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A NEW FREQUENCY FORMULA AND APPLICATIONS TO A SINGULAR PERTURBATION PROBLEM

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ABSTRACT. The present paper contains the announcement and heuristics of results to appear elsewhere. We introduce a new nonlinear frequency formula for a semilinear free boundary problem and use this tool to analyze the singular set in the limit of a singular perturbation problem.

1. INTRODUCTION

Consider the parabolic free boundary problem

$$\Delta u - \partial_t u = 0 \text{ in } \{u > 0\}\,,\ |\nabla u| = 1 \text{ on } \partial\{u > 0\} \tag{1}$$

The problem above has been derived by J.D. Buckmaster (formally) as *singular limit* from the following model for the propagation of equidiffusional premixed flames as $\varepsilon \to 0$, i.e. the activation energy goes to infinity:

$$\Delta u_\varepsilon - \partial_t u_\varepsilon = \beta_\varepsilon(u_\varepsilon) \tag{2}$$

Here $\beta_\varepsilon(z) = \frac{1}{\varepsilon}\beta(\frac{z}{\varepsilon})$, $\beta \in C_0^1([0,1])$, $\beta > 0$ in $(0,1)$ and $\int \beta = \frac{1}{2}$. In the model $u_\varepsilon = \lambda(T_c - T)$, T_c is the flame temperature, which is assumed to be constant, T is the temperature outside the flame and λ is a normalization factor.

Let us shortly summarize the most relevant known results for both the free boundary problem as well as the singular limit: in [1], H.W. Alt and L.A. Caffarelli proved via minimization of the energy $\int(|\nabla u|^2 + \chi_{\{u>0\}})$ – here $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$ – existence of a stationary solution of (1) in the sense of distributions. They also derived regularity of the free boundary $\partial\{u > 0\}$ up to a set of vanishing $n-1$-dimensional Hausdorff measure. [10] shows that

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existence of singular minimizers implies the existence of singular minimizing cones. Those have been excluded in [3], implying full regularity of minimizers in *three* dimensions. *Non-minimizing* singular cones *do* in fact appear for $n = 3$ (see [1, example 2.7]). By [5], singular minimizing cones exist in dimension $7, 9, \ldots$.
Moreover it is known, that solutions of the Dirichlet problem in two space dimensions are not unique (see [1, example 2.6]).
For the time-dependent (1), both "trivial non-uniqueness" (the positive solution of the heat equation is always another solution of (1)) and "non-trivial uniqueness" (see [8]) occur. Even for flawless initial data, classical solutions of (1) develop singularities after a finite time span; consider e.g. the example of two planar traveling waves approaching each other and colliding after a finite time span. Concerning the reaction-diffusion equation, L.A. Caffarelli and J.L. Vazquez proved in [4] uniform estimates for (2) and a convergence result: for initial data u^0 that are strictly mean concave in the interior of their support, a sequence of ε-solutions converges to a solution of (1) in the sense of distributions.
For a convergence result of non-negative solutions to a viscosity solution see [7].
Then, there is the convergence to a solution in the sense of domain variations [9] which seems to contain more information than the viscosity solution. Domain variation solutions are pairs (u, χ) where the order parameter χ shares many properties with the characteristic function $\chi_{\{u>0\}}$ but does not necessarily coincide with it. The most important property of domain variation solutions is the equation

$$\int_{-\infty}^{\infty} \int_{\mathbf{R}^n} [-2\partial_t u \, \nabla u \cdot \xi \; + \; (|\nabla u|^2 \; + \; \chi) \operatorname{div} \xi \; - \; 2\nabla u D\xi \nabla u] \; = \; 0$$

for every $\xi \in C_0^{0,1}(\Omega_T; \mathbf{R}^n)$. By [9], *all limits* of the singular perturbation problem (2) are domain variation solutions. Last, it is known that flatness implies regularity [2]. As a consequence, the regular part of the free boundary is relatively open to the whole free boundary.
A natural question is, whether limits of (2) are solutions in the sense of distributions, i.e.

$$\Delta u(t) - \partial_t u(t) = \mathcal{H}^{n-1} \lfloor \partial \{u(t) > 0\}.$$

Unfortunately the answer is "No". The reason is that "multiplicity 2" solutions like for example $\theta|x_1|$ appear as ϵ-limits for each constant $\theta \in (0, 1]$.
That suggests modifying the above question to the question whether

limits of (2) are solutions in the sense of

$$\Delta u(t) - \partial_t u(t) = \mathcal{H}^{n-1}\lfloor\partial\{u(t) > 0\} \; + \; 2\theta(t,x)\mathcal{H}^{n-1}\lfloor\Sigma(t).$$

Here $\Sigma(t)$ is the part of the singular set, where the rotated solution is close to $\theta|x_1|$. The modified question is still unanswered. [9] gives a partial answer, i.e.

(3) $\Delta u(t) - \partial_t u(t) = \mathcal{H}^{n-1}\lfloor\partial\{u(t) > 0\} + 2\theta(t,x)\mathcal{H}^{n-1}\lfloor\Sigma(t) + \lambda(t)$,

where the density of $\lambda(t)$ with respect to $\mathcal{H}^{n-1}$ vanishes at every point. However existence or non-existence of non-zero *defect measure* $\lambda(t)$ still eludes us.

For the stationary problem — where the difficulties are very similar — there is a relation to *harmonic measures*: it turns out that the harmonic measure of the free boundary and Δu are mutually absolutely continuous. This makes it possible to use in two dimensions a beautiful result by Tom Wolff [11], stating that every harmonic measure *in the plane* lives on a set of σ-finite length. Unfortunately the analogous property in three dimensions does not hold, i.e. there is a finite domain in $\mathbf{R}^3$ such that the harmonic measure puts *all mass on a set of dimension* $2 + \alpha$ with $\alpha > 0$ (see [12]).

Here we announce new tools that lead to a structural analysis of singularities in the limit problem as well as an estimate of the Hausdorff dimension of the topological free boundary corresponding to the result [6] by Peter Jones and Tom Wolff. Everything that follows is described for the stationary problem, but analogous formulas hold for all limits of the parabolic singular perturbation problem (2).

2. Degenerate points

The limit problem possesses the *invariant scaling*

$$u_r(x) = u(x_0 + rx)/r \;,$$

for which there are tools like monotonicity formula etc. The difficulty is that at *degenerate singular points*, i.e. x_0 such that

$$r^{-n-1}\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} \;\to\; 0 \;, \; r \to 0,$$

those tools do not yield information, and the $\mathcal{L}^n$-density of the phase $\{\chi = 0\}$ being zero results in a loss of control.

3. Monotonicity formula and points of highest density

Let us recall a monotonicity formula from [10] related to the monotonicity formula by R. Schoen-K Uhlenbeck for harmonic maps:

Theorem 3.1. *The function*

$$\Phi^u_{x_0}(r) := r^{-n} \int_{B_r(x_0)} (\Delta u^2/2 + \chi) \; - \; r^{-1-n} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} ,$$

satisfies at every x_0 *and for* $r \in (0, \operatorname{dist}(x, \partial\Omega))$ *the monotonicity identity*

$$\Phi^u_{x_0}(\sigma) \; - \; \Phi^u_{x_0}(\rho)$$

$$\geq \int_\rho^\sigma r^{-n} \int_{\partial B_r(x_0)} 2 \left(\nabla u \cdot \nu - \alpha \frac{u}{r}\right)^2 d\mathcal{H}^{n-1} \, dr \; \geq 0 .$$

The density $x \mapsto \Phi^u_x(0+)$ *is an upper semicontinuous function.*

Definition 3.2. *We define* $\Sigma := \{x \in \Omega \; : \; \Phi_x(0+) \; = \; \omega_n\}$, *where* ω_n *is the volume of the unit ball.*

Remark 3.3. *It can be shown that* Σ *contains all degenerate singular points.*

4. Frequency formula

Theorem 4.1. *The function*

$$F_{x_0}(r) := r \, \frac{\int_{B_r(x_0)} (|\nabla u|^2 + \chi - 1)}{\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}}$$

satisfies at every point x_0 *of the closed set* Σ *and for each* $r \in (0, \operatorname{dist}(x_0, \partial\Omega))$ *the identity* $\partial_r F_{x_0}(r)$

$$= \frac{2}{r} \left(\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} \right)^{-2} \Bigg[\int_{\partial B_r(x_0)} (\nabla u \cdot (y - x_0))^2 \, d\mathcal{H}^{n-1} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} -$$

$$\left(\int_{\partial B_r(x_0)} u \nabla u \cdot (y - x_0) \, d\mathcal{H}^{n-1} \right)^2 \Bigg] + 2 \frac{\int_{B_r(x_0)} (1 - \chi)}{\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}} \left(r \frac{\int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}} \; - \; 1 \right)$$

$$\geq 0 .$$

Remark 4.2. *Theorem 4.1 reminds of course of the* frequency formula *by F. Almgren for Q-valued harmonic functions. Let us however point out that the result presented here is not a perturbation of the linear frequency found by F. Almgren, but a* nonlinear frequency, *that can be extended to more general semilinear equations.*

5. Differential inequality

Corollary 5.1. *The functions*

$$D(r) := r\frac{\int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}} - 1$$

and

$$V(r) := r\frac{\int_{B_r(x_0)} (1-\chi)}{\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}}$$

satisfy at every point x_0 of the closed set Σ and for each $r \in (0, \mathrm{dist}(x_0, \partial\Omega))$ the inequalities

$$D - V \geq 0$$

and

$$(D-V)'(r) \geq \frac{2}{r} V^2(r) \, .$$

For $r \to 0$, $V(r) \to 0$, and $F_{x_0}(r) = D(r) - V(r) + 1$ converges to $F_{x_0}(0+) \in [1, +\infty)$. D is on $(0, \mathrm{dist}(x_0, \partial\Omega))$ bounded.

$$x \mapsto F_x(0+)$$

is on Σ an upper semicontinuous function.

Corollary 5.2 (No infinite order vanishing). *For $r \leq r_x$,*

$$\int_{\partial B_r(x)} u^2 \, d\mathcal{H}^{n-1} \geq r^{m(x)} \, ,$$

where $m(x)$ can be arbitrary large, but is always finite.

6. Blow-up limits

Proposition 6.1. *Let $x_0 \in \Sigma$. Then*

$$v_r(y) := \frac{u(x_0 + ry)}{\sqrt{r^{1-n} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}}}$$

is bounded in $W^{1,2}(B_1(0))$ and each weak limit is a homogeneous function v_0 of degree $N(x_0) \geq 1$.

$$N(x_0) = \frac{\int_{B_1} \Delta v_0^2/2}{\int_{\partial B_1} v_0^2 \, d\mathcal{H}^{n-1}} = F_{x_0}(0+) \, .$$

The limit v_0 is harmonic in the sense of domain variations.

7. High frequency points

Theorem 7.1. *The Hausdorff dimension of the set*

$$\Sigma \cap \{N > 1\}$$

is $\leq n - 2$.

Remark 7.2. *N need not be an integer: consider the example*

$$u(r, \theta) = r^{\frac{3}{2}} |\cos \frac{3}{2}\theta|$$

suggested by J. Andersson.

8. Hausdorff Dimension

Theorem 8.1. *The topological free boundary has Hausdorff dimension* $\leq n - 1$.

Remark 8.2. *Frequency formula and Hausdorff dimension estimates extend to the time-dependent case.*

9. Open Problems

Intuitively the frequency 1 set

$$\Sigma \cap \{N = 1\}$$

should be a set of *non-degenerate* singular points, which we know to be of σ-finite $n-1$-dimensional Hausdorff measure (see [9]). To prove this non-degeneracy, however, seems to be a hard issue, and at this stage we cannot exclude frequency 1 points with a growth of, say

$$\frac{r}{|\log r|}.$$

If we knew that $\Sigma \cap \{N = 1\}$ is of σ-finite $n-1$-dimensional Hausdorff measure (and the parabolic counterpart of this statement), we could conclude that the defect measure $\lambda(t)$ in (3) is zero.

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