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RATES OF CONVERGENCE FOR MONOTONE APPROXIMATIONS OF VISCOSITY SOLUTIONS OF FULLY NONLINEAR UNIFORMLY ELLIPTIC PDE

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ABSTRACT. I present here a brief review of the issue of rates of convergence for monotone approximations of viscosity solutions and describe a recent result that settles the problem for uniformly elliptic second-order equations without any convexity assumptions.

1. INTRODUCTION

In this note I describe a recent joint result with L. Caffarelli ([CS1]) concerning rates of convergence for monotone, stable and consistent approximations to viscosity solutions of fully nonlinear uniformly elliptic pde. The general methodology introduced in ([CS1] is also used in [CS2] to obtain error estimates for (periodic and random strongly mixing) homogenization.

Obtaining rates of convergence (error estimates) for approximations to viscosity solutions of fully nonlinear second-order pde has been one of the longstanding problems in the theory of viscosity solutions. In contrast, in the completely degenerate case with no dependence on the Hessian, i.e., for Hamilton-Jacobi equations, rates were obtained from the beginning of the theory (Cradall-Lions [CL] and Souganidis [S]). The convergence of monotone approximation schemes for second-order pde was proved by Barles and the author in [BS]. Establishing rates of convergence for second-order pde proved to be, however, a far more difficult problem. The reason is that, contrary to the first-order case, it was not known how to approximate viscosity solutions of second-order equations so that both the equation is somehow preserved and the approximations share some uniform regularity with respect to their second derivatives. Such regularity is implicit for viscosity solutions of, possibly degenerate, elliptic convex with respect to the Hessian and the gradient equation. This fact was used in a series of papers by Krylov [K1, K2, K3] and Barles and Jacobsen [BJ1, BJ2] who obtained in the convex setting some rates. The results of Krylov are based on stochastic control considerations while Barles and Jacobsen are using more pde-type arguments and, in particular, the approximation of the equations by switching control systems.

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The difficulty for the general nonconvex problem was overcome in [CS1]. The new ingredients are (i) a regularity result about viscosity solutions of uniformly elliptic equations, which roughly speaking, says that, on some "large" subsets of the domain, solution have second order expansions with error of prescribed size that controls, in a universal way, the size of the exceptional sets, and (ii) the introduction of the notion of δ-viscosity solutions, which are "regular" approximations to viscosity solutions at some uniform distance (a small power of δ). The regularity result is used to obtain an error for "quadratic" data, while δ-viscosity solutions allow to "translate" the error for quadratics to an "actual" rate for general solutions.

This short note is organized as follows: In Section 2 I recall the convergence result of [BS] as well as some basic facts about viscosity solutions (sup- and inf-convolutions). In Section 3 I present a new informal proof for the error estimate for Hamilton-Jacobi. Finally, in Section 4 I introduce the regularity result and the δ-solutions and state and prove the result about the rate.

The goal here is to introduce the key ideas. Therefore I will not state all the necessary assumptions and will not describe all the details.

2. Convergence of monotone approximations

I summarize here the main result of [BS] as it applies to the problem

$$F(D^2, Du, u, x) = 0 \quad \text{in} \quad \mathbb{R}^N , \tag{2.1}$$

where, if S^N denotes the space of $N \times N$ symmetric matrices,

$$F \in C(S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N). \ \text{is degenerate elliptic;} \tag{2.2}$$

The nonlinearity must, of course, satisfy more assumptions for (2.1) to admit well posed viscosity solutions (see Crandall, Ishii and Lions [CIL] for such a discussion).

Following the notation of [BS], I consider approximations

$$S([u^h]_x, u^h(x), x, h) = 0 \tag{2.3}$$

of (2.1) which are monotone , i.e.,

$$\text{if } \ u \geqq v \ , \quad \text{then} \quad S([u]_x, t, x, h) \leqq S([v]_x, t, x, h) \ , \quad \text{for all} \quad (t,x) \in \mathbb{R} \times U \ , \tag{2.4}$$

stable, i.e., for some uniform $C > 0$ and all $h \in (0,1)$,

$$\|u^h\| \leqq C \ , \tag{2.5}$$

where $\|\cdot\|$ stands for the sup-norm, and consistent, i.e., for all smooth functions ϕ and locally uniformly with respect to x,

$$S([\phi+\xi]_x, \phi(y)+\xi, y, h) \underset{\substack{h\to 0\\ y\to x\\ \xi\to 0}}{\longrightarrow} F(D^2\phi(x), D\phi(x), \phi(x), x)\ . \tag{2.6}$$

The result proved in [BS], where I refer for more general statements as well as concrete examples, is:

Theorem 1. *Assume* (2.2), (2.4), (2.5) *and* (2.6), *and let* u^h *and* u *be the solutions of* (2.3) *and* (2.1) *respectively. Then, as* $h \to 0$, $u^h \to u$ *uniformly in* $\mathbb{R}^N$.

Sketch of the proof of Theorem 1. The stability assumption (2.5) implies that the classical half-relaxed limits

$$u^*(x) = \operatorname{limsup}_{y\to x, h\to 0} u^h(y) \text{ and } u_* = \operatorname{liminf}_{y\to x, h\to 0} u^h(y) \tag{2.7}$$

are defined. Owing to the comparison of viscosity solutions of (2.1) — here it is implicitly assumed that it holds — it suffices to show that u^* and u_* are respectively sub- and super-solution of (2.1).

Next I sketch the argument leading to the first claim. To this end, assume that, for a smooth function ϕ, $u^* - \phi$ attains a strict global maximum at some $x_0 \in \mathbb{R}^N$. It follows that along subsequences, which for simplicity are still denoted by h, $h \to 0$, $u^h - \phi$ attains a global maximum at $x_h \to x_0$, i.e.,

$$u^h(x) \le \phi(x) + u^h(x_h) - \phi(x_h), \tag{2.8}$$

and, in addition,

$$u^h(x_h) \to u^*(x_0) \quad \text{and} \quad x_h \to x_0. \tag{2.9}$$

The monotonicity of the scheme (2.4) yields

$$S([\phi+\xi_h]_x^h, \phi(y)+\xi_h, y, h) \le 0,$$

where, as $h \to 0$,

$$\xi_h = u^h(x_h) - \phi(x_h) \to h^*(x_0) - \phi(x_0).$$

The claim is now an immediate consequence of the consistency of the scheme. $\square$

3. Sup- and inf-convolutions

To present the main steps of the proofs of the rate estimates as well as to show the new ideas needed for the second-order case it is necessary to recall the by now classical inf- and sup-convolution regularizations of viscosity solutions and their properties. As in the previous section I concentrate for simplicity on problems defines on $\mathbb{R}^N$.

For $u : \mathbb{R}^N \to \mathbb{R}$ and $\varepsilon > 0$ the sup- and inf-convolutions of u are given respectively by

$$\bar{u}_\varepsilon(x) = \sup\{u(y) - (2\varepsilon)^{-1}|x-y|^2\} \tag{3.1}$$

and

$$\underline{u}_\varepsilon(x) = \inf\{u(y) + (2\varepsilon)^{-1}|x-y|^2\} \; . \tag{3.2}$$

The following proposition summarizes the key properties of $\bar{u}_\varepsilon$ and $\underline{u}_\varepsilon$. For the proof I refer to Jensen, Lions and Souganidis [JLS].

Proposition 2. *Assume that $u : \mathbb{R}^N \to \mathbb{R}$ is continuous and bounded and let $\bar{u}_\varepsilon$ and $\underline{\varepsilon}$ be defined by* (3.1) *and* (3.2). *Then*

(i) *$\bar{u}_\varepsilon$ and $\underline{u}_\varepsilon$ are Lipschitz continuous (with Lipschitz constant depending on ε).*

(ii) *As $\varepsilon \to 0$, $\bar{u}_\varepsilon \searrow u$ and $\underline{u}_\varepsilon \nearrow u$ locally uniformly.*

(iii) *For $\varepsilon > 0$ and $x \in \mathbb{R}^N$ let $\bar{y}_\varepsilon(x)$ and $\underline{y}_e p(x)$ be points where the maximum and minimum are achieved in* (3.1) *and* (3.2) *respectively. Then, for some $o(1) \to 0$ as $\varepsilon \to 0$,*

$$|\bar{y}_\varepsilon(x) - x| = o(1) \quad \textit{and} \quad |\underline{y}_e p(x) - x| = o(1).$$

(iv) *$\bar{u}_\varepsilon$ is semiconvex and $\underline{u}_\varepsilon$ is semiconcave, and $D^2\bar{u}_\varepsilon \geqq -I/\varepsilon$ and $D^2\underline{u}_\varepsilon \leqq -I/\varepsilon$, where I is the identity $N \times N$ matrix. Moreover, $\bar{u}_\varepsilon$ and $\underline{u}_\varepsilon$ are twice differentiable a.e.*

(v) *If u is Lipschitz continuous, then the Lipschitz constants of $\bar{u}_\varepsilon$ and $\underline{u}_\varepsilon$ are independent of ε. Moreover,*

$$\|\bar{u}_\varepsilon - u\| \leqq \|Du\|\varepsilon \quad \textit{and} \quad \|\underline{u}_\varepsilon - u\| \leqq \|Du\|\varepsilon \; .$$

(vi) *If u is a subsolution (resp. supersolution) of* (2.1), *then $\bar{u}_\varepsilon$ (resp. $\underline{u}_\varepsilon$) is a subsolution (resp. supersolution) of*

$$F(D^2w, Dw, w, \bar{y}_\varepsilon(x)) = 0 \quad \textit{and} \quad F(D^2w, Dw, w, x) = o(1)$$

and

$$F(D^2w, Dw, w, \underline{y}_e p(x)) = 0 \quad \textit{and} \quad F(D^2w, Dw, w, x) = o(1),$$

respectively for some $o(1) \to 0$.

4. Error estimates for Hamilton-Jacobi equations

Rates of convergence for monotone, consistent and stable approximation schemes for viscosity solutions of Hamilton-Jacobi equations like

$$F(Du, u, x) = 0 \text{ in } U \tag{4.1}$$

were obtained by Crandall and Lions [CL] and the author [S] very early in the development of the theory of viscosity solutions.

I describe next briefly the basic result and a present a new semi-rigorous proof. To obtain an error it is necessary to strengthen the monotonicity hypothesis to (2.4) to

$$\begin{cases} \text{there exists } \lambda > 0 \text{ such that, if } u \leqq v, m \geqq 0, \text{ then, for all } r > 0, \\ \qquad S([u+m]_x, r+m, x, h) \geqq S([v]_x, r, x, h) + \lambda m, \end{cases} \tag{4.2}$$

and to introduce a rate in the consistency condition (2.6). For (2.1) the natural assumption is

$$\begin{cases} \text{there exists a universal constant } C > 0 \text{ such that, for all smooth } phi \text{ and all } x, \\ \qquad |S([\phi]_x, \phi(x), x, h) - F(D\phi(x), \phi(x), x)| \leqq C(1 + |D^2\phi(x)|)h. \end{cases} \tag{4.3}$$

The result of [CL] and [S] is:

Theorem 3. *Let the solution u of* (2.1) *be Lipschitz continuous and assume* (2.5), (4.2) *and* (4.3). *There exists $K > 0$, depending on F and the Lipschitz constant of u, such that*

$$\|u - u^h\| \leqq K h^{1/2} \ .$$

The rate in Theorem 3 is optimal in this generality. When F is convex Capuzzo-Dolcetta and Ishii [CI] the rate be improved from one side to

$$-Kh \leqq u - u^h \leqq K h^{1/2} \ . \tag{4.4}$$

Next I present a formal proof of Theorem 3. The argument, which has not appeared anywhere else, can be made rigorous after some minor modifications.

Sketch of the proof of Theorem 3. Here I show that

$$u^h \leqq u + K h^{1/2} \ ,$$

since the other inequality follows similarly.

Let $\underline{u}_\varepsilon$ be the inf-convolution of u. Since u is Lipschitz continuous, Proposition 2 (v) yields

$$\|u - \underline{u}_\varepsilon\| \leqq \|Du\|\varepsilon \ .$$

Next I compare u^h and $\underline{u}_\varepsilon$. The key idea is to use $\underline{u}_\varepsilon$ as a test function in (4.3). This is, of course, not immediately clear, since test functions are supposed to be C^2-functions and the u_ε's are clearly not. This technical difficulty can be, however, overcome using typical viscosity solution techniques (doubling variables, etc.). To simplify the argument even further I will make the additional (formal) assumption that actually

$$|D^2\underline{u}_\varepsilon| \leqq 1/\varepsilon \ .$$

With all these simplifications, (4.3) yields

$$S([\underline{u}_\varepsilon]_h, \underline{u}_\varepsilon(x), x, h) \geqq F(D\underline{u}_\varepsilon, \underline{u}_\varepsilon, x) - C(1 + |D^2\underline{u}_\varepsilon|)h$$

Finally assume that, for some $c > 0$,

$$F(D\underline{u}_\varepsilon, \underline{u}_\varepsilon, x) \geqq -c\varepsilon \ .$$

Then

$$S([u_\varepsilon]_h, \underline{u}_\varepsilon(x), x, h) \geqq -c\varepsilon - c(1 + \frac{1}{\varepsilon})h \ .$$

Let $m = \max(u^h - \underline{u}_\varepsilon)$. Then

$$u^h \leqq u_\varepsilon + m \ ,$$

and, in view of (4.2),

$$S([\underline{u}_\varepsilon]^h_x, u_\varepsilon(x), x, h) \leqq S([u^h_h - m]_x, u^h(x) - m, x, h) \leqq -\lambda m + S([u_h]^h, u^h(x), x, h) \ .$$

It follows that

$$\lambda m \leqq C(1 + \frac{1}{\varepsilon})h + c\varepsilon$$

and, hence,

$$u^h - u \leqq u^h - \underline{u}_\varepsilon + \underline{u}_\varepsilon - u \leqq (\|Du\| + c)\varepsilon + C(1 + \frac{1}{\varepsilon})h \ .$$

Optimizing in ε yields the rate. □

5. Rates of convergence for (uniformly) elliptic equations

To obtain an error estimate for approximations to (2.1), it is necessary, in addition to (4.2), to introduce again a rate in (2.6). Since (2.1) depends on the Hessian, it is natural to assume that

$$(5.1) \qquad \begin{cases} \text{there exists a universal constant } C > 0 \text{ such that, for all smooth } phi, \\ \qquad |S([\phi]^h_x, \phi(x), x, h) - F(D^2\phi, D\phi, \phi, x)| \leqq C(1 + |D^3\phi|)h \ ; \end{cases}$$

note that instead of $|D^2\phi|$ in the right hand side of the above estimate, it is possible to use some Hölder-seminorm of $D^2\phi$, etc..

No matter what the exact form of (5.1) is, however, the scheme of proof described in Section 4 fails because it is not known how to approximate viscosity solutions of (2.1) by appropriate sub- and super-solutions for which there is a control on the modulus of regularity of the second-derivative.

Consider next, for simplicity, the problem

$$\begin{cases} F(D^2u) = f & \text{in } U\ , \\ u = g & \text{on } \partial U\ . \end{cases} \tag{5.2}$$

If F is uniformly elliptic and convex it is known from the Krylov-Safonov-Evans regularity theory that the solutions are actually in $C^{2,\alpha}$ for some $\alpha \in (0,1)$. In this setting it is then straightforward to obtain an error estimate. For more general equations but still uniformly continuous and convex, Krylov [K1, K2, K3] and Barles and Jacobsen [BJ1, BJ2] were able to find error estimates using stochastic control and pde techniques, respectively. The results of [K3] also apply to degenerate elliptic but always convex nonlinearities.

More recently Caffarelli and the author [CS1] considered (5.2) with F uniformly elliptic but neither convex nor concave. The approach of [CS1] is based on a new regularity result for viscosity solutions of (5.2) and the notion of δ-viscosity solutions which are regularizations/approximations at uniform (δ^α) distance from the solution. Proving the error estimate then reduces to showing that the numerical solutions are $\delta = \delta(h)$-viscosity solutions. The regularity is used to obtain the result about the δ-solutions.

Definition 4. $u \in C(\bar{U})$ is a δ-subsolution (resp. δ-supersolution) of (5.2) if, for all $x \in U$ such that $B(x,\delta) \subset U$ and all quadratics P such that $P = O(\delta^{-\alpha})$, for some universal $\alpha > 0$, $u \leqq P$ (resp. $u \geqq P$) in $B(x,\delta)$ and $u(x) = P(x)$, $F(D^2P) \leqq f(x)$ (resp. $F(D^2P) \geqq f(x)$).

It is easy to check that viscosity solutions are always δ-subsolutions, while δ-solutions are not always solutions.

The main result about δ-viscosity solutions proved in [CS1] and [CS2] is:

Theorem 5. *Let u be a Lipschitz continuous solution of* (5.2) *and assume that u^+ (resp. u^-) is δ-subsolution (resp. delta-supersolution) of* (5.2) *such that, for some $\eta > 0$, $|u^\pm - u| = O(\delta^\eta)$ on ∂U. Then there exists a universal $\theta > 0$ such that $u - u^\pm = O(\delta^\theta)$ in U.*

Before I discuss Theorem 5 we show how it implies error estimates for (monotone and consistent) numerical approximations of (5.2). The key step is

Theorem 6. *Assume* (4.2) *and* (5.1). *Then u^h is a $\delta = Lh$-solution of $F(D^2u) = f \pm Kh$ for some uniform $K > 0$ and $L > 0$.*

Proof. Since the arguments are similar, here I show that u^h is an h-subsolution. To this end assume that, for some $x \in U$ such that $B(x,h) \subset U$ and $P \in S^N$ such that $P = O(h^{-\alpha})$, $u^h \leqq Q$ in $B(x,Lh)$ and $(u^h - P)(x) = 0$. The claim is an immediate consequence of the the monotonicity and the strong consistency. The constant L depends, among other things, on the particular choice of the scheme, i.e., the number of grid points around a fixed point x that enter in the scheme. □

Theorem 5 and Theorem 6 yield the main result about rates, which is proved in [CS1].

Theorem 7. *Let $u \in C^{0,1}(U)$ and u^n be solutions of* (5.2) *and* (2.3) *and assume that* (4.2), (5.1) *and that F is uniformly elliptic. There exist universal $\theta > 0$ and constant K depending on F and the Lipschitz constant of u such that, on $\bar{U}$,*

$$\|u - u^h\| \leqq Kh^\theta \ .$$

To simplify things in the statement of Theorem 7 I omitted any discussion about the boundary conditions. Finally a result similar to Theorem 7 also holds for the general problem (2.1) for F uniformly elliptic. The proof will appear in [CS3].

Next I discuss briefly some of the ingredients of the proof of Theorem 5. For the details I refer to [CS1] and [CS2]. The first key step is

Theorem 8. *Let $u \in C^{0,1}$ be a solution of $F(D^2u) = f$ in B_1 with f Lipschitz and F uniformly elliptic. There exist positive t_0 and $\sigma 0$, depending on the ellipticity constants of F and the dimension, such that, for $t \geqq t_0$, there exist $A_t \subset B_1$ such that $|(B_1 \backslash A_t) \cap B_{1/2}| \leqq t^{-\sigma}$ and, for all $x_0 \in A_t \cap B_{1/2}$, there exist $P^{\pm}_{t,x_0} \in S^N$ such that $F(P^{\pm}_{t,x_0}) = f(x_0)$, $|P^{\pm}_{t,x_0}| \leqq t$ and*

$$u(x) = u(x_0)\frac{1}{2}(P^{\pm}_{t,x_0}(x - x_0), x - x_0) + O(t|x - x_0|^3) \quad in \quad B_1 \ .$$

It turns out that the conclusions of Theorem 8 carry over to the sup- and inf-convolution approximations of Lipschitz continuous solutions of (5.2).

Indeed the following holds:

Theorem 9. *Assume the hypotheses of Theorem 8 and let u^+_ε and u^-_ε be respectively the sup- and inf-convolution of u. There exist universal $t_0, \sigma > 0$ such that, for all $t \geqq t_0$, there exist $A^\varepsilon_t \subset B_1$ with $|(B_1 \setminus A^\varepsilon_t) \cap B_{1/2}| \leqq t^{-\sigma}$ and, for all $x_0 \in A^\varepsilon_t \cap B_{1/2}$, there exist $P^{\pm,\varepsilon}_{t,x_0} \in S^N$ such that $F(^{\pm,\varepsilon}_{t,x_0}) = f(x_0) + o(1)$, $|P^{\pm,\varepsilon}_{t,x_0}| \leqq t$ and*

$$u^{\pm}_\varepsilon(x) = u_\varepsilon(x_0) + \frac{1}{2}(P^{\pm,\varepsilon}_{t,x_0}(x - x_0), (x - x_0)) + O(t|x - x_0|^3) \quad in \quad B_1 \ .$$

The result follows from Theorem 8 provided one uses some additional properties of the sup- and inf-convolutions which hold only for solutions to uniformly elliptic equations. They are stated in

Proposition 10. *Let u_ε^+ and u_ε^- be respectively the sup- and inf-convolution approximations to a Lipschitz continuous solution of* (2.1) *and denote by $y^+(x)$ (resp. $y^-(x)$) a point where the the sup (resp. inf) is achieved in the definition. Then:*

(i) *There exists a universal $C > 0$ such that $|x_1 - x_2| \leqq C|y(x_1) - y(x_2)|$* . (ii) *If P is a paraboloid touching u_ε^+ (resp. u_ε^-) from above (resp. below) at x, then u is touched at $y^+(x)$ (resp. $y^-(x)$) from above (resp. below) by a paraboloid P_ε and*

$$D^2u_\varepsilon^+(x) \geqq D^2u(y^+(x)) + C\varepsilon^2|Du|^2 \quad \textit{and} \quad D^2u_\varepsilon^-(x) \leqq D^2u(y^-(x)) - C\varepsilon^2|Du|^2 \ .$$

I conclude with a heuristic discussion of the proof of Theorem 8. The first step is to change the right hand side by δ^α to have some room for the calculations. Then the solution and the given δ-sub- and super-solutions are regularized by $\varepsilon = \varepsilon(\delta)$ sup- and inf-convolution. These are semi-convex or concave in the right direction, provide appropriate bounds for the Hessian and have second-order expansions (with controlled error) outside small sets with measure estimated by the size of the quadratics in the expansion. The approximations are clearly δ-sub- and super-solutions around points of second-differentiability. What happens on the small exceptional sets is controlled by the classical Alexandrov-Bakelman-Pucci estimate by constructing the convex envelop $\Gamma(w)$ of the difference w of the approximations of u and the δ-solutions. The control on the sizes of the Hessians and the exceptional sets force the contact set $\{\Gamma(w) = w\}$, where the support of $D^2\Gamma(w)$ is concentrated, to be small. The estimate on the Hessian of the approximations then implies that, even in this small exceptional case, the quantity $\det \Gamma^2(w)|\{\Gamma(w) = w\}|$, which controls the size of w, falls within the δ^α margin of error.

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