数理解析研究所講究録
第 1651 巻 2009 年 23-30

Stationary isothermic surfaces and some characterizations of the hyperplane *

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1 Introduction

This is based on the author's recent work with R. Magnanini [MS2, MS3]. Let Ω be a domain in $\mathbb{R}^N$ with $N \geq 3$, and let $u = u(x, t)$ be the unique bounded solution of the following problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{1.2}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}. \tag{1.3}$$

The problem we consider is to characterize the boundary $\partial\Omega$ such that the solution u has a stationary isothermic surface, say Γ. A hypersurface Γ in Ω is said to be a *stationary isothermic surface* of u if at each time t the solution u remains constant on Γ (a constant depending on t). Examples we easily notice are isoparametric hypersurfaces. Namely, Γ and $\partial\Omega$ are either parallel hyperplanes, concentric spheres, or concentric spherical cylinders. This complete classification of isoparametric hypersurfaces was given by Levi-Civita [LC] and Segre [Seg].

Almost complete characterizations of the sphere have already been obtained by [MS1, MS2] with the help of Aleksandrov's sphere theorem [Alek]. In this note,

*This research was partially supported by a Grant-in-Aid for Scientific Research (B) (♯ 20340031) of Japan Society for the Promotion of Science.

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we consider some characterizations of the hyperplane. Assume that Ω satisfies the uniform exterior sphere condition and Ω is given by

$$\Omega = \{ \ x = (x', x_N) \in \mathbb{R}^N \ : \ x_N > \varphi(x') \ \}, \tag{1.4}$$

where $\varphi = \varphi(x')$ $(x' \in \mathbb{R}^{N-1})$ is a continuous function on $\mathbb{R}^{N-1}$. We recall that Ω satisfies the *uniform exterior sphere condition* if there exists a number $r_0 > 0$ such that for every $\xi \in \partial\Omega$ there exists a ball $B_{r_0}(y)$ satisfying $\overline{B_{r_0}(y)} \cap \overline{\Omega} = \{\xi\}$, where $B_{r_0}(y)$ denotes an open ball centered at $y \in \mathbb{R}^N$ and with radius $r_0 > 0$. Then we have

Theorem 1.1 ([MS3]) *Assume that there exists a stationary isothermic surface* $\Gamma \subset \Omega$. *Then, under one of the following conditions* (i), (ii), *and* (iii), $\partial\Omega$ *must be a hyperplane.*

(i) $N = 3$.

(ii) $N \geq 4$ *and* φ *is globally Lipschitz continuous on* $\mathbb{R}^{N-1}$.

(iii) $N \geq 4$ *and there exists a non-empty open subset* A *of* $\partial\Omega$ *such that on* A *either* $H_{\partial\Omega} \geq 0$ *or* $\kappa_j \leq 0$ *for all* $j = 1, \cdots, N-1$, *where* $H_{\partial\Omega}$ *and* $\kappa_1, \cdots, \kappa_{N-1}$ *are the mean curvature of* $\partial\Omega$ *and the principal curvatures of* $\partial\Omega$, *respectively, with respect to the upward normal vector to* $\partial\Omega$.

Remark. When $N = 2$, this problem is easy. Since the curvature of the curve $\partial\Omega$ is constant from (2.3) in Lemma 2.1 in Section 2 of this note, we see that $\partial\Omega$ must be a straight line.

2 Outline of the proof of Theorem 1.1

In this section we give an outline of the proof. For the details, see [MS2, MS3]. Let $d = d(x)$ be the distance function defined by

$$d(x) = \text{ dist } (x, \partial\Omega), \quad x \in \Omega. \tag{2.1}$$

We start with a lemma.

Lemma 2.1 *The following assertions hold:*

(1) $\Gamma = \{ (x', \psi(x')) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1} \}$ *for some real analytic function* $\psi = \psi(x')$ $(x' \in \mathbb{R}^{N-1})$;

(2) *There exists a number* $R > 0$ *such that* $d(x) = R$ *for every* $x \in \Gamma$;

(3) φ *is real analytic and the mapping:* $\partial\Omega \ni \xi \mapsto x(\xi) \equiv \xi + R\nu(\xi) \in \Gamma$ *is a diffeomorphism, where* $\nu(\xi)$ *denotes the upward unit normal vector to* $\partial\Omega$ *at* $\xi \in \partial\Omega$, *that is,* $\partial\Omega$ *and* Γ *are parallel hypersurfaces with distance* R;

(4) *the following inequality holds:*

$$-\frac{1}{r_0} \le \kappa_j(\xi) < \frac{1}{R} \ (j = 1, \cdots, N-1) \ \textit{ for every } \xi \in \partial\Omega, \tag{2.2}$$

where $r_0 > 0$ *is the radius of the uniform exterior sphere condition for* Ω;

(5) *there exists a number* $c > 0$ *satisfying*

$$\prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(\xi) \right) = c \qquad \textit{for every } \xi \in \partial\Omega. \tag{2.3}$$

Proof. The strong maximum principle implies that $\frac{\partial u}{\partial x_N} < 0$, and (1) holds. Since Γ is stationary isothermic, (2) follows from a result of Varadhan [Va]:

$$-\frac{1}{\sqrt{s}} \log W(x, s) \to d(x) \quad \text{as } s \to \infty,$$

where $W(x, s) = s \int_0^\infty u(x, t) e^{-st} dt$ for $s > 0$. The inequality $-\frac{1}{r_0} \le \kappa_j(\xi)$ in (2.2) follows from the uniform exterior sphere condition for Ω. See Lemma 2.2 of [MS2] together with Lemma 3.1 of [MS1] for the remainder. □

Let us proceed to the proof of Theorem 1.1. Set

$$\Gamma^* = \left\{ x \in \Omega : d(x) = \frac{R}{2} \right\}. \tag{2.4}$$

Denote by κ_j^* and $\hat{\kappa}_j$ $(j = 1, \cdots, N-1)$ the principal curvatures of Γ^* and Γ, respectively, with respect to the upward unit normal vectors. Then, the mean curvatures H_{Γ^*} and H_Γ of Γ^* and Γ are given by

$$H_{\Gamma^*} = \frac{1}{N-1} \sum_{j=1}^{N-1} \kappa_j^* \quad \text{and} \quad H_\Gamma = \frac{1}{N-1} \sum_{j=1}^{N-1} \hat{\kappa}_j,$$

respectively. These principal curvatures have the following relationship:

$$\kappa_j = \frac{\kappa_j^*}{1+\frac{R}{2}\kappa_j^*} = \frac{\hat{\kappa}_j}{1+R\hat{\kappa}_j} \quad (j=1,\cdots,N-1). \tag{2.5}$$

Let $\mu = cR^{N-1}$. Then, it follows from (2.3) and (2.5) that

$$\prod_{j=1}^{N-1}(1-R\kappa_j) = \mu, \ \prod_{j=1}^{N-1}(1+R\hat{\kappa}_j) = \frac{1}{\mu}, \ \text{and} \ \prod_{j=1}^{N-1}\frac{1-\frac{R}{2}\kappa_j^*}{1+\frac{R}{2}\kappa_j^*} = \mu. \tag{2.6}$$

We distinguish three cases:

$$\text{(I) } \mu > 1, \ \text{(II) } \mu < 1, \ \text{and (III) } \mu = 1.$$

Let us consider case (I) first. By the arithmetic-geometric mean inequality and the first equation of (2.6) we have

$$1 - RH_{\partial\Omega} = \frac{1}{N-1}\sum_{j=1}^{N-1}(1-R\kappa_j) \geq \left\{\prod_{j=1}^{N-1}(1-R\kappa_j)\right\}^{\frac{1}{N-1}} = \mu^{\frac{1}{N-1}} > 1.$$

This shows that

$$H_{\partial\Omega} \leq -\frac{1}{R}\left(\mu^{\frac{1}{N-1}} - 1\right) < 0. \tag{2.7}$$

Since

$$(N-1)H_{\partial\Omega} = \ \mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) \quad \text{in } \mathbb{R}^{N-1},$$

by using the divergence theorem we get a contradiction as in the proof of Theorem 3.3 in [MS2]. In case (II), by the arithmetic-geometric mean inequality and the second equation of (2.6) we have

$$1 + RH_{\Gamma} = \frac{1}{N-1}\sum_{j=1}^{N-1}(1+R\hat{\kappa}_j) \geq \left\{\prod_{j=1}^{N-1}(1+R\hat{\kappa}_j)\right\}^{\frac{1}{N-1}} = \mu^{-\frac{1}{N-1}} > 1.$$

This shows that

$$H_{\Gamma} \geq \frac{1}{R}\left(\mu^{-\frac{1}{N-1}} - 1\right) > 0, \tag{2.8}$$

which yields a contradiction similarly.

Thus, it remains to consider case (III). By the above arguments we have

$$H_{\partial\Omega} \leq 0 \leq H_{\Gamma}. \tag{2.9}$$

Let us consider case (i) of Theorem 1.1 first. Since $N = 3$ and $\mu = 1$, it follows from the third equation of (2.6) that

$$2H_{\Gamma^*} = \kappa_1^* + \kappa_2^* = 0.$$

We observe that Γ^* is a graph of a function on $\mathbb{R}^2$. Therefore, by the Bernstein's theorem for the minimal surface equation, Γ^* must be a hyperplane. This gives the conclusion desired. (See [GT, Giu] for the Bernstein's theorem.)

Secondly, we consider case (iii) of Theorem 1.1. We have

$$1 - RH_{\partial\Omega} = \frac{1}{N-1}\sum_{j=1}^{N-1}(1 - R\kappa_j) \geq \left\{\prod_{j=1}^{N-1}(1 - R\kappa_j)\right\}^{\frac{1}{N-1}} = 1.$$

Hence, condition (iii) implies that

$$\kappa_j \equiv 0 \ \text{ on } A \ (j = 1, \cdots, N-1).$$

Then by the analyticity of $\partial\Omega$ we get

$$\kappa_j \equiv 0 \ \text{ on } \partial\Omega \ (j = 1, \cdots, N-1),$$

which shows that $\partial\Omega$ must be a hyperplane.

Thus it remains to consider case (ii) of Theorem 1.1. In this case, there exists a constant $L \geq 0$ satisfying

$$\sup_{\mathbb{R}^{N-1}} |\nabla\varphi| = L < \infty.$$

Then, it follows from (1) and (3) of Lemma 2.1 that

$$\sup_{\mathbb{R}^{N-1}} |\nabla\psi| = \sup_{\mathbb{R}^{N-1}} |\nabla\varphi| = L < \infty. \tag{2.10}$$

Hence, in view of this and (3) of Lemma 2.1, we can define a number $K^* > 0$ by

$$K^* = \inf\{K > 0 : \psi \leq \varphi + K \ \text{ in } \mathbb{R}^{N-1}\}. \tag{2.11}$$

Then we have

$$\varphi \leq \psi \leq \varphi + K^* \ \text{ in } \mathbb{R}^{N-1}. \tag{2.12}$$

We define a real analytic function h on $\mathbb{R}^{N-1}$ by

$$h(x') = \varphi(x') + K^* \ \text{ for } x' \in \mathbb{R}^{N-1}.$$

Moreover, by writing

$$M(h) = \text{ div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) \text{ and } M(\psi) = \text{ div}\left(\frac{\nabla \psi}{\sqrt{1+|\nabla \psi|^2}}\right),$$

from (2.9) and (2.12) we have

$$M(h) \leq 0 \leq M(\psi) \text{ and } \psi \leq h \text{ in } \mathbb{R}^{N-1}. \tag{2.13}$$

Hence, the method of sub- and super-solutions with the help of (2.10) yields that there exists $v \in C^\infty(\mathbb{R}^{N-1})$ satisfying

$$M(v) = 0 \text{ and } \psi \leq v \leq h \text{ in } \mathbb{R}^{N-1}, \text{ and } \sup_{\mathbb{R}^{N-1}} |\nabla v| < \infty.$$

(See [MS3] for the details.) Therefore, Moser's theorem [Mo], Corollary, p. 591, implies that v is affine. We set $\eta = \nabla v \in \mathbb{R}^{N-1}$.

On the other hand, by the definition of K^* in (2.11), there exists a sequence $\{z_n\}$ in $\mathbb{R}^{N-1}$ satisfying

$$\lim_{n\to\infty} (h(z_n) - \psi(z_n)) = 0. \tag{2.14}$$

Define a sequence of functions $\{\varphi_n\}$ by

$$\varphi_n(x') = h(x'+z_n) - h(z_n) \ \ (= \varphi(x'+z_n) - \varphi(z_n)).$$

From (2.2) and (2.10) we see that all the second derivatives of φ are bounded in $\mathbb{R}^{N-1}$. Hence we can conclude that there exists a subsequence $\{\varphi_{n'}\}$ of $\{\varphi_n\}$ and a function $\varphi_\infty \in C^1(\mathbb{R}^{N-1})$ such that $\varphi_{n'} \to \varphi_\infty$ in $C^1(\mathbb{R}^{N-1})$ as $n' \to \infty$. Since $M(\varphi_n) \leq 0$ in $\mathbb{R}^{N-1}$, we have that $M(\varphi_\infty) \leq 0$ in $\mathbb{R}^{N-1}$ in the weak sense. Also, since $0 \leq h(x'+z_{n'}) - v(x'+z_{n'})$ in $\mathbb{R}^{N-1}$, with the help of (2.14), letting $n' \to \infty$ yields that

$$0 \leq \varphi_\infty(x') - \eta \cdot x' \text{ in } \mathbb{R}^{N-1}.$$

Consequently, we have

$$M(\varphi_\infty) \leq 0 = M(\eta\cdot x') \text{ and } \varphi_\infty \geq \eta\cdot x' \text{ in } \mathbb{R}^{N-1}, \text{ and } \varphi_\infty(0) = 0 = \eta\cdot 0. \tag{2.15}$$

Hence, the strong comparison principle implies that $\varphi_\infty(x') \equiv \eta\cdot x'$ in $\mathbb{R}^{N-1}$. Here we have used Theorem 10.7 together with Theorem 8.19 in [GT]. Therefore we conclude that

$$\varphi(x'+z_n) - (v(x'+z_n) - K^*) \to 0 \text{ in } C^1(\mathbb{R}^{N-1}). \tag{2.16}$$

Similarly, we can obtain

$$v(x'+z_n)-\psi(x'+z_n)\to 0 \ \text{ in } C^1(\mathbb{R}^{N-1}). \tag{2.17}$$

Therefore, it follows from (3) of Lemma 2.1, (2.16), and (2.17) that the distance between two hyperplanes determined by two affine functions v and $v-K^*$ must be R. Hence, since $v-K^*\le\varphi\le\psi\le v$ in $\mathbb{R}^{N-1}$, we conclude that

$$\psi\equiv v \ \text{ and } \ \varphi\equiv v-K^* \ \text{ in } \mathbb{R}^{N-1},$$

which shows that $\partial\Omega$ is a hyperplane. □

3 Concluding remarks

Let us explain the relationship between Theorem 1.1 and Theorems 3.2, 3.3, and 3.4 in [MS2]. When $\mu=1$, we have

$$1+RH_\Gamma=\frac{1}{N-1}\sum_{j=1}^{N-1}(1+R\hat{\kappa}_j)\ge\left\{\prod_{j=1}^{N-1}(1+R\hat{\kappa}_j)\right\}^{\frac{1}{N-1}}=1.$$

Therefore, the assumption of Theorem 3.2 that $H_\Gamma\le 0$ implies that $\hat{\kappa}_j\equiv 0$ $(j=1,\cdots,N-1)$. This shows that Γ is a hyperplane, and hence $\partial\Omega$ must be a hyperplane. Thus, Theorem 3.2 is contained in Theorem 1.1 with its proof. In the case where Ω is given by (1.4), Theorem 3.3 is contained in Theorem 1.1 with condition (iii). Since Theorem 3.4 does not assume the uniform exterior sphere condition for Ω, Theorem 3.4 is independent of Theorem 1.1.

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