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A survey on Shapovalov determinants of (generalized) quantum groups at roots of 1

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Abstract

This is an informal survey on a joint work [HY08b] with Istvan Heckenberger.

1 A quantum group $U(\chi)$ defined for any bi-character χ

Recently study of Nichols algebras has been achieved very actively for the viewpoint of classification of Hopf algebras, see [AS98], [AS02], [Hec06]. One of their examples is the positive part $U^{+}(\chi)$ of a generalized quantum group $U(\chi)$ defined below.

Let $\Bbbk$ be a field and $\Bbbk^{\times}=\Bbbk\setminus\{0\}$. For $n\in\mathbb{Z}_{\geq 0}$ and $x\in\Bbbk$, let

(1) $$[n]_{x}=\sum_{m=1}^{n}x^{m-1},\quad [n]_{x}!=\prod_{m=1}^{n}[m]_{x}.$$

For two elements X_1 and X_2 of a $\Bbbk$-algebra we use the convention:

(2) $$X_1 \quad X_2 \underset{\text{def}}{\Longleftrightarrow} \exists x\in\Bbbk^{\times}\ X_1=xX_2.$$

Let I be a finite index set. Let $\mathbb{Z}\Pi = \ {}_{i\in I}\mathbb{Z}\alpha_i$ be a rank $|I|$ free $\mathbb{Z}$-module with a basis $\Pi=\{\alpha_i|i\in I\}$. We say that a map $\chi:\mathbb{Z}\Pi\times\mathbb{Z}\Pi\to\Bbbk^{\times}$ is a *bi-character* if $\chi(a+b,c)=\chi(a,c)\chi(b,c)$, and $\chi(a,b+c)=\chi(a,b)\chi(a,c)$ for all a, b, $c\in\mathbb{Z}\Pi$.

Let χ be *any* bi-character. Then, as we explain more precisely in Section 2, Lusztig's definition [L, 3.1.1] of the quantum groups can be applied to define the Hopf $\Bbbk$-algebra $U(\chi)$ with the generators

(3) $$K\ ,\ L\ \ (\lambda\in\mathbb{Z}\Pi),\ E_i,\ F_i\,(i\in I),$$

for which $K\ L\ \ (\lambda,\ \mu \in \mathbb{Z}\Pi)$ are linearly independent and the following equations hold:

(4) $K_0 = L_0 = 1,\quad K_{\ +\ } = K\ K\ ,\quad L_{\ +\ } = L\ L\ ,\quad K\ L\ = L\ K\ ,$

(5) $K\ L\ E_j(K\ L\ )^{-1} = \dfrac{\chi(\lambda,\alpha_j)}{\chi(\alpha_j,\mu)}E_j,\quad K\ L\ F_j(K\ L\ )^{-1} = \dfrac{\chi(\alpha_j,\mu)}{\chi(\lambda,\alpha_j)}F_j,$

(6) $E_iF_j\quad F_jE_i = \delta_{ij}(K_{\ i}\quad L_{\ i}).$

(7) $\Delta(K\ L\ ) = K\ L\quad K\ L\ ,\ \varepsilon(K\ L\ ) = 1,\ S(K\ L\ ) = (K\ L\ )^{-1},$

(8) $\Delta(E_i) = E_i\quad 1 + K_{\ i}\quad E_i,\ \Delta(F_i) = F_i\quad L_{\ i} + 1\quad F_i,$

(9) $\varepsilon(E_i) = \varepsilon(F_i) = 0,\ S(E_i) = \quad K_{\ i}^{-1}E_i,\ S(F_i) = \quad F_iL_{\ i}^{-1}.$

Let $U^0(\chi) := \quad_{\ ,\ \in\mathbb{Z}\Pi}\Bbbk K\ L\ $. Let $U^+(\chi)$ and $U^-(\chi)$ be the subalgebra of $U(\chi)$ generated by E_i and F_i with all $i \in I$ respectively. Then $U(\chi) = U^+(\chi)\quad U^0(\chi)\quad U^-(\chi)$, as a $\Bbbk$-linear space. We have the $\mathbb{Z}_{\geq 0}\Pi$-grading $U^\pm(\chi) = \quad_{\in\mathbb{Z}_{\geq 0}\Pi}U^\pm(\chi)_\pm$ defined by $U^+(\chi)_{\ i} = \Bbbk E_i$, $U^-(\chi)_{-\ i} = \Bbbk F_i$, and $U^\pm(\chi)\ U^\pm(\chi)\ \subset U^\pm(\chi)_{\ +\ }$. We also have $\dim U^-(\chi)_-\ = \dim U^+(\chi)\ $ for all $\lambda \in \mathbb{Z}_{\geq 0}\Pi$.

2 Drinfeld pairing of $U(\chi)$

Here we will explain how to define $U(\chi)$ more precisely. By abuse of notation, we use the same symbols as above for the generators of the algebras introduced in this paragraph. Let $\widetilde{U}^+(\chi)$ and be $\widetilde{U}^-(\chi)$ the free $\Bbbk$-algebras (with 1) with the generators $\{E_i|i \in I\}$ and $\{F_i|i \in I\}$ respectively. Let $\widetilde{U}^0(\chi)$ be the $\Bbbk$-linear space with the basis $\{K\ L\ |\lambda,\mu \in \mathbb{Z}\Pi\}$. Let $\widetilde{U}(\chi) = \widetilde{U}^+(\chi)\quad_{\Bbbk}\widetilde{U}^0(\chi)\quad_{\Bbbk}\widetilde{U}^-(\chi)$. Identify $X \in \widetilde{U}^+(\chi)$, $Z \in \widetilde{U}^0(\chi)$ and $Y \in \widetilde{U}^-(\chi)$ with $X\quad 1\quad 1$, $1\quad Z\quad 1$ and $1\quad 1\quad Y$ respectively, and regard $\widetilde{U}^+(\chi)$, $\widetilde{U}^0(\chi)$ and $\widetilde{U}^-(\chi)$ as subspaces of $\widetilde{U}(\chi)$ in this way. Then $\widetilde{U}(\chi)$ can be regarded as the $\Bbbk$-algebra (with 1) presented by the same generators as the ones for $U(\chi)$ and the relations (4), (5) and (6) (cf. [L, Prop. 3.2.4]). Further $\widetilde{U}(\chi)$ can be regarded as the Hopf $\Bbbk$-algebra with the same equalities as (7), (8) and (9). Let $\widetilde{U}^{+,K}(\chi)$ be the subalgebra of $\widetilde{U}(\chi)$ generated by E_i's and $K\ $'s. Let $\widetilde{U}^{L,-}(\chi)$ be the subalgebra of $\widetilde{U}(\chi)$ generated by F_i's and $L\ $'s. Then there exists a unique $\Bbbk$-bilinear form

(10) $$\langle\,,\rangle : \widetilde{U}^{+,K}(\chi) \times \widetilde{U}^{L,-}(\chi) \to \Bbbk$$

with

$$(11) \qquad \langle 1, Y\rangle = \varepsilon(Y),\ \langle X, 1\rangle = \varepsilon(X),\ \langle S(X), Y\rangle = \langle X, S^{-1}(Y)\rangle,$$

$$(12) \qquad \langle X_1X_2, Y\rangle = \sum_g \langle X_2, Y_g^{(1)}\rangle\langle X_1, Y_g^{(2)}\rangle,$$

$$(13) \qquad \langle X, Y_1Y_2\rangle = \sum_h \langle X_h^{(1)}, Y_1\rangle\langle X_h^{(2)}, Y_2\rangle,$$

$$(14) \qquad \langle E_i, F_j\rangle = \quad \delta_{ij},\ \langle K \ , L \ \rangle = \chi(\lambda, \mu),\ \langle E_i, L \ \rangle = \langle K \ , F_j\rangle = 0$$

for X, X_1, $X_2 \in \widetilde{U}^{+,K}(\chi)$ with $\Delta(X) = \sum_h X_h^{(1)} \quad X_h^{(2)}$, and Y, Y_1, $Y_2 \in \widetilde{U}^{L,-}(\chi)$ with $\Delta(Y) = \sum_g Y_g^{(1)} \quad Y_g^{(2)}$ and for i, $j \in I$ and λ, $\mu \in \mathbb{Z}\Pi$. We see

$$(15) \qquad \langle \widetilde{E}K \ , \widetilde{F}L \ \rangle = \langle \widetilde{E}, \widetilde{F}\rangle\langle K \ , L \ \rangle$$

for $\widetilde{E} \in \widetilde{U}^+(\chi)$ and $\widetilde{F} \in \widetilde{U}^-(\chi)$. Further, letting $\widetilde{U}^\pm(\chi) = \quad {}_{\in\mathbb{Z}_{\geq 0}\Pi}\widetilde{U}^\pm(\chi)_\pm$ be the $\mathbb{Z}_{\geq 0}\Pi$-grading on $\widetilde{U}^\pm(\chi)$ defined in a way similar to the one on $U^\pm(\chi)$, we have $\langle \widetilde{U}^+(\chi) \ , \widetilde{U}^-(\chi)_- \ \rangle = \{0\}$ if $\lambda \neq \mu$. Let

$$(16) \qquad \widetilde{J}^+(\chi) = \{\widetilde{E} \in \widetilde{U}^+(\chi) | \langle \widetilde{E}, \widetilde{U}^-(\chi)\rangle = \{0\}\},$$

$$(17) \qquad \widetilde{J}^-(\chi) = \{\widetilde{F} \in \widetilde{U}^-(\chi) | \langle \widetilde{U}^+(\chi), \widetilde{F}\rangle = \{0\}\},$$

$$(18) \qquad \widetilde{J}(\chi) = \mathrm{Span}_{\Bbbk}(\widetilde{J}^+(\chi)\widetilde{U}^0(\chi)\widetilde{U}^-(\chi) + \widetilde{U}^+(\chi)\widetilde{U}^0(\chi)\widetilde{J}^-(\chi)).$$

Then $\widetilde{J}(\chi)$ is the kernel of the Hopf algebra epimorphism from $\widetilde{U}(\chi)$ to $U(\chi)$ sending the generators to the ones denoted by the same symbols.

Theorem 1. (Kharchenko [Kha99]) *There exist* $M \in \mathbb{N} \cup \{\infty\}$ *and elements* $\widehat{E}_i \in U^+(\chi)_{\ i}$ $(1 \quad i \quad M)$ *for some* $\beta_i \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$ *such that we have the* $\Bbbk$*-basis of* $U^+(\chi)$ *formed by the elements*

$$(19) \qquad \begin{cases} \widehat{E}_1^{m_1}\widehat{E}_2^{m_2} \quad \widehat{E}_M^{m_M} & \text{if } M \text{ is finite, that is } M \in \mathbb{N}, \\ \widehat{E}_1^{m_1}\widehat{E}_2^{m_2} \quad \widehat{E}_{M'}^{m_{M'}} & \text{for some } M' \in \mathbb{N} \text{ if } M = \infty \end{cases}$$

with $0 \quad m_i \quad h_{\ i}$, *and* $h_{\ i} := \mathrm{Max}\{n | [n]_{(\ i, \ i)}! \neq 0\} \in \mathbb{N} \cup \{+\infty\}$.

Let

$$(20) \qquad R_+ := \{\beta_i | 1 \quad i \quad M\}.$$

Note that $|R_+| \quad M$, that is, β_i and β_j may be the same for some $i \neq j$.

We say that χ is *finite-type* if $|R_+| < +\infty$. See [H09] for the classification.

Note that if $\dim U(\chi) < \infty$, then χ is finite-type.

Theorem 2. (*see* [HY08b, Theorems 4.8, 4.9]) *Assume that χ is finite-type. Then $|R_+| = M$ as for* (20). *We write $E_{\ \ i} = \widehat{E}_i$ if $\widehat{E}_i \in U(\chi)_{\ \ i}$. Then after re-choosing $E_{\ \ i}$ (as in* (51)*), we may assume that $E^{h^{\chi}_{\beta}+1} = 0$ if $h \quad < +\infty$ and that $E_{\ \ i}E_{\ \ j} \quad \chi(\beta_i,\beta_j)E_{\ \ j}E_{\ \ i} \in \langle E_{\ \ r} | i < r < j\rangle$ for any $i < j$, so*

$$\{E^{m_{f(1)}}_{f(1)} E^{m_{f(2)}}_{f(2)} \quad E^{m_{f(M)}}_{f(M)} | 0 \quad m_i \quad h_{\ \ i}\} \tag{21}$$

is a $\Bbbk$-basis of $U(\chi)$ for any bijective map $f : \{1, 2, \dots, M\} \to \{1, 2, \dots, M\}$.

Convention. Let χ_1, χ_2 : $\mathbb{Z}\Pi \times \mathbb{Z}\Pi \to \Bbbk^\times$ be two bi-characters. Let f_1, $f_2 : U(\chi_1) \to U(\chi_2)$ be two $\Bbbk$-algebra homomorphisms. Then we write

$$f_1 = f_2 \tag{22}$$

if

$$f_1(K \ L \ ) = f_2(K \ L \ ),\ f_1(E_i) \quad f_2(E_i),\ f_1(F_i) \quad f_2(F_i) \tag{23}$$

for all λ, $\mu \in \mathbb{Z}\Pi$ and $i \in I$.

3 Heckenberger's Lusztig-type isomorphisms

Here we explain a generalization [H07] of Lusztig-type isomorphisms [L].

Assume χ to be any bi-character. Let

$$[\![X \ , Y \ ]\!]^+ = X \ Y \quad \chi(\lambda,\mu)Y \ X \ , \tag{24}$$

$$[\![X \ , Y \ ]\!]^- = X \ Y \quad \chi(\lambda,\mu)^{-1}Y \ X \ , \tag{25}$$

$$[\![X \ , Y \ ]\!]^{\vee,+} = X \ Y \quad \chi(\mu,\lambda)Y \ X \ , \tag{26}$$

$$[\![X \ , Y \ ]\!]^{\vee,-} = X \ Y \quad \chi(\mu,\lambda)^{-1}Y \ X \tag{27}$$

for $X \ \in U(\chi)$ and $Y \ \in U(\chi)$ with λ, $\mu \in \mathbb{Z}\Pi$. Let i, $j \in I$ be such that $i \neq j$. Let

$$\begin{aligned} &E^+_{\ j} = E_j, \quad E^-_{\ j} = E_j, \\ &E^+_{\ j+m \ i} = [\![E_i, E^+_{\ j+(m-1) \ i}]\!],\ E^-_{\ j+m \ i} = [\![E_i, E^-_{\ j+(m-1) \ i}]\!]^{\vee,-}, \\ &F^+_{\ j+m \ i} = [\![F_i, F^+_{\ j+(m-1) \ i}]\!]^{\vee,+},\ F^-_{\ j+m \ i} = [\![F_i, F^-_{\ j+(m-1) \ i}]\!]^- \end{aligned} \tag{28}$$

for $m \in \mathbb{N}$. For $m \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} &[m]_{(\ i,\ i)}! \prod_{s=1}^{m} (1 \quad \chi(\alpha_i,\alpha_i)^{s-1}\chi(\alpha_i,\alpha_j)\chi(\alpha_j,\alpha_i)) \neq 0 \\ &\Longleftrightarrow E^+_{\ j+m \ i} \neq 0 \Longleftrightarrow E^-_{\ j+m \ i} \neq 0 \Longleftrightarrow F^+_{\ j+m \ i} \neq 0 \Longleftrightarrow F^-_{\ j+m \ i} \neq 0 \\ &\Longleftrightarrow \alpha_j + m\alpha_i \in R_+. \end{aligned} \tag{29}$$

We also have
(30)
$$[E^+_{j+m\ i}, F^+_{j+m\ i}] = (\chi(\alpha_i,\alpha_i)^{m-1}\chi(\alpha_i,\alpha_j)\chi(\alpha_j,\alpha_i))^m[E^-_{j+m\ i}, F^-_{j+m\ i}]$$
$$= (\ \ 1)^m([m]_{(\ i,\ i)}!\prod_{s=1}^{m}(1\ \ \ \chi(\alpha_i,\alpha_i)^{s-1}\chi(\alpha_i,\alpha_j)\chi(\alpha_j,\alpha_i))(K_{\ j+m\ i}\ \ \ L_{\ j+m\ i}).$$

Theorem 3. ([H07]) *Let $i \in I$. Assume that for all $j \in I \setminus \{i\}$, there exist $m_{ij} \in \mathbb{Z}_{\geq 0}$ such that $E^+_{j+m^{\chi}_{ij}\ i} \neq 0$ and $E^+_{j+(m^{\chi}_{ij}+1)\ i} = 0$.*

(1) *There exist a bi-character $r_i(\chi) : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to \Bbbk^{\times}$ and $\Bbbk$-algebra isomorphisms*

$$T_i = T_i^+ : U(r_i(\chi)) \to U(\chi), \quad T_i^- : U(r_i(\chi)) \to U(\chi) \tag{31}$$

such that

$$T_i^{\pm}(K_{\ i}) = K_{-\ i},\ T_i^{\pm}(L_{\ i}) = L_{-\ i}, \tag{32}$$
$$T_i^{\pm}(K_{\ j}) = K_{\ j+m^{\chi}_{ij}\ i},\ T_i^{\pm}(L_{\ j}) = L_{\ j+m^{\chi}_{ij}\ i}, \tag{33}$$
$$T_i(E_i) = F_iL_{-\ i},\ T_i(F_i) = K_{-\ i}E_i, \tag{34}$$
$$T_i^-(E_i) = K_{-\ i}F_i,\ T_i^-(F_i) = E_iL_{-\ i}, \tag{35}$$
$$T_i^{\pm}(E_j)\ \ \ E^{\pm}_{\ j+m^{\chi}_{ij}\ i},\ T_i^{\pm}(F_j)\ \ \ F^{\pm}_{\ j+m^{\chi}_{ij}\ i}, \tag{36}$$

where $j \in I \setminus \{i\}$.

(2) *$r_i(r_i(\chi))$ exists in the same way as above with $r_i(\chi)$ in place of χ. Further $r_i(r_i(\chi)) = \chi$, $m^{r_i(\ )}_{ij} = m_{ij}$ for all $j \in I \setminus \{i\}$.*

(3) *Let $T_i : U(r_i(\chi)) \to U(\chi)$ be as in (31). Let $T_i^- : U(\chi) \to U(r_i(\chi))$ be the one as in (31) defined with $r_i(\chi)$ in place of χ. Then $T_i^-T_i = \mathrm{id}_{U(r_i(\ ))}$ and $T_iT_i^- = \mathrm{id}_{U(\ )}$.*

(4) *Define the $\mathbb{Z}$-module isomorphism $\sigma_i^{r_i(\ )} = \sigma_i : \mathbb{Z}\Pi \to \mathbb{Z}\Pi$ by $T_i^{\pm}(U(r_i(\chi))\ \ ) = U(\chi)_{\ i(\ )}$ for all $\lambda \in \mathbb{Z}\Pi$. Then*

$$\sigma_i^{r_i(\ )} = \sigma_i\ ,\quad \sigma_i\,\sigma_i^{r_i(\ )} = \mathrm{id}_{\mathbb{Z}\Pi} \tag{37}$$

and

$$\sigma_i^{r_i(\ )}(R_+^{r_i(\ )} \setminus \{\alpha_i\}) = R_+ \setminus \{\alpha_i\},\quad \sigma_i^{r_i(\ )}(\alpha_i) = \ \ \alpha_i. \tag{38}$$

Theorem 4. ([H07]) *Assume χ to be finite-type. Let $i, j \in I$ to be such that $i \neq j$. Let $M = |R_+ \cap (\mathbb{Z}_{\geq 0}\alpha_i\ \ \ \mathbb{Z}_{\geq 0}\alpha_j)|$. For $n \in \{1, 2, \ldots, M\}$, define two bi-characters χ_n, χ'_n, two $\mathbb{Z}$-module automorphism $\bar{\sigma}_n$, $\bar{\sigma}'_n$ of $\mathbb{Z}\Pi$ and two $\Bbbk$-algebra* *isomorphisms* $\bar{T}_n : U(\chi_n) \to U(\chi)$, $\bar{T}'_n : U(\chi'_n) \to U(\chi)$ *in the way that* $\chi_1 = \chi'_1 = \chi$, $\bar{\sigma}_1 = \bar{\sigma}'_1 = \mathrm{id}_{\mathbb{Z}\Pi}$, $\bar{T}_1 = \bar{T}'_1 = \mathrm{id}_{U(\ )}$, *and*

(39) $\chi_{2n} = r_i(\chi_{2n-1})$, $\chi_{2n+1} = r_j(\chi_{2n})$, $\chi'_{2n} = r_j(\chi'_{2n-1})$, $\chi'_{2n+1} = r_i(\chi'_{2n})$,

(40) $\bar{\sigma}_{2n} = \bar{\sigma}_{2n-1}\sigma_i^{\ 2n}$, $\bar{\sigma}_{2n+1} = \bar{\sigma}_{2n}\sigma_j^{\ 2n+1}$, $\bar{\sigma}'_{2n} = \bar{\sigma}'_{2n-1}\sigma_j^{\ 2n}$, $\bar{\sigma}'_{2n+1} = \bar{\sigma}'_{2n}\sigma_i^{\ 2n+1}$,

(41) $\bar{T}_{2n} = \bar{T}_{2n-1}T_i$, $\bar{T}_{2n+1} = \bar{T}_{2n}T_j$, $\bar{T}'_{2n} = \bar{T}'_{2n-1}T_j$, $\bar{T}'_{2n+1} = \bar{T}'_{2n}T_i$.

Then we have

(42) $$\chi_M = \chi'_M,$$

(43) $$\bar{\sigma}_M = \bar{\sigma}'_M$$

and

(44) $$\bar{T}_M = \bar{T}'_M.$$

4 Longest elements of Weyl groupoids

In this section we always assume χ to be finite-type, and refer to [CH08] for categorical definitions of Weyl groupoids.

Convention. For a category $\mathcal{C}$, we denote the product of the morphisms by . That is, for two morphism $f_1 \in \mathrm{Mor}(a_1, b_1)$ and $f_2 \in \mathrm{Mor}(a_2, b_2)$ with a_1, b_1, a_2 and $b_2 \in \mathrm{Ob}(\mathcal{C})$, we denote their product by

(45) $$f_1 \quad f_2 \quad \text{if } b_2 = a_1.$$

Set

(46) $$\mathcal{C}(\chi) = \{\chi\} \cup \bigcup_{n=1}^{\infty} \{r_{i_1} \quad r_{i_n}(\chi) | i_1, \dots, i_n \in I\}.$$

Let $W = W(\chi)$ be the category with $\mathrm{Ob}(W) = \mathcal{C}(\chi)$ and generated by the maps $\sigma_i^{\ \prime} \in \mathrm{Mor}_W(\chi', r_i(\chi'))$ with $\chi' \in \mathrm{Ob}(W)$ and $i \in I$. Let $\mathcal{W} = \mathcal{W}(\chi)$ be the (abstract) category with $\mathrm{Ob}(\mathcal{W}) = \mathcal{C}(\chi)$ defined by generators $s_i^{\ \prime} \in \mathrm{Mor}_{\mathcal{W}}(\chi', r_i(\chi'))$ with $\chi' \in \mathrm{Ob}(W)$ and $i \in I$ and relations

(47) $$s_i^{\ \prime} \quad s_i^{r_i(\ ')} = 1_{r_i(\ ')},$$

$$(48) \qquad s_i^{\;\;'} \; s_j^{r_i(\;')} \; s_i^{r_j r_i(\;')} \quad = s_j^{\;\;'} \; s_i^{r_j(\;')} \; s_j^{r_i r_j(\;')}$$

(both sides are composed of $|R_+ \cap (\mathbb{Z}\alpha_i \quad \mathbb{Z}\alpha_j)|$-factors).

We call $\mathcal{W}$ the *Weyl groupoid.* Define the morphism $\phi : \mathcal{W} \to W$ by $\phi(s_i^{\;\;'}) = \sigma_i^{\;\;'}$. Then ϕ is bijective, see [HY08a, Theorem 1]. Let $\ell(1_{\;'}) = 0$ for $\chi' \in \mathcal{C}(\chi)$. Let $\ell(s_i^{\;\;'}) = 1$. For $w \in \mathrm{Mor}_{\mathcal{W}}(\chi_1, \chi_2)$, let $\ell(w)$ be the least number $\ell(w') + \ell(w'')$ with $w = w' \quad w''$ for some $\chi_3 \in \mathcal{C}(\chi)$, and some $w' \in \mathrm{Mor}_{\mathcal{W}}(\chi_3, \chi_2)$, some $w'' \in \mathrm{Mor}_{\mathcal{W}}(\chi_1, \chi_3)$. By [HY08a, Lemma 8(iii)], we have

$$(49) \qquad \ell(w) = |\{\alpha \in R_+^{\;1} | \phi(w)(\alpha) \in \quad R_+^{\;2}\}|.$$

Moreover for each $\chi_1 \in \mathcal{C}(\chi)$, there exists unique $\chi_2 \in \mathcal{C}(\chi)$ and ${}^{1}w_0 \in \mathrm{Mor}_{\mathcal{W}}(\chi_2, \chi_1)$ such that $\phi({}^{1}w_0)(R_+^{\;2}) = \quad R_+^{\;1}$. We call ${}^{1}w_0$ the *longest element* since $\ell({}^{1}w_0)$ $\ell(w')$ for any $w' \in \mathrm{Mor}_{\mathcal{W}}(\chi_3, \chi_4)$ for any χ_3, $\chi_4 \in \mathcal{C}(\chi)$.

Let $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(\chi)$ be the (abstract) category with $\mathrm{Ob}(\widetilde{\mathcal{W}}) = \mathcal{C}(\chi)$ defined by generators $\widetilde{s}_i^{\;\;'} \in \mathrm{Mor}_{\widetilde{\mathcal{W}}}(\chi', r_i(\chi'))$ with $\chi' \in \mathrm{Ob}(W)$ and $i \in I$ and relations

$$(50) \qquad \widetilde{s}_i^{\;\;'} \; \widetilde{s}_j^{r_i(\;')} \; \widetilde{s}_i^{r_j r_i(\;')} \quad = \widetilde{s}_j^{\;\;'} \; \widetilde{s}_i^{r_j(\;')} \; \widetilde{s}_j^{r_i r_j(\;')}$$

(both sides are composed of $|R_+ \cap (\mathbb{Z}\alpha_i \quad \mathbb{Z}\alpha_j)|$-factors).

Let $\widetilde{1}_{\;'} \in \mathrm{Mor}_{\widetilde{\mathcal{W}}}(\chi', \chi')$ denote the identity morphism. Define the morphism $\widetilde{\phi} : \widetilde{\mathcal{W}} \to \mathcal{W}$ by $\widetilde{\phi}(\widetilde{s}_i^{\;\;'}) = s_i^{\;\;'}$.

Let $\widetilde{\ell}(\widetilde{1}_{\;'}) = 0$ for $\chi' \in \mathcal{C}(\chi)$. Let $\widetilde{\ell}(\widetilde{s}_i^{\;\;'}) = 1$. For $\widetilde{w} \in \mathrm{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_2)$, let $\widetilde{\ell}(\widetilde{w})$ be the least number $\widetilde{\ell}(\widetilde{w}') + \widetilde{\ell}(\widetilde{w}'')$ with $\widetilde{w} = \widetilde{w}' \quad \widetilde{w}''$ for some $\chi_3 \in \mathcal{C}(\chi)$, and some $\widetilde{w}' \in \mathrm{Mor}_{\widetilde{\mathcal{W}}}(\chi_3, \chi_2)$, some $\widetilde{w}'' \in \mathrm{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_3)$.

Theorem 5. ([HY08a, Theorem 5, Corollary 6]) *Let χ_1, $\chi_2 \in \mathcal{C}(\chi)$. For $w \in \mathrm{Mor}_{\mathcal{W}}(\chi_1, \chi_2)$ and $\widetilde{w}_1$, $\widetilde{w}_2 \in \widetilde{w} \in \mathrm{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_2)$ with $\widetilde{\phi}(\widetilde{w}_1) = \widetilde{\phi}(\widetilde{w}_2) = w$ and $\ell(w) = \widetilde{\ell}(\widetilde{w}_1) = \widetilde{\ell}(\widetilde{w}_2)$, we have $\widetilde{w}_1 = \widetilde{w}_2$. Further, if $\widetilde{w} \in \mathrm{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_2)$ is such that $\widetilde{\ell}(\widetilde{w}) > \ell(\widetilde{\phi}(\widetilde{w}))$, then $\widetilde{w} = \widetilde{w}' \quad \widetilde{s}_i^{r_i(\;3)} \quad \widetilde{s}_i^{\;3} \quad \widetilde{w}''$ for some $i \in I$, $\widetilde{w}' \in \mathrm{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_3)$ and $\widetilde{w}'' \in \mathrm{Mor}_{\widetilde{\mathcal{W}}}(\chi_3, \chi_2)$ with $\widetilde{\ell}(\widetilde{w}') + \widetilde{\ell}(\widetilde{w}'') = \widetilde{\ell}(\widetilde{w}) \quad 2$.*

Assume w_0 to be $s_{j_1}^{\;\;1} \quad s_{j_2}^{\;\;2} \qquad s_{j_M}^{\;\;M}$, where $M = |R_+|$, $r_1(\chi_1) = \chi$, and $r_j(\chi_j) = \chi_{j-1}$. Let $\bar{T}_1 = \mathrm{id}_{U(\;)}$. For $2 \quad n \quad M$, define the $\Bbbk$-algebra isomorphism $\bar{T}_n : U(\chi_{n-1}) \to U(\chi)$ by $\bar{T}_n = \bar{T}_{n-1} T_{j_{n-1}}$. Then as for $E_{\;\;i}$ of Theorem 2, we may put

$$(51) \qquad E_{\;\;i} = \bar{T}_i(E_{j_i})$$

for $1 \quad j \quad M$.

5 Shapovalov determinants

Let χ be a bi-character. We define the *Shapovalov matrix* Sh in the natural way for each $\alpha \in \mathbb{Z}_{\geq 0}\Pi$. More precisely, Sh is a $\dim U^+(\chi) \quad \times \dim U^+(\chi)$ -matrix whose components are elements of $U^0(\chi)$. Let $\rho \quad : \mathbb{Z}\Pi \to \Bbbk^\times$ be the (abelian) group homomorphism defined by $\rho \ (\alpha_i) = \chi(\alpha_i, \alpha_i)$. We use the *Kostant partition function* $P \ (\alpha, \beta, t) := \dim E^t U^+(\chi) \ {}_{-t}$, where we define $P \ (\alpha, \beta, t) = 0$ in case $\alpha \quad t\beta \notin \mathbb{Z}_{\geq 0}\Pi$.

Theorem 6. ([HY08b, Theorem 7.3]) *Let χ be finite-type. Assume that $\chi(\beta, \beta) \neq 1$ for all $\beta \in R_+$. Then for $\alpha \in \mathbb{Z}_{\geq 0}\Pi$, we have*

$$\det \mathrm{Sh} \quad = c \prod_{\in R_+^\chi} \prod_{t=1}^{h_\beta^\chi} (\rho \ (\beta)K \qquad \chi(\beta, \beta)^t L \ )^{P^\chi(\ , \ ,t)}$$

for some $c \in \Bbbk^\times$.

As stated below, for $U(\chi)$ which is the (ordinary or small) quantum group of a finite dimensional Lie algebra $\mathfrak{g}$, we have the generalization of (1) the one [dDK90] for $q \in \mathbb{C}^\times$ which is not a root of unity, and (2) the one [KL97] for $q \in \mathbb{C}^\times$ which is a primitive p-th root of unity for some prime number p.

Corollary 7. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of type A-G or a finite dimensional simple Lie superalgebra of type A-G. Then the Shapovalov determinant of the quantum group $U_q(\mathfrak{g})$ when q is not root of unity or the small quantum group $u_q(\mathfrak{g})$ when q is a primitive r-th root of unity for some positive integer $r \quad 2$ is given by*

$$c \prod_{\in R_+} \prod_{t=1}^{h_\beta^\chi} (q^{2(\ , \ )}K \qquad q^{(\ , \ )t} K^{-1})^{P^\chi(\ , \ ,t)}$$

for some $c \in \mathbb{C}^\times$.

We even recover the original ones due to Shapovalov [Sha72], and Kac [Kac77] (super cases):

Corollary 8. *Let $\mathfrak{g}$ be as above. Then the Shapovalov determinant of the enveloping algebra $U(\mathfrak{g})$ is given by*

$$c \prod_{\in R_+} \prod_{t=1}^{\infty} (H \quad + (\rho, \beta) \qquad \frac{(\beta, \beta)t}{2})^{P^\chi(\ , \ ,t)}$$

for some $c \in \mathbb{C}^\times$.

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