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Large-time behavior of solutions to Hamilton-Jacobi equations with time-dependent boundary data

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1 Introduction and Preliminaries

We consider the Cauchy-Dirichlet problem for Hamilton-Jacobi equations

$$\text{(CD)} \qquad \begin{cases} u_t(x,t) + H(x, Du(x,t)) = 0 & \text{in } \Omega \times (0,\infty), & (1.1) \\ u(x,t) = g(x,t) & \text{on } \partial\Omega \times (0,\infty), & (1.2) \\ u(x,0) = f(x) & \text{in } \Omega, & (1.3) \end{cases}$$

under the following standing assumptions on the Hamiltonian $H = H(x,p) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$, $f : \overline{\Omega} \to \mathbb{R}$, $g : \partial\Omega \times [0,\infty) \to \mathbb{R}$ and the bounded domain $\Omega \subset \mathbb{R}^n$:

(A1) $H \in C(\overline{\Omega} \times \mathbb{R}^n)$,

(A2) the function $p \mapsto H(x,p)$ is strictly convex for each $x \in \overline{\Omega}$,

(A3) the function H is coercive, i.e., $\lim_{r\to\infty} \inf\{H(x,p) \mid x \in \overline{\Omega}, p \in \mathbb{R}^n \backslash U(0,r)\} = \infty$, where $U(x,r) := \{y \in \mathbb{R}^n \mid |x-y| < r\}$,

(A4) $f \in C(\overline{\Omega})$, $g \in C(\partial\Omega \times [0,\infty))$ and $f(x) \le g(x,0)$ for any $x \in \partial\Omega$,

(A5) for each $z \in \partial\Omega$, there are a constant $r > 0$, a C^1-diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and a function $b \in C(\mathbb{R}^{n-1})$ such that

$$\Phi(\Omega \cap U(z,r)) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > b(x')\} \cap \Phi(U(z,r)).$$

Here $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is the unknown function and we set $u_t := \partial u/\partial t$ and $Du := (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$.

We are concerned with the large time behavior of solutions of (CD) in the viscosity sense. We are dealing with two cases.

Case (A): the function g is asymptotically time periodic, i.e.,

$$g(x,t) - g_1(x,t) \to 0 \quad \text{uniformly on } \partial\Omega \text{ as } t \to \infty,$$

where $g_1 \in C(\partial\Omega \times \mathbb{R})$ is a time-periodic function with period 1.

Case (B): the function $g(x,t) = g_2(x) + g_3(t)$ is diverging as $t \to \infty$. More precisely, we assume that

$$(\mathrm{g})_+ \quad g_3(t) \to \infty \text{ as } t \to \infty \quad \text{or} \quad (\mathrm{g})_- \quad g_3(t) \to -\infty \text{ as } t \to \infty.$$

The study of the large-time behavior of solutions of the Cauchy problem for Hamilton-Jacobi equations goes back to the works of Kružkov, Lions and Barles [1]. Since the works by Namah-Roquejoffre [16] and Fathi [5], there has been much interest on the subject by many authors. We refer to the literatures [3, 17, 4, 10, 2, 7, 8, 9, 13, 14] and so on and references therein. This article reviews recent results on the asymptotic behavior of viscosity solutions of (CD).

Before closing the introduction, we state some basic propositions. We define the function $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ by

$$u(x,t) = \inf\{\int_\tau^t L(\gamma(s),\dot\gamma(s))\,ds + h(\gamma(\tau),\tau) \mid \gamma \in \mathcal{C}(x,t), \tau \in [0,t], (\gamma(\tau),\tau) \in \partial Q_p\}, \tag{1.4}$$

where $h : \partial_p Q \to \mathbb{R}$ denotes the function given by $h(x,0) = f(x)$ for $x \in \Omega$ and $h(x,t) = g(x,t)$ for $(x,t) \in \partial\Omega \times (0,\infty)$ and $\partial_p Q := \partial\Omega \times (0,\infty) \cup \Omega \times \{0\}$.

Theorem 1.1. *The function $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is continuous on $\overline{\Omega} \times [0,\infty)$ and is a viscosity solution of* (CD).

Theorem 1.2. *Let $T > 0$ and set $Q_T := \Omega \times (0,T)$. Let $u, v \in C(\overline{Q_T})$ be a viscosity subsolution and a viscosity supersolution of* (1.1) *and* (1.2), *respectively. Assume that $u \le v$ on $\overline{\Omega} \times \{0\}$. Then $u \le v$ on $\overline{Q_T}$.*

We define the constant c_H by

$$c_H := \inf\{a \in \mathbb{R} \mid \text{there exists a viscosity solution } v \in C(\Omega) \text{ of } H(x,Du) \le a \text{ in } \Omega\}.$$

Proposition 1.3. *Let $a \in \mathbb{R}$ be a constant.* (i) *There exists a viscosity solution in $C(\overline{\Omega})$ of*

$$(\mathrm{SC})_a \qquad \begin{cases} H(x,Du(x)) \le a & \text{in } \Omega, \\ H(x,Du(x)) \ge a & \text{on } \overline{\Omega}, \end{cases}$$

if and only if $a = c_H$. (ii) *For any $h \in C(\partial\Omega)$, there exists a viscosity solution in $C(\overline{\Omega})$ of*

$$\begin{cases} H(x,Du(x)) = a & \text{in } \Omega, \\ u(x) = h(x) & \text{on } \partial\Omega, \end{cases}$$

if and only if $a \ge c_H$.

We refer to [13, 15] for the proof of the above results.

2 Case (A)

We describe briefly some results obtained in [15].

Theorem 2.1 ([15, Theorem 4.1]). *There exists a viscosity solution in* $C(\overline{\Omega}\times\mathbb{R})$ *of*

$$\text{(P)}\qquad \begin{cases} u_t(x,t)+H(x,Du(x,t))=0 & \text{in } \Omega\times\mathbb{R},\\ u(x,t)=g_1(x,t) & \text{on } \partial\Omega\times\mathbb{R},\\ u(x,t+1)=u(x,t) & \text{on } \overline{\Omega}\times\mathbb{R} \end{cases}$$

if and only if $c_H\le 0$.

We state one of our main theorems.

Theorem 2.2. *Let* $u\in C(\overline{\Omega}\times[0,\infty))$ *be the viscosity solution of* (CD). *If* $c_H>0$, *then*

$$u(x,t)-(\min\{d_{c_H}(x,y)+v_f(y)\wedge v_{\underline{g}_{c_H}}(y)\mid y\in\mathcal{A}_{c_H}\}-c_Ht)\to 0$$

uniformly for $x\in\overline{\Omega}$ *as* $t\to\infty$, *where*

$$d_{c_H}(x,y):=\sup\{v(x)-v(y)\mid v\in C(\overline{\Omega}), H(x,Dv)\le c_H \text{ in } \Omega\},$$
$$v_f(x):=\min\{d_{c_H}(x,y)+f(y)\mid y\in\overline{\Omega}\},$$
$$v_{\underline{g}_{c_H}}(x):=\inf\{d_{c_H}(x,y)+\underline{g}_{c_H}(y)\mid y\in\partial\Omega\}\quad \textit{for all } x,y\in\overline{\Omega},$$
$$\underline{g}_{c_H}(x):=\inf_{s\ge 0}\{g(x,s)+c_Hs\}\quad \textit{for all } x\in\partial\Omega.$$

If $c_H<0$, *then*

$$u(x,t)-w_p(x,t)\to 0 \quad \textit{uniformly for } x\in\overline{\Omega} \textit{ as } t\to\infty,$$

where

$$w_p(x,t):=\inf\{\int_\tau^t L(\gamma(\lambda),\dot\gamma(\lambda))\,d\lambda+g_1(\gamma(\tau),\tau)\mid$$
$$\tau\in(-\infty,t),\ \gamma\in \mathrm{AC}((-\infty,t],\overline{\Omega}),\gamma(t)=x,\gamma(\tau)\in\partial\Omega\}.$$

If $c_H=0$, *then*

$$u(x,t)-\min\{d_{c_H}(x,y)+v_f(y)\wedge v_{\underline{g}_{c_H}}(y)\mid y\in\mathcal{A}_{c_H}\}\wedge w_p(x,t)\to 0$$

uniformly for $x\in\overline{\Omega}$ *as* $t\to\infty$.

Here we write $a\wedge b=\min\{a,b\}$ for $a,b\in\mathbb{R}$. The set $\mathcal{A}_{c_H}$ is defined by

$$\mathcal{A}_{c_H}=\{y\in\overline{\Omega}\mid d_{c_H}(\cdot,y) \text{ is a viscosity solution of } (\mathrm{SC})_{c_H}\}$$

and this set is called the *Aubry set.* We remark that $\min\{d_{c_H}(\cdot,y)+v_f(y)\wedge v_{\underline{g}_{c_H}}(y)\mid y\in\mathcal{A}_{c_H}\}$ is a viscosity solution of $(\mathrm{SC})_{c_H}$ and that w_p is a viscosity solution of (P) if $c_H\le 0$. We refer to Section 4 in [15] for the proof of convergence of u and to Section 5 in [15] for representations for asymptotic solutions.

3 Case (B)

We first consider the case where $(\mathrm{g})_+$ is assumed and we assume for simplicity that g_3 is increasing. We will use the following assumptions.

(g1) The function g_3 has a super-linear growth, i.e., $\lim_{t\to\infty} g_3(t)/t = +\infty$.

(g2) $g_3 \in C^1([0,\infty))$ and the function g_3 satisfies that $\lim_{t\to\infty} \dot{g}_3(t) = 0$.

Here, we write $\dot{g}(t) = dg(t)/dt$ for any $g \in C^1([0,\infty))$. Let u be the viscosity solution of (CD).

Proposition 3.1. *Assume that* (g1) *or* (g2) *holds. Then there exists* $M_1 > 0$ *such that* $|u(x,t) - (-c_H t) \wedge g_3(t)| \le M_1$ *for all* $(x,t) \in \overline{\Omega} \times [0,\infty)$.

Proof. We first consider the cases where (g1) is assumed or where (g2) and $c_H \ge 0$ are assumed. Note that $-c_H t \le g_3(t)$ for some $t_1 > 0$, all $t \ge t_1$. In view of [13, Theorem 3.1], there exists a viscosity solution $\psi \in C(\overline{\Omega})$ of $(\mathrm{SC})_{c_H}$. Then $\psi(x) - c_H t \pm C_1$ are a viscosity supersolution and a viscosity subsolution of (1.1) and (1.2) on $\partial\Omega \times (t_1,\infty)$ for some $C_1 > 0$, respectively. By Theorem 1.2, we get

$$\psi(x) - c_H t - C_1 \le u(x,t) \le \psi(x) - c_H t + C_1 \quad \text{on } \overline{\Omega} \times [t_1,\infty)$$

if C_1 is sufficiently large. Therefore, we have $|u(x,t) + c_H t| \le M_1$ for some $M_1 > 0$ and all $(x,t) \in \overline{\Omega} \times [0,\infty)$.

We next consider the case where (g2) and $c_H < 0$ are assumed. Then it is easily seen that $-c_H t \ge g_3(t)$ for some $t_2 > 0$ and all $t \ge t_2$ Set $v(x,t) := u(x,t) - g_3(t)$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$. Then v satisfies

$$\begin{cases} v_t(x,t) + H(x, Dv(x,t)) = -\dot{g}_3(t) & \text{in } \Omega \times (0,\infty), & (3.1)\\ v(x,t) = g_2(x) & \text{on } \partial\Omega \times (0,\infty), & (3.2)\\ v(x,0) = f(x) - g_3(0) & \text{in } \Omega & (3.3)\end{cases}$$

in the viscosity sense. Note that $-\dot{g}_3(t) \le 0$ and also that by (g2), we may assume that $c_H \le -\dot{g}_3(t)$ for all $t \ge t_2$ by replacing t_2 by a sufficiently large constant if necessary. Let ϕ_1 and ϕ_2 be a viscosity solution of $H(x,Du) = c_H$ in Ω, $u = g_2$ on $\partial\Omega$ and a viscosity solution of $H(x,Du) = 0$ in Ω, $u = g_2$ on $\partial\Omega$, respectively. Then for some constant $C_2 > 0$, $\phi_1 - C_2$ and $\phi_2 + C_2$ are a viscosity subsolution and a viscosity supersolution of (3.1) and (3.2) in $\Omega \times (t_2,\infty)$, respectively. By Theorem 1.2, we have $\phi_1(x) - C_2 \le v(x,t) \le \phi_2(x) + C_2$ on $\overline{\Omega} \times [t_2,\infty)$. Therefore, we may assume that $|u(x,t) - g_3(t)| \le M_1$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$, by replacing M_1 by a sufficiently large constant. □

Theorem 3.2. (i) *Assume that* (g1) *holds. Then* $u(x,t)-(v_\infty(x)-c_Ht) \to 0$ *uniformly on* $\overline{\Omega}$ *as* $t \to \infty$, *where* $v_\infty(x) := \min\{d_{c_H}(x,y)+v_f(y) \mid y \in \mathcal{A}_{c_H}\}$. (ii) *Assume that* (g2) *holds. If* $c_H \geq 0$, *then* $u(x,t)-(v_\infty(x)-c_Ht) \to 0$ *uniformly on* $\overline{\Omega}$ *as* $t \to \infty$, *and if* $c_H < 0$, *then* $u(x,t)-(v_{g_2}(x)+g_3(t)) \to 0$ *uniformly on* $\overline{\Omega}$ *as* $t \to \infty$, *where* $v_{g_2}(x) := \min\{d_0(x,y)+g_2(y) \mid y \in \partial\Omega\}$ *and* $d_0(x,y) := \sup\{v(x)-v(y) \mid v \in C(\overline{\Omega}), H(x,Dv) \leq 0 \text{ in } \Omega\}$.

Proof. Set $u_{c_H}(x,t) = u(x,t)+c_Ht$ for $(x,t) \in \overline{\Omega}\times[0,\infty)$. Then, u_{c_H} satisfies

$$\begin{cases} (u_{c_H})_t(x,t)+H(x,Du_{c_H}(x,t)) = c_H & \text{in } \Omega\times(0,\infty), \\ u_{c_H}(x,t) = g_2(x)+g_3(t)+c_Ht & \text{on } \partial\Omega\times(0,\infty) \end{cases}$$

in the viscosity sense. If we assume (g1) or we assume (g2) and $c_H \geq 0$, then it is clear that u_{c_H} is bounded on $\overline{\Omega}\times[0,\infty)$ in view of Proposition 3.1 and that $g_2(x)+g_3(t)+c_Ht \to \infty$ uniformly for $x \in \partial\Omega$ as $t \to \infty$. From this, we find a constant $\bar{t} > 0$ such that $g_2(x)+g_3(t)+c_Ht > u_{c_H}(x,t)$ for all $(x,t) \in (\bar{t},\infty)$. Therefore, we see easily that

$$\begin{cases} (u_{c_H})_t(x,t)+H(x,Du_{c_H}(x,t)) \leq c_H & \text{in } \Omega\times(\bar{t},\infty), \\ (u_{c_H})_t(x,t)+H(x,Du_{c_H}(x,t)) \geq c_H & \text{on } \overline{\Omega}\times(\bar{t},\infty). \end{cases}$$

Thus, Theorems 2.1 and 6.3 in [13] guarantees (i).

In the case where (g2) and $c_H < 0$ are assumed, by Proposition 3.1, $u(\cdot,t)-g_3(t)$ is bounded on $\overline{\Omega}\times[0,\infty)$. Thus we may define the functions $w^+_{g_3}, w^-_{g_3} \in C(\overline{\Omega})$ by $w^+_{g_3}(x) := \limsup^*_{t\to\infty}(u(x,t)-g_3(t))$, $w^-_{g_3}(x) := \liminf^*_{t\to\infty}(u(x,t)-g_3(t))$, where $\limsup^*_{t\to\infty}$ and $\liminf^*_{t\to\infty}$ are half relaxed limits. Due to the stability property of viscosity solutions and the convexity of $H(x,\cdot)$, $w^+_{g_3}$ and $w^-_{g_3}$ are a viscosity subsolution and a viscosity solution of

$$\text{(D)} \qquad \begin{cases} H(x,Du(x)) = 0 & \text{in } \Omega, \\ u(x) = g_2(x) & \text{on } \partial\Omega. \end{cases}$$

If $c_H < 0$, the comparison principle of viscosity solutions of (D) holds (see [14, Theorem 5.3]). Therefore, $w^+_{g_3}(x) \leq w^-_{g_3}(x)$ for all $x \in \overline{\Omega}$. We also see that v_{g_2} is a viscosity solution of (D). We may conclude from these that $u(x,t)-g_3(t) \to v_{g_2}(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$. □

We next consider the case where $(g)_-$ is assumed and we assume for simplicity that g_3 is decreasing.

Proposition 3.3. *Assume that* (g2) *holds. Then there exists* $M_2 > 0$ *such that* $|u(x,t)-(-c_Ht)\wedge g_3(t)| \leq M_2$ *for all* $(x,t) \in \overline{\Omega}\times[0,\infty)$.

Theorem 3.4. *Assume that* (g2) *holds. If $c_H > 0$, then $u(x,t)-(\min_{y\in\mathcal{A}_{c_H}}\{d_{c_H}(x,y)+v_f(y)\wedge v_{\underline{g}_{c_H}}(y)\}-c_Ht) \to 0$ uniformly on $\overline{\Omega}$ as $t\to\infty$. If $c_H \le 0$, then $u(x,t)-(v_{g_2}(x)+g_3(t)) \to 0$ uniformly on $\overline{\Omega}$ as $t\to\infty$.*

The proofs of Proposition 3.3 and Theorem 3.4 are almost same as that of Proposition 3.1 and Theorem 3.2, respectively, except for the case where $c_H = 0$. In the case where $c_H = 0$, uniqueness of viscosity solutions of (D) does not hold. Therefore we cannot apply the argument of Theorem 3.2 (b) to a proof of Theorem 3.4 (b) in the case $c_H = 0$. We give a proof of it here.

Proof of Theorem 3.4 *in the case* $c_H = 0$. Fix any $\epsilon > 0, x \in \overline{\Omega}$ and $t > 0$. There exist $\gamma_\epsilon \in \mathcal{C}(x,t), \tau_\epsilon \in [0,t]$ such that $(\gamma_\epsilon(\tau_\epsilon),\tau_\epsilon) \in \partial_p Q$ and

$$u(x,t)+\epsilon > \int_{\tau_\epsilon}^t L(\gamma_\epsilon(\lambda),\dot\gamma_\epsilon(\lambda))\,d\lambda + h(\gamma(\tau_\epsilon),\tau_\epsilon).$$

In view of Proposition 3.3 and [14, Lemma 6.1], we have

$$\begin{aligned} C &\ge u(x,t)-g_3(t)+\epsilon \\ &> \int_{\tau_\epsilon}^t L(\gamma_\epsilon(\lambda),\dot\gamma_\epsilon(\lambda))\,d\lambda + h(\gamma(\tau_\epsilon),\tau_\epsilon) - g_3(t) \\ &\ge v_{g_2}(\gamma_\epsilon(t)) - v_{g_2}(\gamma_\epsilon(\tau_\epsilon)) + h(\gamma(\tau_\epsilon),\tau_\epsilon) - g_3(t). \end{aligned}$$

for some $C > 0$. If we suppose that $\tau_\epsilon > 0$, then we have $-g_3(t) \le C - (v_{g_2}(\gamma_\epsilon(t)) - v_{g_2}(\gamma_\epsilon(\tau_\epsilon)) + h(\gamma(\tau_\epsilon),\tau_\epsilon)) \le C'$, which implies a contradiction if $t > 0$ is sufficiently large. From this, we may assume that $\tau_\epsilon > 0$ for such a $t > 0$. Note that $v_{g_2}(\gamma_\epsilon(\tau_\epsilon)) \le g_2(\gamma_\epsilon(\tau_\epsilon))$ and $g_3(t) \le g_3(\tau_\epsilon)$. Therefore,

$$\begin{aligned} &u(x,t)-g_3(t)+\epsilon \ge v_{g_2}(x) - v_{g_2}(\gamma_\epsilon(\tau_\epsilon)) + g_2(\gamma_\epsilon(\tau_\epsilon)) + g_3(\tau_\epsilon) - g_3(t) \\ &\ge v_{g_2}(x) - g_2(\gamma_\epsilon(\tau_\epsilon)) + g_2(\gamma_\epsilon(\tau_\epsilon)) + g_3(t) - g_3(t) = v_{g_2}(x). \end{aligned}$$

From this, we get $w^-_{g_3}(x) \ge v_{g_2}(x)$ for all $x \in \overline{\Omega}$.

Since we have $H(x, Dw^+_{g_3}(x)) \le 0$ in Ω and $w^+_{g_3}(x) \le g_2(x)$ on $\partial\Omega$, we get $w^+_{g_3}(x) \le d_{c_H}(x,y) + w^+_{g_3}(y) \le d_{c_H}(x,y) + g_2(y)$ for all $x \in \overline{\Omega}, y \in \partial\Omega$. Therefore, $w^+_{g_3}(x) \le v_{g_2}(x)$ for all $x \in \overline{\Omega}$. Thus, we have $v_{g_2}(x) = w^+_{g_3}(x) = w^-_{g_3}(x)$ on $\overline{\Omega}$. □

We next use the following assumption:

(g3) $g_3 \in C^1((0,\infty))$ and the function g_3 satisfies that $\lim_{t\to\infty}\dot g_3(t) = -\infty$.

Proposition 3.5. *Assume that* (g3) *holds. Then*

$$|u(x,t)-g_3(t)| \le C \quad \textit{for all } (x,t) \in \partial\Omega\times[0,\infty), \tag{3.4}$$

$$u(x,t)-g_3(t) \to \infty \quad \textit{for all } x\in\Omega \textit{ as } t\to\infty. \tag{3.5}$$

Lemma 3.6. *Set $H_0(r) := \max_{x\in\overline{\Omega},|q|\leq r} H(x,q)+r$ for $r>0$ and $H_0(r) := 0$ for $r=0$. There exist a constant $T_1>0$ and a positive increasing function $f \in C([T_1,\infty))$ such that*

$$\dot{f}(t) + H_0(f(t)) \leq -\dot{g}_3(t) \quad \textit{for a.e. } t \in (T_1,\infty), \tag{3.6}$$

$$f(t) \to \infty \quad \textit{as } t \to \infty. \tag{3.7}$$

Proof. Set $a_1 := 0$ and $b := H_0(a_1)$. Note that H_0 is strictly increasing and $H_0(r) \to \infty$. Define the sequence $\{a_n\}_{n\in\mathbb{N},n\geq 2}$ by $a_n := H_0^{-1}(b+(n-1))$. We choose a sequence $\{T_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ such that for each $n \in \mathbb{N}$,

$$-\dot{g}_3(t) \geq b+(n+1) \quad \text{for } a.e.\ t \geq T_n,$$
$$T_{n+1} \geq T_n + a_{n+1} - a_n + 1.$$

We define the function $f : [T_1,\infty) \to [0,\infty)$ by

$$f(t) := \begin{cases} a_n + t - T_n & \text{for } \ t \in [T_n, T_n + a_{n+1} - a_n), \\ a_{n+1} & \text{for } \ t \in [T_n + a_{n+1} - a_n, T_{n+1}) \end{cases}$$

for any $n \in \mathbb{N}$. Then f satisfies required properties. Indeed, we see that

$$\dot{f}(t) + H_0(f(t)) = 1 + H_0(a_n + t - T_n) \leq 1 + H_0(a_{n+1}) = b+n+1 \leq -\dot{g}_3(t)$$

for $a.e.$ $t \in (T_n, T_n + a_{n+1} - a_n)$ and,

$$\dot{f}(t) + H_0(f(t)) = 0 + H_0(a_{n+1}) = b+n \leq -\dot{g}_3(t)$$

for $a.e.$ $t \in (T_n + a_{n+1} - a_n, T_{n+1})$. Moreover, since $f(t) \geq a_{n+1} = H_0^{-1}(b+n)$ for all $t \geq T_{n+1}$, we see that $f(t) \to \infty$ as $n \to \infty$. $\square$

Proof of Proposition 3.5. It is clear that (3.4) holds, so we only prove (3.5). Let H_0 and f be the functions and T_1 be the constant given in Lemma 3.6. We extend the function f as a positive continuous function on $\mathbb{R}$ and by abuse of notation we denote the resulting function by f again. Let $\rho \in C^\infty(\mathbb{R})$ be a standard mollification kernel, i.e., $\rho \geq 0$, $\operatorname{supp}\rho \subset [-1,1]$ and $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$, where $\operatorname{supp}\rho := \overline{\{t\in\mathbb{R} \mid \rho(t)\neq 0\}}$. Set $\rho_n(t) := n\rho(nt)$ and $f_n(t) := \rho_n * f(t)$ for $t\in\mathbb{R}$. Note that $f_n \in C^\infty(\mathbb{R})$, $f_n \to f$ locally uniformly on $\mathbb{R}$ as $n\to\infty$ and by Jensen's inequality,

$$\dot{f}_n(t) + H_0(f_n(t)) \leq \rho_n * (\dot{f} + H_0(f))(t) \leq -\rho_n * \dot{g}_3(t)$$

for any $t \geq T_1$.

Set $w_n(x,t) := a^{-1}d(x)f_n(t)$ for all $(x,t) \in \overline{\Omega}\times[0,\infty)$, where $d(x) := \min\{|x-y| \mid y \in \partial\Omega\}$ and $a := \max\{|x-y| \mid x,y\in\overline{\Omega}\} \vee 1$. Let $\phi \in C^1(\overline{\Omega}\times[0,\infty))$ and $w_n - \phi$ take a local maximum at $(x_0,t_0) \in \Omega\times(T_1,\infty)$. Note that the function d satisfies

$|Dd(x)| \leq 1$ in Ω in the viscosity sense and $x \mapsto d(x) - a\phi(x,t_0)/f_n(t_0)$ takes a local maximum at x_0. Thus we have $|D\phi(x_0,t_0)| \leq a^{-1}f_n(t_0) \leq f_n(t_0)$. Moreover we have $\phi_t(x_0,t_0) = a^{-1}d(x_0)\dot{f}_n(t_0)$.

We calculate that

$$\begin{aligned}&\phi_t(x_0,t_0) + H(x_0, D\phi(x_0,t_0)) \leq a^{-1}d(x_0)\dot{f}_n(t_0) + H_0(|D\phi(x_0,t_0)|)\\&\leq \dot{f}_n(t_0) + H_0(f_n(t_0)) \leq -\rho_n * \dot{g}_3(t_0).\end{aligned}$$

In view of the stability property of viscosity solution, we see that $w(x,t) = a^{-1}d(x)f(t)$ satisfies $w_t(x,t) + H(x, Dw(x,t)) \leq -\dot{g}_3(t)$ in $\Omega \times (T_1,\infty)$ in the viscosity sense. Noting that $w(x,t) = 0$ for any $x \in \partial\Omega$, we have $w(x,t) - C \leq g_2(x)$ for all $(x,t) \in \partial\Omega \times (T_1,\infty)$. Due to Theorem 1.2, by replacing C by a larger number if necessary, $w(x,t) - C \leq u(x,t)$ for all $(x,t) \in \overline{\Omega} \times [T_1,\infty)$, which guarantees that $u(x,t) \to +\infty$ for each $x \in \Omega$. $\square$

We finally give an example. The following example illustrates the fact that in the case (g)$_+$, if we do not assume (g1) or (g2), the statements of Theorem 3.2 do not hold in general.

Example. Let the function H and the domain Ω be any function and bounded domain which satisfy (A1)–(A3) and (A5). We define the function $g \in C([0,\infty))$ by

$$g(t) := \begin{cases} (n^2-(n-1)^2)(t-(n^4-1)) + (n-1)^2 & \text{for} \quad t \in [n^4-1, n^4), \\ n^2 & \text{for} \quad t \in [n^4, (n+1)^4-1) \end{cases}$$

for any $n \in \mathbb{N}$. It is easily seen that the function g goes to plus infinity, and does not satisfy (g1) and (g2).

We consider the problem,

$$\begin{cases} u_t(x,t) + H(x, Du(x,t)) = 0 & \text{in } \Omega \times (0,\infty), \quad (3.8)\\ u(x,t) = g(t) & \text{on } \partial_p Q. \quad (3.9)\end{cases}$$

Then the viscosity solution of this problem is given by the function

$$u(x,t) = \inf\{\int_\tau^t L(\gamma(s),\dot{\gamma}(s))\,ds + g(\tau) \mid \gamma \in \mathcal{C}(x,t), \tau \in [0,t], (\gamma(\tau),\tau) \in \partial Q_p\}, \tag{3.10}$$

for $(x,t) \in \overline{\Omega} \times [0,\infty)$.

Recall (see [10, Proposition 2.1]) that there exist constants $\rho > 0$ and $C_1 > 0$ such that $L(x,\xi) \leq C_1$ for all $(x,\xi) \in \overline{\Omega} \times B(0,\rho)$. Set $C_2 := (a/2\rho) \vee 1$, where $a := \max\{|x-y| \mid x,y \in \overline{\Omega}\}$.

Fix $x \in \overline{\Omega}$. Take $z_x \in \partial\Omega$ such that $|x - z_x| = \min\{|x-z| \mid z \in \partial\Omega\} =: d(x)$. Set $\gamma(\lambda) := z_x$ for $\lambda \in [0, t - \rho^{-1}d(x))$, $\gamma(\lambda) := z_x + \rho(\lambda - t + \rho^{-1}d(x))|x - z_x|^{-1}(x - z_x)$

for $\lambda \in [t-\rho^{-1}d(x), t]$ if $x \in \Omega$ and $\gamma(\lambda) \equiv x$ if $x \in \partial\Omega$. Then we have $\gamma \in \mathcal{C}(x,t)$. By (3.10), we get

$$u(x,t) \leq \int_{t-C_2}^{t} L(\gamma(\lambda), \dot{\gamma}(\lambda))\, d\lambda + g(t-C_2) \leq C_2(C_1 \vee \max_{x\in\overline{\Omega}} |L(x,0)|) + g(t-C_2).$$

Set $C_3 := C_2(C_1 \vee \max_{x\in\overline{\Omega}} |L(x,0)|)$. In particular, we have for any $n \in \mathbb{N}$,

$$u(x,n^4) - g(n^4) \leq C_3 + g(n^4 - C_2) - g(n^4) \leq -2n + C_3 + 1,$$

which implies that $u(x,n^4) - g(n^4) \to -\infty$ as $n \to \infty$. This observation tells us that Proposition 3.1 does not hold if $c_H < 0$.

Moreover, we set $H(x,p) := |p| - 1$, $\Omega := U(0, 2^{-1})$ and consider the problem (3.8) and (3.9). Then the viscosity solution is given by the function $u(x,t) = \min\{t-\tau+g(\tau) \mid \gamma \in \mathcal{C}(x,t), |\dot{\gamma}| \leq 1, \tau \in [0,t], (\gamma(\tau),\tau) \in \partial_p Q\}$. If $t = n^4 - 1$, the optimal exit time τ^* is $n^4 - 1 - (2^{-1} - |x|)$, which implies

$$u(x,n^4-1) - g(n^4-1) = \frac{1}{2} - |x| + g(n^4 - 1 - |x|) - g(n^4-1) = \frac{1}{2} - |x|.$$

Thus $\{u(\cdot, n^4-1) - g(n^4-1)\}_{n\in\mathbf{N}}$ is bounded. Therefore, we see that there exist diverging sequences $\{a_n\}_{n\in\mathbf{N}}, \{b_n\}_{n\in\mathbf{N}} \subset [0,\infty)$ such that $u(x,a_n) - g(a_n) \to -\infty$ as $n \to \infty$ and $u(x,b_n) - g(b_n)$ is bounded for any $x \in \Omega$.

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