correspondence, which is one of the most prominent subjects in the recent developments of the string theory. The basic idea is that, when the number of branes in an open string background becomes large, the effect of the branes is equivalent to an appropriate deformation of the background. The case of our interests is when the resulting background has no branes, namely when it gives a closed string background. This intuitively means that the existence of the D brane can be replaced with the curvature of the space-time. At the technical level this is achieved by summing up all the possible boundary states of open string in a given background with D branes. When we make a perturbative expansion of the free energy $F = \log Z$ of topological open string by the genus g and the number h of boundaries of the world sheet, we see the structure $F_{\text{open}} = \sum_{g=0}^{\infty}\sum_{h=1}^{\infty} F_{g,h}(z) g_s^{2g-2+h} N^h$, where z is the moduli parameters of the open string background. N is the number of D branes and the string coupling g_s plays the role of the parameter of the genus expansion. Then the open/closed string correspondence means schematically

$$F_{\text{open}} \simeq F_{\text{closed}} = \sum_{g=0}^{\infty} F_g(z;t) g_s^{2g-2} , \quad t := g_s N , \tag{2.9}$$

where a new parameter t is regarded as a modulus of the closed string. In the present case of the deformed conifold T^*S^3 with N Lagrangian branes wrapping on S^3, such a correspondence actually occurs and it is called geometric transition (or large N duality) [26, 7, 23]. It turns out that the corresponding closed string background is nothing but the resolved conifold, where the volume (the Kähler parameter) of the rational curve $\mathbf{P}^1$ is given by $t_B = g_s N$. The correspondence is given by the coincidence of the partition function in sense of large N expansion as we will explain shortly.

We can compute the topological (closed) string amplitude on the resolved conifold $X : \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$ by counting the BPS states which are $D2$ branes wrapping over a two-cycle $\beta \in H_2(X, \mathbb{Z})$ [8]. We may put an arbitrary number of $D0$ branes bound to the $D2$ brane. Summing up all the contributions from the BPS bound states of such $D2/D0$ brane system, we obtain the free energy

$$F(g_s) = -\sum_{k=1}^{\infty} \frac{1}{k} \frac{e^{-kt_B}}{\left(2\sinh\frac{kg_s}{2}\right)^2} , \tag{2.10}$$

and the partition function

$$Z = \exp F = \prod_{n=1}^{\infty} (1 - Qq^n)^n , \tag{2.11}$$

where $Q := e^{-t_B}$ and $q := e^{-g_s}$. On the other hand, the partition function of the Chern-Simons theory on S^3 is

$$\begin{aligned} S_{\bullet\bullet}(q,N) &= \prod_{1\le i<j\le N} \left(q^{\frac{1}{2}(j-i)} - q^{-\frac{1}{2}(j-i)}\right) , \\ &= \exp\left[-\sum_{1\le i<j\le N} \left(\frac{j-i}{2}\log q - \log(1-q^{j-i})\right)\right] , \end{aligned} \tag{2.12}$$

for $SU(N)$ theory. Using the strange formula for $SU(N)$

$$\frac{1}{2}\sum_{1\leq i<j\leq N}(j-i)=\frac{1}{12}N(N^2-1)=\rho_N^2\ , \tag{2.13}$$

and

$$\sum_{1\leq i<j\leq N}\log(1-q^{(j-i)})=-\sum_{m=1}^{\infty}\left[\frac{Nq^m}{m(1-q^m)}-\frac{q^m-q^{m(N+1)}}{m(1-q^m)^2}\right]\ , \tag{2.14}$$

we find

$$q^{\rho_N^2+\frac{N}{24}}S_{\bullet\bullet}(q,N)=M(q)\eta(q)^N N_0(q,\lambda)\ , \tag{2.15}$$

where $\lambda=q^N$ and

$$N_0(q,\lambda)=\exp\left(-\sum_{n=1}^{\infty}\frac{q^n}{n(1-q^n)^2}\lambda^n\right)=\prod_{n=1}^{\infty}\left(1-\lambda^{-1}q^n\right)^n\ . \tag{2.16}$$

The function

$$M(q):=\prod_{k>0}(1-q^k)^{-k}\ , \tag{2.17}$$

is known as the MacMahon function. Note that the eta function

$$\eta(q)=q^{1/24}\prod_n(1-q^n)\ , \tag{2.18}$$

appears as an overall normalization factor. We find that the function $N_0(q,\lambda)$ is nothing but the partition function of the resolved conifold! (See (2.11).) If we identify the 't Hooft coupling $t=g_sN$ with the Kähler parameter t_B of the resolved conifold geometry, we find an agreement of two partition functions. As mentioned above when we realize the Chern-Simons theory as topological open string theory, we have another Lagrangian branes intersection with S^3 along a link. In the geometric transition these branes survive and we have holomorphic branes (2 cycles) in the resolved conifold as a remnant of the original link in S^3.

3 Topological vertex and instanton counting

The topological vertex $C_{\lambda_1\lambda_2\lambda_3}(q)$ is a topological open string amplitude on the flat Calabi-Yau manifold $\mathbb{C}^3$ with three Lagrangian D brane insertions [1]. The boundaries of open string should be ended on the branes. The boundary state $|\vec{k}\rangle$ on each brane is labeled by $\vec{k}=(k_1,k_2,\cdots,k_n,\cdots)$, where k_n denotes the number of boundaries with winding number n. They are naturally identified with the conjugacy classes $C(\vec{k})$ of the symmetric group. By the Frobenius relation (or in terms of the character χ_{R_μ} of the symmetric group), we can change the winding number base $|\vec{k}\rangle$ to the representation base $|R_\mu\rangle$ labeled by partitions;

$$|R_\mu\rangle = \sum_{\vec{k}} \frac{\chi_{R_\mu}(C(\vec{k}))}{z_{\vec{k}}} |\vec{k}\rangle, \qquad z_{\vec{k}} := \prod_j k_j! j^{k_j} . \tag{3.1}$$

The topological vertex has three indices of partitions $(\lambda_1, \lambda_2, \lambda_3)$ which specify three boundary states on the branes and the topological open string amplitude on $\mathbb{C}^3$ is expanded in this base;

$$Z_{\mathbb{C}^3}(q) = \sum_{\lambda_1,\lambda_2,\lambda_3} C_{\lambda_1\lambda_2\lambda_3}(q) |R_{\lambda_1}\rangle \otimes |R_{\lambda_2}\rangle \otimes |R_{\lambda_3}\rangle , \tag{3.2}$$

where, as before, q is related to the string coupling g_s by $q = \exp(-g_s)$.

In the last section we have reviewed that the invariants of the colored Hopf link are given by topological open string amplitudes on T^*S^3 with Lagrangian branes. By the large N duality the amplitudes agree with topological string amplitude on the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$ with appropriate brane insertions. Furthermore, since the volume of the base $\mathbf{P}^1$ is given by $t_B = g_s N$, the leading term of N^{-1} expansion corresponds to the limit where the volume of $\mathbf{P}^1$ becomes infinite, which can be approximated by the flat space $\mathbb{C}^3$. Thus, we can relate the topological vertex to the Hopf link invariants in large N limit

$$W_{\mu\nu}(q) = \lim_{N\to\infty} \lambda^{-\frac{|\mu|}{2} - \frac{|\nu|}{2}} \mathcal{W}_{\mu\nu}(q, \lambda) = s_\mu(q^{\nu+\rho}) s_\nu(q^\rho) . \tag{3.3}$$

After these rather long arguments [1], we find

$$C_{\lambda_1\lambda_2\lambda_3}(q) = q^{\frac{\kappa(\lambda_2)}{2} + \frac{\kappa(\lambda_3)}{2}} \sum_{\mu_1,\mu_3,\nu} N_{\mu_1\nu}{}^{\lambda_1} N_{\mu_3\nu}{}^{\lambda_3^\vee} \frac{W_{\mu_1\lambda_2^\vee} W_{\lambda_2\mu_3}}{W_{\lambda_2\bullet}} , \tag{3.4}$$

where $\bullet$ means the trivial representation and $N_{\mu\nu}{}^{\lambda}$ is the Littlewood-Richardson coefficients (the branching coefficients of the tensor product). The integer $\kappa(\lambda)$ is defined by $\kappa(\lambda) := 2\sum_{(i,j)\in\lambda}(j-i)$. Substituting (3.3) to (3.5) we obtain

$$C_{\lambda_1\lambda_2\lambda_3}(q) = q^{\frac{\kappa(\lambda_3)}{2}} s_{\lambda_2}(q^\rho) \sum_\mu s_{\lambda_1/\mu}(q^{\lambda_2^\vee+\rho}) s_{\lambda_3^\vee/\mu}(q^{\lambda_2+\rho}), \tag{3.5}$$

where the skew Schur function is defined by $s_{\lambda/\mu}(x) = \sum_\nu N_{\mu\nu}{}^{\lambda} s_\nu(x)$.

The topological vertex has a quite suggestive interpretation as the generating function of the number of plane partitions with a fixed boundary condition [21]. We consider the plane partitions π whose asymptotic behaviors at infinity are fixed by three partitions $\lambda_1, \lambda_2, \lambda_3$ and define the generating function by

$$Z_{\lambda_1\lambda_2\lambda_3}(q) \sim \sum_{\pi} q^{|\pi|} , \tag{3.6}$$

where $|\pi|$ denotes the number of cubes or the volume of π. It was shown that [21]

$$Z_{\lambda_1\lambda_2\lambda_3}(q) = M(q) \cdot C_{\lambda_1\lambda_2\lambda_3}(q) , \tag{3.7}$$

where the MacMahon function $M(q) = \prod_{k>0}(1 - q^k)^{-k}$ has already appeared in the partition function of the Chern-Simons theory on S^3. We see that the deviation of the generating function from the MacMahon function due to the asymptotic conditions λ_i is given by the topological vertex. Though it is not obvious at all in (3.5), the relation to the enumerative problem of plane partitions tells us that the topological vertex enjoys the cyclic symmetry $C_{\lambda_1\lambda_2\lambda_3}(q) = C_{\lambda_2\lambda_3\lambda_1}(q) = C_{\lambda_3\lambda_1\lambda_2}(q)$.

The topological vertex is a building block of topological string amplitudes on toric Calabi-Yau threefold in the following sense. We consider the dual toric diagram to the Newton polyhedron of a toric (non-compact) Calabi-Yau manifold X. It is a tri-valent diagram in $\mathbb{R}^3$ and encodes the toric geometry of X. It shows the degeneration loci of the toric action on X. The face, the edge and the vertex in the diagram represent invariant 4-cycle (divisor), 2-cycle (curve), 0-cycle (point) of $(\mathbb{C}^\times)^3$ action, respectively. Due to the Calabi-Yau condition the vertices of the diagram other than the origin lie on a common plane, say $z = 1$, in $\mathbb{R}^3$. Usually we only show the projection of the polyhedron on the plane. A typical example of toric Calabi-Yau threefold is the total space of the canonical bundle K_S of a toric (Fano) surface S. In this case the projection is nothing but the toric diagram of S itself. One may consider a T^2 fibration on the plane, which is regarded as a Lagrangian submanifold of X. Along the edge of the diagram one of the cycles of T^2 shrinks and the dual cycle is left as S^1 fiber. The direction of the edge shows which cycle collapses. At the vertex the fiber is completely degenerate. The basic idea of the topological vertex formalism is simple. We divide the diagram into tri-valent vertices connected by edges, which correspond to invariant rational curves. For each vertex we can associate an affine local coordinate patch of the toric Calabi-Yau manifold. The decomposition of the diagram physically means that we put D-brane/anti-D-brane pair to cut the manifold into affine spaces which are homeomorphic to $\mathbb{C}^3$. Since the vertex is tri-valent, there are three D-brane/anti-D-brane pairs in each local coordinate patch and this is nothing but the configuration where the topological vertex computes the amplitude. If we fix the boundary states $|R_{\lambda_i}\rangle$ on three D-brane/anti-D-brane pairs, then the topological string amplitude on each affine space is given by the topological vertex $C_{\lambda_1\lambda_2\lambda_3}(q)$. According to the localization theorem, if we consider holomorphic maps from the world sheet to the target space with toric action, the map localizes to the invariant loci of the toric action. This means the map localizes on the edges and vertices of the diagram. Thus we expect that total amplitude is obtained by appropriately gluing the topological vertex along the edges of the toric diagram.

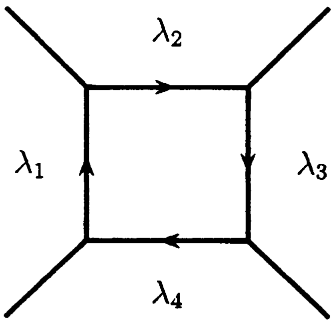
The rule of computation is as follows;

- Decompose the toric diagram into tri-valent vertices connected by internal (compact) edges. Some of the vertices have external (non-compact) edges. For each edge assign an integer vector v_i describing the direction of shrinking cycle and a partition μ_i.

- For each vertex order three edges, for example, clockwise and associate the topological vertex $C_{\lambda_1,\lambda_2,\lambda_3}(q)$, where $\lambda = \mu$ if the vector v_i is incoming and $\lambda = \mu^\vee$ if it is outgoing.

- Glue all the topological vertices along their common edges with the factor (propagator)
$$(-1)^{(n_i+1)|\mu_i|} q^{-\frac{n_i}{2}\kappa(\mu_i)} e^{-|\mu_i| t_i}, \tag{3.8}$$
and take the summation over all the partitions assigned to internal edges. The index n_i is computed for each edges from the date of integer vectors v_i at the two vertices which the edge connects. We refer to the original reference [1] for details. The factor $q^{-\frac{n_i}{2}\kappa(\mu_i)}$ accommodates the framing at two vertices. Finally the parameter t_i is the Kähler parameter of the rational curve that corresponds to the edge.

To illustrate the rule let us look at the topological string amplitude on the local Hirzebruch surface $K_{\mathbf{F}_0}$. Since $\mathbf{F}_0 \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is covered by four local coordinate patches, the toric diagram consists of four vertices which are glued together as shown in Fig.1. To compute the amplitude, we assign partitions λ_2, λ_4 to the base $\mathbf{P}^1$ and λ_1, λ_3 to the fiber $\mathbf{P}^1$. The external edges have the trivial representations. Hence we can put one of the indices of the topological vertex trivial and the amplitude is obtained by gluing four vertices of type $C_{\bullet\mu\nu^\vee}(q) = q^{-\frac{\kappa(\nu)}{2}} s_\mu(q^\rho) s_\nu(q^{\mu+\rho}) = q^{-\frac{\kappa(\nu)}{2}} W_{\mu\nu}(q)$ as follows;
$$Z^{(\mathbf{F}_0)}_{top\ str}(t_B, t_F; q) = \sum_{\lambda_1 \cdots \lambda_4} W_{\lambda_4\lambda_1}(q) W_{\lambda_1\lambda_2}(q) W_{\lambda_2\lambda_3}(q) W_{\lambda_3\lambda_4}(q) \cdot e^{-t_F\cdot(|\lambda_1|+|\lambda_3|) - t_B\cdot(|\lambda_2|+|\lambda_4|)} . \tag{3.9}$$



$v_1 = (0,1),\ v_2 = (1,0)$

$v_3 = (0,-1),\ v_4 = (-1,0)$

$n_1 = n_2 = n_3 = n_4 = -1$

Figure 1: Dual toric diagram of $\mathbf{F}_0$

The index in (3.8) is $n_i = -1$ for all the internal edges in Fig. 1 and the factors $\kappa(\lambda_i)$ of the topological vertex are canceled by the gluing factors. Introducing

$$K_{\lambda_2\lambda_4}(Q_F;q) := \sum_{\lambda} W_{\lambda_2\lambda} W_{\lambda_4\lambda} Q_F^{\ell_\lambda} = s_{\lambda_2}(q^\rho)s_{\lambda_4}(q^\rho)\sum_{\lambda} s_\lambda(q^{\mu^{\lambda_2}+\rho})s_\lambda(q^{\mu^{\lambda_4}+\rho})Q_F^{\ell_\lambda}\ , \quad (3.10)$$

we can express

$$Z^{(\mathbf{F}_0)}_{top\ str}(Q_B,Q_F;q) = \sum_{\lambda_2,\lambda_4} (K_{\lambda_2\lambda_4})^2\, Q_B^{(\ell_{\lambda_2}+\ell_{\lambda_4})}\ , \quad (3.11)$$

where we have defined $Q_B := e^{-t_B}, Q_F := e^{-t_F}$ and used the symmetry $W_{\mu\nu}(q) = W_{\nu\mu}(q)$. By the summation formula for the Schur function [17];

$$\sum_{\lambda} s_\lambda(x)s_\lambda(y) = \prod_{i,j\geq 1}(1-x_iy_j)^{-1}\ , \quad (3.12)$$

we find

$$K_{\lambda_2\lambda_4} = s_{\lambda_2}(q^\rho)s_{\lambda_4}(q^\rho)\prod_{i,j\geq 1}\left(1-q^{h_{\lambda_2\lambda_4}(i,j)}Q_F\right)^{-1}\ , \quad (3.13)$$

where the "relative" hook length is defined by $h_{\mu,\nu}(i,j) := \mu_i - j + \nu_j - i + 1$. Note that $h_{\mu,\mu^\vee}(i,j)$ gives the hook length of μ at (i,j). We finally obtain

$$Z^{(\mathbf{F}_0)}_{top\ str}(Q_B,Q_F;q) = \sum_{\lambda_2,\lambda_4} Q_B^{(|\lambda_2|+|\lambda_4|)} s^2_{\lambda_2}(q^\rho)s^2_{\lambda_4}(q^\rho)\prod_{i,j\geq 1}\left(1-q^{h_{\lambda_2\lambda_4}(i,j)}Q_F\right)^{-2}\ . \quad (3.14)$$

It turns out that with appropriate identifications of the parameters $(Q_B,Q_F;q)$, $Z^{(\mathbf{F}_0)}_{top\ str}$ agrees to the Nakrasov's partition function for $SU(2)$ Seiberg-Witten theory [10, 11, 5, 6], which is expected by the idea of geometric engineering [14, 15].

4 Refinement of the topological vertex and the preferred direction

The Nekrasov's partition function $Z_{\rm Nek}(Q_\ell; q, t)$ arises from the instanton counting of four dimensional gauge theory and gives the generating function of the equivariant Donaldson invariants of $\mathbb{C}^2$, [18, 19, 20]. The parameters $q = q_1, t = q_2^{-1}$ are equivariant parameters of the toric action $(z_1, z_2) \longrightarrow (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$ on $\mathbb{C}^2$. As we remarked in the example of the amplitude on the local Hirzebruch surface, there is a correspondence of the Nekrasov's partition function and topological string amplitudes on local toric Calabi-Yau manifolds. As the topological vertex is a building block of the topological string amplitude on toric Calabi-Yau manifolds, it is a natural question that if the Nekrasov's partition function has a similar building block. As an answer to this question we have proposed the following refinement of the topological vertex [2, 3];

$$
\begin{aligned}
C_{\mu\lambda}{}^{\nu}(q,t) = {} & f_\nu\,(q,t)^{-1} P_\lambda(t^\rho; q, t) \\
& \times \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|-|\nu|}{2}} \iota P_{\mu^\vee/\eta^\vee}(-t^{\lambda^\vee} q^\rho; t, q) P_{\nu/\eta}(q^\lambda t^\rho; q, t) \ , \qquad (4.1)
\end{aligned}
$$

where roughly speaking we have promoted the Schur functions in the topological vertex to the Macdonald functions $P_\lambda(x; q.t)$. The notation ι in (4.1) denotes an involution on the space of symmetric functions which is defined on the power sum p_n by $\iota(p_n) = -p_n$. The factor $f_\nu(q,t)$ defined by

$$
f_\nu(q,t) := \prod_{s\in\nu} (-1) q^{a(s)+\frac{1}{2}} t^{-\ell(s)-\frac{1}{2}} \ , \qquad (4.2)
$$

is responsible for the framing of the vertex. A slightly different version of the refined topological vertex has been introduced [13];

$$
\begin{aligned}
C^{(IKV)}_{\mu\nu\lambda}(t,q) = {} & \left(\frac{q}{t}\right)^{\frac{||\nu||^2+||\lambda||^2}{2}} t^{\frac{\kappa(\nu)}{2}} P_{\lambda^\vee}(t^{-\rho}; q, t) \qquad (4.3) \\
& \times \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\mu|-|\nu|}{2}} s_{\mu^\vee/\eta}(q^{-\lambda} t^{-\rho}) s_{\nu/\eta}(t^{-\lambda^\vee} q^{-\rho}) \ . \qquad (4.4)
\end{aligned}
$$

As we have emphasized in the last section, the topological vertex has a highly non-trivial cyclic symmetry among three partitions. However both of the above refinements cannot keep the cyclic symmetry. They have a special direction which we call preferred. Consequently, it is convenient to introduce the following four types of the vertices, though they are related each other [3];

$$
\begin{aligned}
C^{\mu}{}_{\lambda\nu}(q,t) &:= P_{\lambda}(t^{\rho};q,t)\sum_{\sigma}\iota P_{\mu^\vee/\sigma^\vee}(-t^{\lambda^\vee}q^{\rho};t,q)\;P_{\nu/\sigma}(q^{\lambda}t^{\rho};q,t)v^{|\sigma|+|\mu|}f_{\mu}(q,t)\,,\\
C_{\mu}{}^{\lambda\nu}(q,t) &:= P_{\lambda^\vee}(-q^{\rho};t,q)\sum_{\sigma}P_{\nu^\vee/\sigma^\vee}(-t^{\lambda^\vee}q^{\rho};t,q)\;\iota P_{\mu/\sigma}(q^{\lambda}t^{\rho};q,t)v^{-|\sigma|-|\mu|}f_{\mu}(q,t)^{-1}\,,\\
C_{\mu\lambda}{}^{\nu}(q,t) &:= C^{\mu}{}_{\lambda\nu}(q,t)v^{-|\mu|-|\nu|}f_{\mu}(q,t)^{-1}\,f_{\nu}(q,t)^{-1}\,,\\
C^{\mu\lambda}{}_{\nu}(q,t) &:= C_{\mu}{}^{\lambda\nu}(q,t)v^{|\mu|+|\nu|}f_{\mu}(q,t)\,f_{\nu}(q,t)\,,
\end{aligned}
\tag{4.5}
$$

where $v := (q/t)^{1/2}$. When we associate the refined topological vertex to each vertex of the toric diagram, we order three edges clockwise so that the second edge is along the preferred direction. The lower and the upper indices of the refined topological vertex correspond to the incoming and the outgoing partitions at the vertex, respectively.

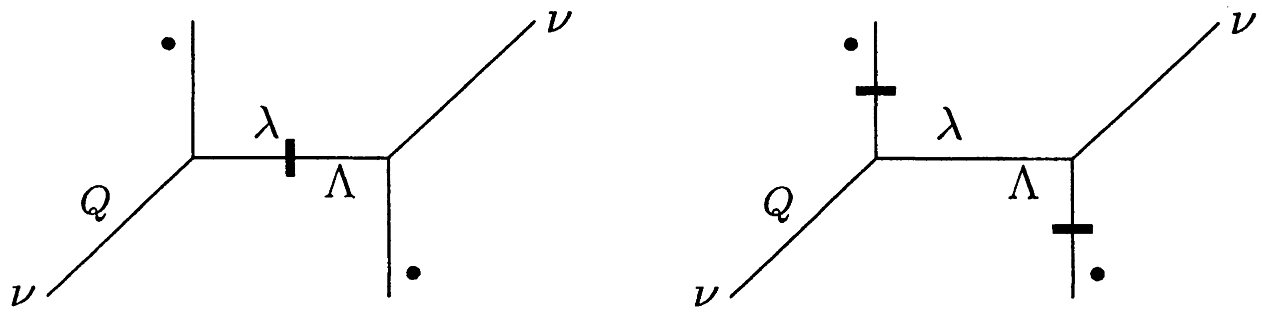


Figure 2: Change of the preferred direction

Changing the preferred direction in the computation of the refined topological vertex often gives a highly non-trivial combinatorial equality that involves a summation over partitions. Let us consider the toric diagram of Fig.2, which appears in the geometric engineering of five dimensional $U(1)$ gauge theory with adjoint matter [9]. These are one loop diagrams where we identify two external vertical edges with ν. As has been discussed in [12, 3], the refined topological string amplitude for these diagrams gives the generating function of the equivariant χ_y genus of the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points in $\mathbb{C}^2$. In the left diagram the preferred direction is along the internal line, while it is along the external lines in the right diagram. The gluing rule of the refined topological vertex gives the amplitude for the left diagram

$$
Z_L := \sum_{\lambda,\nu} Q^{|\nu|}\Lambda^{|\lambda|}C_{\bullet\lambda}{}^{\nu}(q,t)C^{\bullet\lambda}{}_{\nu}(q,t)\,\cdot
$$

$$= \sum_{\lambda,\nu} Q^{|\nu|}\Lambda^{|\lambda|} P_\lambda\left(t^\rho; q, t\right) P_{\lambda^\vee}\left(-q^\rho; t, q\right) P_\nu\left(q^\lambda t^\rho; q, t\right) P_{\nu^\vee}\left(-t^{\lambda^\vee} q^\rho; t, q\right). \quad (4.6)$$

On the other hand the right diagram gives us the following partition function;

$$\begin{aligned} Z_R := & \sum_{\mu,\nu} Q^{|\nu|}\Lambda^{|\lambda|} C^{\nu}{}_{\bullet\lambda}(q,t) C_{\nu}{}^{\bullet\lambda}(q,t) \\ = & \sum_{\mu,\nu,\sigma_1,\sigma_2} Q^{|\nu|}\Lambda^{|\lambda|} P_{\nu^\vee/\sigma_1{}^\vee}\left(-\iota q^\rho; t, q\right) P_{\lambda/\sigma_1}\left(t^\rho; t, q\right) P_{\lambda^\vee/\sigma_2{}^\vee}\left(-q^\rho; t, q\right) P_{\nu/\sigma_2}\left(\iota t^\rho; t, q\right) v^{|\sigma_1|-|\sigma_2|}. \end{aligned} \quad (4.7)$$

The computation of Z_L is made by the Cauchy formula for the Macdonald function

$$\sum_\lambda P_{\lambda/\mu}(x;q,t) P_{\lambda^\vee/\nu^\vee}(y;t,q) = \Pi_0(x,y) \sum_\eta P_{\mu^\vee/\eta^\vee}(y;t,q) P_{\nu/\eta}(x;q,t), \quad (4.8)$$

where

$$\Pi_0(-x,y) := \exp\left\{-\sum_{n>0} \frac{1}{n} p_n(x) p_n(y)\right\} = \prod_{i,j}(1 - x_i y_j). \quad (4.9)$$

We also use the following adding formula

$$\sum_\mu P_{\lambda/\mu}(x;q,t) P_{\mu/\nu}(y;q,t) = P_{\lambda/\nu}(x,y;q,t). \quad (4.10)$$

From the formula of the principal specialization (1.5) and the Cauchy formula (4.8) for $\mu = \nu = \bullet$, we have

$$Z_L = \sum_\lambda \Pi_0(-Qq^\lambda t^\rho,\ t^{\lambda^\vee} q^\rho) \prod_{s\in\lambda} v^{-1}\Lambda \frac{1}{(1 - q^{a(s)} t^{\ell(s)+1})(1 - q^{-a(s)-1} t^{-\ell(s)})}. \quad (4.11)$$

If we define the perturbative part by $Z_L^{\mathrm{pert}} := \sum_\nu Q^{|\nu|} C_{\bullet\bullet}{}^{\nu}(q,t) C^{\bullet\bullet}{}_{\nu}(q,t) = \Pi_0(-Qt^\rho,\ q^\rho)$, which is independent of Λ, then the instanton part $Z_L^{\mathrm{inst}} := Z_L / Z_L^{\mathrm{pert}}$ is

$$Z_L^{\mathrm{inst}} = \sum_\lambda \prod_{s\in\lambda} v^{-1}\Lambda \frac{1 - \widetilde{Q} q^{a(s)} t^{\ell(s)+1}}{1 - q^{a(s)} t^{\ell(s)+1}} \frac{1 - \widetilde{Q} q^{-a(s)-1} t^{-\ell(s)}}{1 - q^{-a(s)-1} t^{-\ell(s)}}, \quad (4.12)$$

where $\widetilde{Q} = (q/t)^{\frac{1}{2}} Q$.

The computation of Z_R is more involved and we have to employ the trace formula (B.26) of [2], which was obtained by successively using (4.8). The result is

$$Z_R = \prod_{k\geq 0} \frac{\Pi_0\left(t^\rho, -\Lambda c^k q^\rho\right) \Pi_0\left(t^\rho, -Q c^k q^\rho\right)}{\Pi_0\left(t^\rho, -v c^{k+1} q^\rho\right) \Pi_0\left(t^\rho, -v^{-1} c^{k+1} q^\rho\right)} \frac{1}{1 - c^{k+1}}$$

$$= \exp\left\{ -\sum_{n>0} \frac{1}{n} \frac{1}{1-c^n} \left(\frac{(\Lambda^n + Q^n) - (v^n + v^{-n})c^n}{(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(q^{\frac{n}{2}} - q^{-\frac{n}{2}})} - c^n \right) \right\}, \tag{4.13}$$

where $c := Q\Lambda$. As before we define the perturbative part by $Z_R^{\text{pert}} := Z_R(\Lambda = 0)$. Then the instanton part $Z_R^{\text{inst}} := Z_R / Z_R^{\text{pert}}$ is

$$Z_R^{\text{inst}} = \exp\left\{ -\sum_{n>0} \frac{1}{n} \frac{\Lambda^n}{1-c^n} \frac{(Q^n - (qt)^{\frac{n}{2}})(Q^n - (qt)^{-\frac{n}{2}})}{(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(q^{\frac{n}{2}} - q^{-\frac{n}{2}})} \right\}. \tag{4.14}$$

If we require that the partition function is independent of the choice of the preferred direction, we have $Z_L^{\text{inst}} = Z_R^{\text{inst}}$, which means

$$\begin{aligned} &\sum_\lambda \Lambda^{|\lambda|} \prod_{s\in\lambda} \frac{1 - yq^{a(s)}t^{\ell(s)+1}}{1 - q^{a(s)}t^{\ell(s)+1}} \frac{1 - yq^{-a(s)-1}t^{-\ell(s)}}{1 - q^{-a(s)-1}t^{-\ell(s)}} \\ &\qquad = \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{\Lambda^n}{1 - \Lambda^n y^n} \frac{(1 - t^n y^n)(1 - q^{-n} y^n)}{(1 - t^n)(1 - q^{-n})} \right\}. \end{aligned} \tag{4.15}$$

Thus the consistency of the refined topological vertex formalism implies the identity (1.1) we gave in the beginning. It is highly desirable that we can prove the identity (1.1).

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