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Denjoy-Schwartz and Hamilton-Jacobi

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Abstract

This expository paper explains some of the restrictions imposed by the theory of Dynamical Systems on the Aubry set, the solutions and the critical subsolutions of a Tonelli Hamiltonian on a surface.

This article is essentially expository, most of what we say is well-known (at least to the specialists in Dynamical Systems) with the exception of some results on C^3 critical subsolutions of the Hamilton-Jacobi Equation on surfaces. We hope that it will help people with no serious background in Dynamics understand the strong restrictions put on Mather sets, solutions and critical subsolutions of the Hamilton-Jacobi Equation by the theory of flows on surfaces.

1 The setting

We will consider a Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, (x,p) \mapsto H(x,p)$. Here x represents position and p represents momentum.

We need some compactness assumptions and we want to avoid boundary effects. Therefore we will assume that $H(x,p)$ is $\mathbb{Z}^n$-periodic in x

$$\forall x \in \mathbb{R}^n, \forall z \in \mathbb{Z}^n, \forall p \in \mathbb{R}^n,\ H(x,p) = H(x+z,p).$$

Hence H is well defined on $\mathbb{T}^n \times \mathbb{R}^n$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Two good examples to keep in mind are

$$H^V(x,p) = \frac{1}{2}\|p\|^2_{\mathrm{euc}} + V(x), \tag{E1}$$

where $V : \mathbb{T}^n \to \mathbb{R}$, and $\|\cdot\|^2_{\mathrm{euc}}$ is the usual Euclidean norm on $\mathbb{R}^n$, and

$$H^X(x,p) = \frac{1}{2}\|p\|^2_{\mathrm{euc}} + \langle X, p\rangle_{\mathrm{euc}}, \tag{E2}$$

where $X : \mathbb{T}^n \to \mathbb{R}^n$ is a vector field and $\langle \cdot, \cdot \rangle_{\mathrm{euc}}$ is the usual scalar product on $\mathbb{R}^n$.

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What we will say is more generally true for a Hamiltonian $H : T^*M \to \mathbb{R}$, on the cotangent space T^*M of the compact boundaryless manifold M.

We will assume in the sequel that the Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Tonelli Hamiltonian, i.e. it satisfies the following conditions:

1) The Hamiltonian H is of class C^r with $r \geq 2$.

2) The Hamiltonian H is C^2-strictly convex in p, i.e. for every $(x,p) \in \mathbb{T}^n \times \mathbb{R}^n$, the second partial derivative $\partial^2 H/\partial^2 p(x,p)(\cdot,\cdot)$ is positive definite as a quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$.

3) The Hamiltonian H is superlinear
$$\frac{H(x,p)}{\|p\|} \to +\infty \text{ as } \|p\| \to +\infty,$$
uniformly in $x \in \mathbb{T}^n$.

2 The Hamilton-Jacobi Equation: solutions and subsolutions

The (stationary) Hamilton-Jacobi equation associated to H is
$$H(x, d_x u) = c \tag{HJE$_c$}$$
where $c \in \mathbb{R}$ is fixed.

Although we will say something about viscosity solutions, we will be mainly interested in C^1 solutions and subsolutions. Let us recall that a *(global, classical) solution* of (HJE$_c$) is a C^1 map $u : \mathbb{T}^n \to \mathbb{R}$ such that
$$\forall x \in \mathbb{T}^n,\ H(x, d_x u) = c,$$
where $d_x u$ is the differential or derivative of u at x (here identified with its gradient at x since the second variable of p is in $\mathbb{R}^n$ rather than in the dual space $(\mathbb{R}^n)^*$ which is where $d_x u$ really lies). A *(global, classical) subsolution* of (HJE$_c$) is a C^1 map $v : \mathbb{T}^n \to \mathbb{R}$ such that
$$\forall x \in \mathbb{T}^n,\ H(x, d_x v) \leq c.$$

If one is allowed to choose c then subsolutions always exist. In fact, if $v : \mathbb{T}^n \to \mathbb{R}$ is C^1 and we define the *Hamiltonian constant* $\mathbb{H}(v)$ of v by
$$\mathbb{H}(v) = \sup_{x \in \mathbb{T}^n} H(x, d_x v),$$
then by compactness $\mathbb{H}(v) = \sup_{x\in\mathbb{T}^n} H(x, d_x v) < +\infty$, and v is a subsolution of (HJE$_c$) for any $c \geq \mathbb{H}(v)$. The following quantity
$$\bar{H}(0) = \inf\{\mathbb{H}(v) \mid v : \mathbb{T}^n \to \mathbb{R} \text{ of class } C^1\}$$

is therefore the challenge. Note that we can also take the infimum on the class of C^∞ functions, since this last subset is dense in the class of C^1 functions for the uniform C^1 topology.

We will say that a function $v : \mathbb{T}^n \to \mathbb{R}$ is a *critical subsolution* of the Hamilton-Jacobi equation if it is C^1 and satisfies

$$\forall x \in \mathbb{T}^n,\ H(x, d_x v) \leq \bar{H}(0).$$

The importance of $\bar{H}(0)$ is also illustrated by the following simple proposition.

Proposition 2.1. *If* (HJE_c) *admits a (global* C^1*) solution then* $c = \bar{H}(0)$.

In fact, the argument to prove the proposition above, yields the following more general result.

Lemma 2.2. *If* $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ *is a continuous Hamiltonian, for every* C^1 *function* $u : \mathbb{T}^n \to \mathbb{R}$, *we have*

$$\mathbb{H}(u) = \sup_{x\in\mathbb{T}^n} H(x, d_x u) \geq \bar{H}(0) \geq \inf_{x\in\mathbb{T}^n} H(x, d_x u).$$

In particular, we can find a point $x_0 \in \mathbb{T}^n$ *such that* $H(x_0, d_{x_0} u) = \bar{H}(0)$.

Proof. The left hand side inequality

$$\mathbb{H}(u) = \sup_{x\in\mathbb{T}^n} H(x, d_x u) \geq \bar{H}(0)$$

comes from the definition of $\bar{H}(0)$. To show the right hand side inequality, we consider a C^1 function $v : \mathbb{T}^n \to \mathbb{R}$. Since $\mathbb{T}^n$ is compact $u - v$ reaches a maximum at some point $x_v \in \mathbb{T}^n$. At such a point, we have $d_{x_v}(u - v) = 0$, therefore $d_{x_v} u = d_{x_v} v$. This implies

$$\mathbb{H}(v) \geq H(x_v, d_{x_v} v) = H(x_v, d_{x_v} u) \geq \inf_{x\in\mathbb{T}^n} H(x, d_x u).$$

Therefore $\bar{H}(0) = \inf_v \mathbb{H}(v) \geq \inf_{x\in\mathbb{T}^n} H(x, d_x u)$.

The existence of x_0 is now a consequence of the connectedness of $\mathbb{T}^n$. □

To get some familiarity with this constant $\bar{H}(0)$, we compute it for Examples (E1) and (E2) given above.

If we consider

$$H^X(x,p) = \frac{1}{2}\|p\|^2_{\mathrm{euc}} + \langle X, p\rangle_{\mathrm{euc}},$$

where $X : \mathbb{T}^n \to \mathbb{R}^n$ is a vector field, it is obvious that any constant function is a solution of the Hamilton-Jacobi Equation

$$H^X(x, d_x u) = 0.$$

Therefore $\bar{H}^X(0) = 0$ by Proposition 2.1.

Let us show $\bar{H}^V(0) = \max V$ for the Hamiltonian

$$H^V(x,p) = \frac{1}{2}\|p\|^2_{\text{euc}} + V(x),$$

where $V : \mathbb{T}^n \to \mathbb{R}$, and $\|\cdot\|^2_{\text{euc}}$ is the usual Euclidean norm on $\mathbb{R}^n$.

If $u : \mathbb{T}^n \to \mathbb{R}$ is C^1, we have

$$H^V(x, d_x u) = \frac{1}{2}\|d_x u\|^2_{\text{euc}} + V(x) \geq V(x).$$

Therefore $\mathbb{H}(u) = \max_{x\in\mathbb{T}^n} H^V(x, d_x u) \geq \max V$, and $\bar{H}^V(0) = \inf_u \mathbb{H}(u) \geq \max V$. But if u is constant function $d_x u \equiv 0$, and $H^V(x, d_x u) = V(x)$, which implies $\mathbb{H}(u) = \max V$. Hence $\bar{H}^V(0) = \max V$. Notice that in this case we have C^∞ critical subsolutions, namely the constant functions.

3 Existence of C^1 solutions. Viscosity solutions

One does not always have a C^1 solution to the Hamilton-Jacobi Equation. To show this, let us consider again the example of the Hamiltonian

$$H^V(x,p) = \frac{1}{2}\|p\|^2_{\text{euc}} + V(x),$$

where $V : \mathbb{T}^n \to \mathbb{R}$, and $\|\cdot\|^2_{\text{euc}}$ is the usual Euclidean norm on $\mathbb{R}^n$. Assume that there exists a C^1 solution $u : \mathbb{T}^n \to \mathbb{R}$. By what we have seen in the previous section, we have

$$\forall x \in \mathbb{T}^n,\ H^V(x, d_x u) = \bar{H}^V(0) = \max V.$$

Since u is a C^1 function on the compact $\mathbb{T}^n$, it has at least two distinct critical points, i.e. two points $x_1 \neq x_2$ with $d_{x_1} u = d_{x_2} u = 0$. In fact, if x_{max} and x_{min} are points where u reaches its maximum and minimum, they are critical. If they are distinct, we have obtained our two points. If $x_{\text{max}} = x_{\text{min}}$, then u is constant since its minimum and maximum are equal, therefore all points are critical. At the distinct critical points $x_i, i = 1, 2$, we have

$$\max V = H^V(x_i, d_{x_i} u) = \frac{1}{2}\|d_{x_i} u\|^2_{\text{euc}} + V(x_i) = V(x_i).$$

This implies that V reaches its maximum at two distinct points. In particular, if we choose V such that V achieves its maximum at a unique point, there is no C^1 (global) solution of the Hamilton-Jacobi Equation. Such an example is given by the Hamiltonian $H_0 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ defined by

$$H_0(x,p) = \frac{1}{2}p^2 + \cos(2\pi x),$$

since $\cos(2\pi x)$ reaches its maximum 1 only at $x = 0 \mod 1$.

P.-L. Lions and M. Crandall, see [9], have introduced the notion of viscosity solution (for an introduction to viscosity solutions see [5], and for more complete treatments consult [1, 2] or [6]). In 1987, in an unpublished preprint [10], P.-L. Lions, G. Papanicolaou, and S.R.S. Varadhan have shown that there always exists viscosity solutions for the Hamilton-Jacobi Equation.

Theorem 3.1 (Lions-Papanicolaou-Varadhan). *There exists a Lipschitz function $u : \mathbb{T}^n \to \mathbb{R}$ and a $c \in \mathbb{R}$ such that u is a viscosity solution of (HJE$_c$). Moreover, the $c \in \mathbb{R}$ for which there exists a viscosity solution of (HJE$_c$) is unique.*

It can be shown that the c given in the theorem above is equal to $\bar{H}(0)$ even if the viscosity solution is not C^1.

We now deal with the question of existence of a C^1 solution for the Hamilton-Jacobi Equation.

In 2000, the author realized that the theory of viscosity solutions easily gives an answer to this question. For this, we define the Hamiltonian defect $\mathrm{HamDef}(u)$ of the C^1 function $u : \mathbb{T}^n \to \mathbb{R}$ by

$$\mathrm{HamDef}(u) = \sup_{x \in \mathbb{T}^n} H(x, d_x u) - \inf_{x \in \mathbb{T}^n} H(x, d_x u) \geq 0,$$

and the defect $\mathcal{D}(H)$ of H by

$$\mathcal{D}(H) = \inf\{\mathrm{HamDef}(u) \mid u : \mathbb{T}^n \to \mathbb{R} \text{ of class } C^1\}.$$

Again we could restrict in the definition above u to C^∞ functions without changing the value of $\mathcal{D}(H)$.

Proposition 3.2. *The Hamilton-Jacobi Equation associated to the Tonelli Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ has a C^1 solution if and only if the defect $\mathcal{D}(H)$ of the Hamiltonian is equal to 0.*

Proof. Obviously, if u_0 is a C^1 solution we have $\mathrm{HamDef}(u_0) = 0$. But

$$0 = \mathrm{HamDef}(u_0) \geq \mathcal{D}(H) \geq 0.$$

To prove the converse, we will need to assume that the reader has a good knowledge of viscosity solutions. Since $\mathcal{D}(H) = 0$, we can find a sequence of C^1 function $u_\ell : \mathbb{T}^n \to \mathbb{R}$ such that $\epsilon_\ell = \mathrm{HamDef}(u_\ell) \searrow 0$. Note that by Lemma 2.2, we have

$$\forall x \in \mathbb{T}^n, \bar{H}(0) + \epsilon_\ell \geq H(x, d_x u_\ell) \geq \bar{H}(0) - \epsilon_\ell.$$

This implies that u_ℓ is a (viscosity) supersolution of (HJE$_{\bar{H}(0)-\epsilon_\ell}$) and a (viscosity) subsolution of (HJE$_{\bar{H}(0)+\epsilon_\ell}$). In particular, since ϵ_ℓ is convergent (hence bounded), by the coercivity of H (which is in fact superlinear) we obtain that the derivatives of the functions u_ℓ are bounded in norm independently of ℓ. Therefore the sequence u_ℓ is equi-Lipschitzian. By the Arzela-Ascoli theorem subtracting constants from the u_ℓ and extracting a subsequence of the u_ℓ, we can assume that u_ℓ

converge uniformly to $u : \mathbb{T}^n \to \mathbb{R}$. By the stability theorem for viscosity subsolutions and supersolutions, since $\bar{H}(0) \pm \epsilon_\ell \to 0$, we obtain that u is a viscosity subsolution and supersolution of $(\mathrm{HJE}_{\bar{H}(0)})$. Hence it is a viscosity solution of the Hamilton-Jacobi Equation. Since H is a Tonelli Hamiltonian, such a viscosity solution has to be semi-concave.

Now we can consider the Tonelli Hamiltonian $\check{H}$ defined by

$$\check{H}(x,p) = H(x,-p).$$

If we define the C^1 function $v_\ell = -u_\ell$, we have

$$\forall x \in \mathbb{T}^n, \bar{H}(0) + \epsilon_\ell \geq \check{H}(x, d_x v_\ell) \geq \bar{H}(0) - \epsilon_\ell.$$

Since v_ℓ converges uniformly to $v = -u$, we obtain as above that $v = -u$ is a viscosity solution of the Hamilton Jacobi Equation associated to $\check{H}$. Therefore $-u$ is semi-concave, and u is semi-convex. Hence u is both semi-concave and semi-convex. This implies that u is $\mathrm{C}^{1,1}$. Since u is a viscosity solution of $(\mathrm{HJE}_{\bar{H}(0)})$, it is a classical solution. □

We end this section by the following proposition which can be proved by the same argument as the proposition above.

Proposition 3.3. *Let U be an open subset of some Euclidean space $\mathbb{R}^n$. Suppose the sequence of C^1 maps $u_n : U \to \mathbb{R}$ converges uniformly on U to the function $u : U \to \mathbb{R}$. Suppose further that the sequence of maps $x \mapsto \|d_x u\|_{\mathrm{euc}}$ converges uniformly on U to the constant 1. Then u is C^1 and $\|d_x u\|_{\mathrm{euc}} = 1$, for every $x \in U$.*

It would be nice to find an elementary proof of this result, i.e. a proof that does not use a viscosity type argument (or disguised in an equivalent Optimal Control/Calculus of Variations argument).

4 More regular critical subsolutions

Although there is in general no C^1 solution of the Hamilton-Jacobi Equation, one can find C^1 critical subsolutions, see [7], or even better, by a result of Patrick Bernard see [3], a $\mathrm{C}^{1,1}$ critical sub-solution of the Hamilton-Jacobi Equation, i.e. a C^1 critical subsolution whose derivative is Lipschitz.

Theorem 4.1 (Patrick Bernard). *If $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Tonelli Hamiltonian then there exists a $C^{1,1}$ function $\mathbb{T}^n \to \mathbb{R}$ such that*

$$\forall x \in \mathbb{T}^n,\ H(x, d_x u) \leq \bar{H}(0).$$

By this theorem, we can characterize $\bar{H}(0)$ as the only constant $c \in \mathbb{R}$ which satisfies the following two conditions:

(i) We have $\mathbb{H}(u) \geq c$ for every $\mathrm{C}^{1,1}$ function $\mathbb{T}^n \to \mathbb{R}$.

(ii) There exists $u_0 : \mathbb{T}^n \to \mathbb{R}$ which is $C^{1,1}$ and satisfies

$$\forall x \in \mathbb{T}^n, H(x, d_x u_0) \leq c.$$

In general Theorem 4.1 is sharp, one cannot find a C^2 critical subsolution as we will presently show.

Consider again the Hamiltonian $H_0 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ defined by

$$H_0(x,p) = \frac{1}{2}p^2 + \cos(2\pi x).$$

The maximum value of the cosine is 1, therefore $\bar{H}_0(0) = 1$. The equation $H_0(x,p) = 1$ can be solved by $p = \pm v_0(x)$, where $v_0 : \mathbb{T} \to \mathbb{R}$ is the continuous function defined by

$$v_0(x) = \sqrt{2 - 2\cos(2\pi x)}.$$

Moreover, we have

$$H_0(x,p) \leq 1 \iff |p| \leq v_0(x). \tag{*}$$

We note that v_0 is piecewise C^1, therefore it is Lipschitz. However, the function v_0 is not C^1, because it is equivalent to $\sqrt{2}\pi|x|$ near 0.

We set $p_0 = \int_{\mathbb{T}} v_0(x)\, dx$. The function $v_0 - p_0$ is obviously the derivative of the 1-periodic function

$$u_0(x) = \int_0^x v_0(s) - p_0\, ds.$$

Therefore u_0 is a function from $\mathbb{T}$ to $\mathbb{R}$. By the properties of v_0, it is $C^{1,1}$ but not C^2. Of course, we have

$$v_0 = p_0 + u_0'.$$

We now define the Tonelli Hamiltonian $H : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ by

$$H(x,p) = H_0(x, p_0 + p).$$

We show that $\bar{H}(0) = 1$. In fact, since

$$H(x, u_0'(x)) = H_0(x, p_0 + u_0'(x)) = H(x, v_0(x)) = 1,$$

the Hamilton-Jacobi Equation associated to H admits u_0 a solution therefore $\bar{H}(0) = 1$. Suppose now that $u : \mathbb{T} \to \mathbb{R}$ is a critical C^1 subsolution for the Hamiltonian H, we must have

$$\forall x \in \mathbb{T}, H_0(x, p_0 + u'(x)) = H(x, u'(x)) \leq 1.$$

Therefore by (*) above, we obtain $p_0 + u'(x) \leq v_0(x) = p_0 + u_0'(x)$. This yields $u' \leq u_0'$. In fact this inequality must be an equality because u, u_0 being 1-periodic we have

$$\int_0^1 u'(s)\, ds = u(1) - u(0) = 0 = \int_0^1 u_0'(s)\, ds.$$

Since both functions u' and u'_0 are continuous the inequality $u' \leq u'_0$ cannot be strict at a point. Hence u differs from u_0 by a constant and is $C^{1,1}$ but not C^2.

Notice that the example above gives a case where all critical subsolutions are solutions.

In the rest of the paper we want to investigate the case when there exists critical subsolutions which are smoother than $C^{1,1}$. For this we first need to define $\bar{H}$ as a function from $\mathbb{R}^n$ to $\mathbb{R}$.

5 The homogenized Hamiltonian $\bar{H} : \mathbb{R}^n \longrightarrow \mathbb{R}$

If $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Tonelli Hamiltonian, then it is easy to check that for a given $P \in \mathbb{R}^n$, the Hamiltonian $H_P : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$H_P(x,p) = H(x, P+p)$$

is also a Tonelli Hamiltonian. Therefore we can define the homogenized Hamiltonian $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ by

$$\bar{H}(P) = \bar{H}_P(0).$$

Proposition 5.1. *The homogenized Hamiltonian $\bar{H} : \mathbb{R}^n \to \mathbb{R}$ is convex and superlinear.*

Proof. We have

$$\bar{H}(P) = \inf_u \max_{x \in \mathbb{T}^n} H(x, P + d_x u),$$

where the inf is taken over all C^1 maps $u : \mathbb{T}^n \to \mathbb{R}$.

Suppose now that $P, Q \in \mathbb{R}^n$ and $t \in [0,1]$ are fixed. Given C^1 functions $u_1, u_2 : \mathbb{T}^n \to \mathbb{R}$, we set $u = tu_1 + (1-t)u_2$. By convexity of H in the p variable, we obtain

$$\begin{aligned} H(x, [tP + (1-t)Q] + d_x u) &= H(x, [tP + (1-t)Q] + [td_x u_1 + (1-t) d_x u_2]) \\ &= H(x, t[P + d_x u_1] + (1-t)[Q + d_x u_2]) \\ &\leq tH(x, P + d_x u_1) + (1-t) H(x, Q + d_x u_2). \end{aligned}$$

This yields

$$\max_x H(x, [tP+(1-t)Q]+d_x u) \leq t \max_x H(x, P+d_x u_1) + (1-t) \max_x H(x, Q+d_x u_2).$$

Therefore

$$\bar{H}(tP + (1-t)Q) \leq t \max_x H(x, P + d_x u_1) + (1-t) \max_x H(x, Q + d_x u_2),$$

for every pair of C^1 functions $u_1, u_2 : \mathbb{T}^n \to \mathbb{R}$. Taking the infimum over u_1 and u_2, we conclude that

$$\bar{H}(tP + (1-t)Q) \leq t\bar{H}(P) + (1-t)\bar{H}(Q).$$

This proves the convexity for $\bar{H}$. To prove the superlinearity, we again use that a C^1 function $u : \mathbb{T}^n \longrightarrow \mathbb{R}$ has a critical point x_0 (i.e. a point x_0 where $d_{x_0}u = 0$), to obtain

$$\inf_{x\in\mathbb{T}^n} H(x,P) \leq H(x_0,P) = H(x_0, P + d_{x_0}u) \leq \max_{x\in\mathbb{T}^n} H(x, P + d_x u).$$

Taking the infimum over u, we obtain

$$\inf_{x\in\mathbb{T}^n} H(x,P) \leq \bar{H}(P).$$

Since H is superlinear in P (uniformly in $x \in \mathbb{T}^n$), we have

$$\frac{\inf_{x\in\mathbb{T}^n} H(x,P)}{\|P\|} \longrightarrow +\infty \text{ as } \|P\| \longrightarrow +\infty.$$

Together with the inequality above this implies the superlinearity of $\bar{H}$. □

Like every convex function, the function $\bar{H} : \mathbb{R}^n \longrightarrow \mathbb{R}$ has a subgradient $\partial\bar{H}(P) \subset \mathbb{R}^n$ at each $P \in \mathbb{R}^n$. Let us recall that this subgradient is defined by

$$\partial\bar{H}(P) = \{v \in \mathbb{R}^n \mid \forall p \in \mathbb{R}^n,\ \langle v, p\rangle_{\mathrm{euc}} \leq \bar{H}(P+p) - \bar{H}(P)\},$$

where $\langle\cdot,\cdot\rangle_{\mathrm{euc}}$ is the usual scalar product on $\mathbb{R}^n$. As is well-known by Hahn-Banach Theorem $\partial\bar{H}(P)$ is non-empty. It is also convex and compact.

We would now like to give a geometrical-dynamical description of $\partial\bar{H}(0)$ (hence also of $\partial\bar{H}(P) = \partial\bar{H}_P(0)$).

To do this let us consider a $C^{1,1}$ critical subsolution $u : \mathbb{T}^n \longrightarrow \mathbb{R}$ for H, given by Patrick Bernard's Theorem 4.1. The derivative $x \mapsto d_x u$ is Lipschitz. Since the Hamiltonian H is C^2, it follows that the vector field

$$\chi_u(x) = \frac{\partial H}{\partial p}(x, d_x u),$$

called the Hamiltonian gradient of u, is also Lipschitz on $\mathbb{T}^n$. By compactness of $\mathbb{T}^n$, this Lipschitz vector field x defines a global flow $(\varphi_t^u)_{t\in\mathbb{R}}$ on $\mathbb{T}^n$.

We introduce for such a $C^{1,1}$ critical subsolution $u : \mathbb{T}^n \longrightarrow \mathbb{R}$, its tangency set $\mathcal{T}(u) \subset \mathbb{T}^n$ defined by

$$\mathcal{T}(u) = \{x \in \mathbb{T}^n \mid H(x, d_x u) = \bar{H}(0)\}.$$

Note that by Lemma 2.2, the subset $\mathcal{T}(u) \subset \mathbb{T}^n$ is not empty. It is also compact. We now define the compact subset $\mathcal{I}(u) \subset \mathcal{T}(u)$ by

$$\mathcal{I}(u) = \bigcap_{t\in\mathbb{R}} \varphi_t^u[\mathcal{T}(u)].$$

This subset $\mathcal{I}(u)$ is the maximal subset of $\mathcal{T}(u)$ which is invariant by the flow φ_t^u.

Another description of this subset $\mathcal{I}(u)$ can be given. The Hamiltonian H generates a Hamiltonian flow ϕ_t^H on $\mathbb{T}^n \times \mathbb{R}^n$ which solves the ODE

$$\dot{x} = \frac{\partial H}{\partial p}(x,p)$$
$$\dot{p} = -\frac{\partial H}{\partial x}(x,p).$$

The set $\mathcal{I}(u)$ is the projection of the orbits of ϕ_t^H contained in $\mathrm{graph}(du) \cap H^{-1}(\bar{H}(0))$, where

$$\mathrm{graph}(du) = \{(x, d_x u) \mid x \in \mathbb{T}^n\} \subset \mathbb{T}^n \times \mathbb{R}^n.$$

This set $\mathcal{I}(u)$ is also the so-called projected Aubry set of u, see [6] or [7]. The following theorem is therefore a consequence of a result of John Mather, see [11] or [6].

Theorem 5.2 (Mather). *The subset $\mathcal{I}(u)$ is non-empty.*

As is usual in Dynamical Systems, we will consider invariant probability measures. More precisely, we will denote by $\mathcal{P}(u)$ the set of probability measures on $\mathbb{T}^n$ which are invariant under the flow φ_t^u and *whose support is contained in* $\mathcal{I}(u)$. These measures on $\mathbb{T}^n$ are just the projections of the minimizing measures introduced in the work of John Mather, see [11, 12] (it might be helpful to use some facts shown in [6] or [7]).

Obviously, the set $\mathcal{P}(u)$ is convex and compact for the weak topology on measures. Since $\mathcal{I}(u)$ is not empty and compact, by the Krylov-Bogoliouboff theorem, see [8, Theorem **4.1.1** page 135], the set $\mathcal{P}(u)$ is also non-empty.

We can now give the geometrical-dynamical description of $\partial\bar{H}(0)$.

Theorem 5.3. *We have $\partial\bar{H}(0) = \{\int_{\mathbb{T}^n} \chi_u(x)\, d\mu(x) \mid \mu \in \mathcal{P}(u)\}$.*

In fact, since the measures in $\mathcal{P}(u)$ are projections of minimizing measures, the theorem above is a simple reformulation of the fact that the convex functions α and β functions defined in Mather's work, see [11, 12], are dual functions.

We will use the notation

$$\mathcal{S}(\mu) = \int_{\mathbb{T}^n} \chi_u(x)\, d\mu(x),$$

for $\mu \in \mathcal{P}(u)$ with this notation we have $\partial\bar{H}(0) = \mathcal{S}(\mathcal{P}(u))$.

We give a couple of classical examples of invariant measures. If the point $x \in \mathcal{I}(u)$ is a fixed point of the flow φ_t^u, then $\chi_u(x) = 0$, and the Dirac mass δ_x is invariant by the flow φ_t^u. We have

$$\mathcal{S}(\mu) = \int_{\mathbb{T}^n} \chi_u\, d\delta_x = 0.$$

If $x \in \mathcal{I}(u)$ is a periodic point of the flow φ_t^u, with period $T > 0$ we define the probability measure μ_x on $\mathbb{T}^n$ by

$$\int_{\mathbb{T}^n} \theta \, d\mu_x = \frac{1}{T} \int_0^T \theta(\varphi_t^u(x)) \, dt,$$

for $\theta : \mathbb{T}^n \to \mathbb{R}$ continuous. It is not difficult to check that this measure is invariant, and we have

$$\mathcal{S}(\mu_x) = \frac{1}{T} \int_0^T \chi_u(\varphi_t^u(x)) \, dt.$$

The orbit of x under φ_t^u is the closed curve $\gamma_x(t) = \varphi_t^u(x), t \in [0, T]$. Moreover, we have $\dot{\gamma}_x(t) = \chi_u(\varphi_t^u(x))$. Therefore

$$\int_0^T \chi_u(\varphi_t^u(x)) \, dt = \int_0^T \dot{\gamma}_x(t) \, dt.$$

But this last quantity is in $\mathbb{Z}^n$ because

$$\int_0^T \dot{\gamma}_x(t) \, dt = \varphi_T^u(x) - x = x - x = 0 \mod \mathbb{Z}^n.$$

This quantity $\int_0^T \dot{\gamma}_x(t) \, dt$ will be denoted by $[\gamma_x] \in \mathbb{Z}^n$. This notation is to remind us that the first homology group $H_1(\mathbb{T}^n, \mathbb{R})$ is canonically isomorphic to $\mathbb{R}^n$, and in this canonical isomorphism the homology class of the closed curve γ_x is precisely $\int_0^T \dot{\gamma}_x(t) \, dt$. Therefore we get

$$\mathcal{S}(\mu_x) = \frac{1}{T} [\gamma_x].$$

We conclude this section by treating the Examples (E1) and (E2) given above. For example (E1), we can take the function 0 as a critical subsolution.

$$\chi_0(x) = \frac{\partial H^V}{\partial p}(x, 0) = 0,$$

Therefore the flow φ_t^0 is the identity. We have

$$\mathcal{T}(0) = \{x \in \mathbb{T}^n \mid H^V(x, 0) = \bar{H}^V(0)\} = \{x \in \mathbb{T}^n \mid V(x) = \max V\}.$$

Since, the flow φ_t^0 is the identity, we get $\mathcal{I}(0) = \mathcal{T}(0)$. In that case $\mathcal{P}(0)$ is simply the set of probably measures supported in $\mathcal{I}(0) = \{x \in \mathbb{T}^n \mid V(x) = \max V\}$. Moreover, since $\chi_0 \equiv 0$, we obtain

$$\partial \bar{H}^V(0) = \{0\}.$$

In particular, the convex function $\bar{H}^V$ is differentiable at 0.

For example (E2), the function 0 is a solution. This implies

$$\mathcal{T}(0) = \{x \in \mathbb{T}^n \mid H^X(x,0) = \bar{H}^X(0)\}$$

is the whole of $\mathbb{T}^n$. Hence $\mathcal{I}(0) = \mathcal{T}(0) = \mathbb{T}^n$. Since

$$\frac{\partial H^X}{\partial p}(x,p) = p + X,$$

we get

$$\chi_0(x) = \frac{\partial H^V}{\partial p}(x,0) = X(x).$$

Therefore the flow φ_t^0 is simply the flow φ_t^X of X, and $\mathcal{P}(0)$ is the set of probability measures μ which are invariant under φ_t^X. In PDE terms, this is the set of probability measures such that $\operatorname{div}(X\mu) = 0$. Let us now consider, on $\mathbb{T}^n$, a constant vector field $X = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$. In that case, since $\chi_0 = X$, we get $\mathcal{S}(\mu) = (\alpha_1, \dots, \alpha_n)$ for every probability measure. Therefore $\partial H^{(\alpha_1,\dots,\alpha_n)}(0) = \{(\alpha_1, \dots, \alpha_n)\}$. In this case, the convex function is also differentiable at 0.

We will now consider $\mathbb{T}^n = \mathbb{T} \times \mathbb{T}^{n-1}$, and we will denote a point in $\mathbb{T}^n$ by (s,x) with $s \in \mathbb{T}$ and $x \in \mathbb{T}^{n-1}$. We fix a C^2 function $\theta : \mathbb{T}^{n-1} \to \mathbb{R}$. We consider the vector field $X(s,x) = (\theta(x), 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$. For the Hamiltonian H^X, we get $\chi_0(s,x) = X(s,x) = (\theta(x), 0)$. Therefore, $\varphi_t^0(s,x) = \varphi_t^X(s,x) = (s + \theta(x)t, x)$. We have already shown that $\mathcal{I}(0) = \mathbb{T}^n$. Moreover, one can show that the invariant measures for φ_t^X are precisely the measures of the form $ds \otimes \nu$, where ν is a probability measure on $\mathbb{T}^{n-1}$. For such a measure $ds \otimes \nu$, we have

$$\mathcal{S}(ds \otimes \nu) = \int_{\mathbb{T}^n} X(s,x)\, ds \otimes d\nu = (\int_{\mathbb{T}^{n-1}} \theta\, d\nu, 0).$$

Since ν is an arbitrary probability measure on $\mathbb{T}^{n-1}$, we obtain

$$\partial \bar{H}^X(0) = \{(t,0) \mid t \in [\min\theta, \max\theta]\} \subset \mathbb{R} \times \mathbb{R}^{n-1}.$$

In this case, unless θ is constant, the convex function ∂H^X is not differentiable at 0.

6 Dynamics of flows on $\mathbb{T}^2$

In this section, we recall the classical results on flows on $\mathbb{T}^2$. Going back to Poincaré there has been a long tradition in studying flows on plane domains, for example the Poincaré-Bendixson Theorem [8, §1 Chapter14], and the two-dimensional torus $\mathbb{T}^2$, [8, Chapter14]. Denjoy has extended this work and shown that there are strong restrictions on closed subsets invariant under a C^1 flow on $\mathbb{T}^2$. These restrictions follow from the strong restrictions on the possible dynamics of homeomorphisms on $\mathbb{T}$, as was also shown by Denjoy, see for example [8, Chapters 11 & 14].

We start by recalling that a continuous flow φ_t acting on the topological space S is a continuous map $S\times\mathbb{R} \to S$, $(x,t) \mapsto \varphi_t(x)$ which satisfies the flow properties $\varphi_0 = \mathrm{Id}_S$, and $\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$, for all $t, t' \in \mathbb{R}$. The flow properties imply that the φ_t are homeomorphisms. The orbit of $x \in S$ for φ_t is $\{\varphi_t(x) \mid t \in \mathbb{R}\}$. The point $x \in S$ is said to be recurrent for φ_t if we can find a sequence $t_n \in \mathbb{R}$, with $|t_n| \to +\infty$ such that $\varphi_{t_n}(x) \to x$ when $n \to +\infty$. Obviously, by continuity of φ_t and its flow properties, if this condition is satisfied for x it is satisfied at any other point $\varphi_t(x)$ of the orbit of x. The α-limit $\alpha(x)$ (resp. ω-limit $\omega(x)$) of the orbit of x is the set of limit points of $\varphi_t(x)$ as $t \to -\infty$ (resp. $t \to +\infty$). Obviously by the flow property of φ_t, the α-limit and ω-limit sets are the same along an orbit of φ_t. Moreover, by the continuity of φ_t and its flow properties, the α-limit and ω-limit sets of a an orbit are both invariant under the flow φ_t. With this definition, the point x is recurrent under the flow φ_t if and only if $x \in \alpha(x) \cup \omega(x)$. If x is fixed under the flow, then we have $\alpha(x) = \omega(x) = \{x\}$. If x is periodic under φ_t, i.e. there exists $t_0 > 0$ with $\varphi_{t_0}(x) = x$, then $\alpha(x)$ and $\omega(x)$ are both equal to the orbit $\{\varphi_t(x) \mid t \in \mathbb{R}\} = \{\varphi_t(x) \mid t \in [0, t_0]\}$ of x. Note that $\alpha(x)$ and $\omega(x)$ are never empty if the space S is compact (or even if the orbit of x is relatively compact).

If a point x is periodic but not fixed there exists $t_x > 0$ such that $\varphi_t(x) = x$ if and only if $t = nt_x$, with $n \in \mathbb{Z}$. This t_x is called the minimal (or smallest) period of x. The orbit of x is in that case the simple closed curve $t \in [0, t_x] \mapsto \varphi_t(x)$.

If x is neither fixed nor periodic for φ_t, then it is convenient to say that its orbit is nonperiodic.

We now recall the Poincaré-Bendixson Theorem as it is stated for example in [8, Theorem 14.1.1, page 452]. Although it is usually stated for C^1 flows, it is also true for a flow φ_t generated by a continuous field vector X. Recall that the flow φ_t is generated by the continuous field vector X, if for each x the curve $s \mapsto \varphi_s(x)$ is C^1 and

$$X(x) = \frac{d\varphi_s(x)}{ds}_{s=0}.$$

By the flow property it follows that

$$\forall x \in \mathbb{T}^n, \forall t \in \mathbb{R}, X(\varphi_t(x)) = \frac{d\varphi_s(x)}{ds}_{s=t}.$$

The proof in [8] applies also to a flow generated by the continuous field vector X, since it only uses the existence of a local transversal at a regular point x_0. We recall that, since we are on a surface, a C^1 curve $\gamma : I \to S$ on the surface S is transversal to the continuous vector field X on S if for each $t \in I$ the speed $\dot\gamma(t)$ is not collinear with $X(\gamma(t))$. Note that by continuity of X any non singular smooth curve $\gamma : [-\epsilon, \epsilon]$ which is such that $\gamma(0) = x_0$ and $\dot\gamma(0) \notin \mathbb{R}X(x_0)$ will be transversal to X on a smaller appropriate interval $[-\delta, \delta]$.

Theorem 6.1 (Poincaré-Bendixson). *Let U be an open subset of the plane $\mathbb{R}^2$. If φ_t is a flow generated by the continuous vector field X on U, then any recurrent orbit of φ_t is either a fixed point or a periodic orbit. Moreover, if $x \in U$ and $\alpha(x)$*

(resp. $\omega(x)$) is not empty and does not contain a fixed point, then it is necessarily reduced to a single closed orbit.

In fact the theorem above is also true for continuous flows but it requires construction of topological transversals which are arcs. This is a more delicate matter.

An important well-known proposition to study recurrent non-periodic orbits is the following one.

Proposition 6.2. *If φ_t is a flow generated by the continuous vector field X on the surface M, and the orbit of a given $x \in M$ is nonperiodic and recurrent, then there exists a simple smooth closed curve $\gamma \subset M$ which is everywhere transversal to the vector field X tangent to the flow φ_t, and such that $x \in \gamma$.*

The proof of the proposition above is essentially given in the proof of [8, Proposition 14.1.3, page 453] at least for when X is C^1. One can make adjustments to have it work for flows generated by continuous vector fields.

To state our next proposition, we need to recall the following known topological facts:

- If γ is simple closed curve in $\mathbb{T}^2$ which disconnects $\mathbb{T}^2$ then one of the components is diffeomorphic to $\mathring{B}^2$ the open Euclidean ball of center 0 and radius 1 in $\mathbb{R}^2$.
- If γ does not disconnect $\mathbb{T}^2$ then $\mathbb{T}^2 \setminus \gamma$ is diffeomorphic to the open annulus $\mathring{\mathbb{A}}^2 = \{x \in \mathbb{R}^2 \mid 1 < \|x\|_{\mathrm{euc}} < 2\}$, where $\|\cdot\|_{\mathrm{euc}}$ is the usual Euclidean norm on $\mathbb{R}^2$.

Proposition 6.3. *Let φ_t be a flow on $\mathbb{T}^2$ generated by the continuous vector field X. Suppose x_0 is a nonperiodic point for the flow φ_t which satisfies $x_0 \in \alpha(x_0) \cap \omega(x_0)$ (i.e. x_0 is both positively and negatively recurrent). If γ is a smooth simple closed curve transversal to the vector field X which does intersect the orbit of x_0, then γ does not disconnect $\mathbb{T}^2$, and any other recurrent nonperiodic orbit intersects γ.*

Moreover, if $x \in \mathbb{T}^2$ is such that $\{t \geq 0 \mid \varphi_t(x) \in \gamma\}$ (resp. $\{t \leq 0 \mid \varphi_t(x) \in \gamma\}$) is finite then $\omega(x)$ (resp. $\alpha(x)$) is disjoint from γ and $\omega(x)$ (resp. $\alpha(x)$) either contains a fixed point or is reduced to a periodic orbit.

Proof. In fact, if the curve γ bounds a disc D and is transversal to X, we can assume for example that X points along γ toward the interior of D (if not replace φ_t by the flow $\hat{\varphi}_t = \varphi_{-t}$). Now the orbit of x enters D because it cuts γ, but it can never get out again from D because X is transversal to γ and points to the interior of D along γ. Therefore we can apply the Poincaré-Bendixson Theorem 6.1 to conclude that x is not positively recurrent which is a contradiction.

Therefore γ separates $\mathbb{T}^2$. If an orbit does not cut γ then it is entirely contained in $\mathbb{T}^2 \setminus \gamma$ which is diffeomorphic to the annulus $\mathring{\mathbb{A}}^2$. By the Poincaré-Bendixson Theorem 6.1, it cannot be recurrent.

To prove the last statement of the Proposition, note that by transversality we can find a neighborhood U of γ such that any orbit $\varphi_t(z)$ entering U at time t_0 cuts γ at a time in $[t_0 - 1, t_0 + 1]$. Now if $\{t \geq 0 \mid \varphi_t(x) \in \gamma\}$ is finite, we can find $T > 0$ such that $\varphi_t(x) \notin \gamma$, for every $t \geq T$, therefore $\varphi_t(x) \notin U$ for every $t \geq T + 1$. This implies that $\omega(x) \subset \mathbb{T}^2 \setminus U$. But this last subset is contained in $\mathbb{T}^2 \setminus \gamma$ which is diffeomorphic to the annulus $\mathring{\mathbb{A}}^2$. By Poincaré-Bendixson Theorem 6.1, the set $\omega(x)$ either contains a fixed point or is precisely one periodic orbit. □

Now we come to the consequences of the famous Poincaré-Denjoy theory of circle homeomorphisms, see [8, Chapter 11].

Theorem 6.4 (Poincaré-Denjoy). *Let φ_t be a flow on $\mathbb{T}^2$ generated by the continuous vector field X. Suppose A is a closed non-empty subset of $\mathbb{T}^2$ which is invariant under φ_t and does not contain any fixed point or periodic orbit. Consider the set A_0 of non-periodic recurrent orbits contained in A. We have:*

1) *The set A_0 is closed and not empty. Every orbit of a point in A_0 is dense in A_0.*

2) *There is exactly one probability measure μ carried by A and invariant under φ_t. Moreover, this measure has support in A_0.*

3) *We have $\int_{\mathbb{T}^2} X\,d\mu \notin \mathbb{R}\cdot\mathbb{Z}^2$, i.e. the components of the planar vector $\int_{\mathbb{T}^2} X\,d\mu$ are rationally independent.*

Proof. By the Krylov-Bogoliouboff theorem, see [8, Theorem **4.1.1** page 135], there exists a probability measure μ invariant under φ_t with support contained in A. By the Poincaré Recurrence Theorem, see [8, Page 142], the set A_0 of points in A which are both positively and negatively invariant has full measure for μ. In particular it is not empty, and we can find a closed smooth curve γ transversal to the vector field X which intersects one of the orbits in A_0. By the previous proposition, since the closed invariant set A contains neither a fixed point nor a periodic orbit, every point in A has an orbit which intersects γ infinitely often both in positive and negative time. Therefore if we set $K = A \cap \gamma \subset \gamma$, we have $A = \cup_{t\in\mathbb{R}}\varphi_t(K)$. Moreover the Poincaré return map R_γ on γ is well defined on K by

$$R_\gamma(x) = \varphi_{T_\gamma(x)}(x),$$

where $T_\gamma(x) = \inf\{t > 0 \mid \varphi_t(x) \in \gamma\}$. Both T_γ and R_γ are continuous on K. Moreover, the Poincaré return map R_γ is a bijection of the compact set K, because every orbit of a point K intersects γ in negative time at a point which is also in $K = A \cap \gamma$. Therefore R_γ is even a homeomorphism of K on itself. Note that γ is homeomorphic to $\mathbb{T}$ and K is a compact subset which is infinite. Therefore, if C is a connected component of $\gamma \setminus K$ it is homeomorphic to an open interval, and its $\bar{C}$ is equal to $C \cup \{a_C, b_C\}$, where a_C and b_C are two distinct points in K. Since for $x \neq x' \in K$ the pieces of orbits $\varphi_t(x), t \in [0, T_\gamma(x)]$ and $\varphi_t(x'), t \in [0, T_\gamma(x')]$ do not intersect, the two points $R_\gamma(a_C), R_\gamma(b_C)$ are the endpoints of a connected component $\tilde{C}$ of $\gamma \setminus K$. We can extend the map R_γ to C by extending it homeomorphically to a map from C to $\tilde{C}$. Doing this to every connected component of $\gamma \setminus K$, we find a map $\tilde{R}_\gamma$ from γ to itself which is an orientation preserving homeomorphism of $\gamma \approx \mathbb{T}$ to itself, which is equal to R_γ on K. We can apply to this homeomorphism $\tilde{R}_\gamma$ the theory of Poincaré and Denjoy, see [8, Chapter 11]. Since K is invariant non-empty and has only infinite orbits the rotation number of $\tilde{R}_\gamma$ is irrational, and K contains the non-wandering set Ω of $\tilde{R}_\gamma$. Moreover, the dynamics on Ω is minimal, i.e. every orbit in Ω is dense in Ω, and every other orbit has an α-limit set and an ω-limit set equal to Ω. It is then not difficult to see that $A_0 \cap \gamma \subset \Omega \subset A_0 \cap \gamma$. Hence, we obtain that $A_0 = \cup_{x\in\Omega}\{\varphi_t(x) \mid t \in [0, T_\gamma(x)]\}$ is compact, the action of φ_t on A_0 is minimal (i.e. every orbit in A_0 is dense in A_0), and every φ_t orbit of a point in A has its α-limit set and its ω-limit set equal to A_0. We again apply the Poincaré-Denjoy theory to the homeomorphism with irrational rotation number $\tilde{R}_\gamma$ to obtain that it has a unique invariant probability measure, and its support is in Ω, see [8, Theorem 11.2.9 page 399]. This implies that the action of φ_t on A has a unique probability measure μ invariant under φ_t and its support is A_0.

It remains to show that $\int X\, d\mu$ is not in $\mathbb{R}\cdot\mathbb{Z}^2$. This is a consequence of the fact that the rotation number α of $\tilde{R}_\gamma$ is irrational. We will explain it in the case where the curve γ is $\mathbb{T} = \mathbb{T} \times \{0\} \subset \mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$. Since there exists a diffeomorphism that carries the simple closed curve γ, which is not homologous to 0, to $\mathbb{T} = \mathbb{T} \times \{0\}$, one can reduce the general case to the case above. Therefore assuming $\gamma = \mathbb{T} = \mathbb{T} \times \{0\}$, let us call (x, y) the canonical coordinates of $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$. The vector field X can be written $X(x, y) = (X_x(x, y), X_y(x, y))$ where X_x and X_y are real valued continuous functions. Transversality of X to $\gamma = \mathbb{T} = \mathbb{T} \times \{0\}$ means that $X_y(x, 0)$ is never 0. By connectedness, we can assume for example that $X_y(x, 0) > 0$ everywhere. Note that the return time $T(x) = T_\gamma(x, 0) = T_\mathbb{T}(x, 0)$ of a point $(x, 0) \in K$ to $\gamma = \mathbb{T} = \mathbb{T} \times \{0\}$ is the first time $t > 0$ for which $\varphi_t(x, 0)$ has its y coordinate equal to 0 in $\mathbb{T}$. The y coordinate of $\varphi_t(x, 0)$ is equal to $\int_0^t X_y(\varphi_s(x, 0))\, ds \mod 1$. Therefore $T(x) = T_\mathbb{T}(x, 0)$ is the first time $t > 0$ for which $\int_0^t X_y(\varphi_s(x, 0))\, ds \in \mathbb{Z}$. Since X_y is > 0 on $\mathbb{T} = \mathbb{T} \times \{0\}$, we see that we necessarily have

$$\forall (x, 0) \in K, \int_0^{T(x)} X_y(\varphi_s(x, 0))\, ds = 1.$$

How can one interpret the quantity $d(x) = \int_0^{T(x)} X_x(\varphi_s(x, 0))\, ds$ for $(x, 0) \in K$? It is in fact the the quantity $\bar{R}_\mathbb{T}(x, 0) - x$ for an appropriate lift $\bar{R}_\mathbb{T}$ to $\mathbb{R}$ of the extended Poincaré return map $\tilde{R}_\mathbb{T} = \tilde{R}_\gamma : \mathbb{T} \to \mathbb{T}$. Therefore if α is the rotation number of that lift, we have

$$\alpha = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\tilde{R}^i_\mathbb{T}(x)),$$

for every $(x, 0) \in K$. If we set $T_n(x) = T(x) + T(\tilde{R}_\mathbb{T}(x)) + \cdots T(\tilde{R}^{n-1}_\mathbb{T}(x))$, we then have

$$\varphi_{T_n(x)}(x,0) = (\tilde{R}^n_{\mathbb{T}}(x),0)$$
$$\int_0^{T_n(x)} X_x(\varphi_s(x,0))\,ds = \sum_{i=0}^{n-1} d(\tilde{R}^i_{\mathbb{T}}(x))$$
$$\int_0^{T_n(x)} X_y(\varphi_s(x,0))\,ds = \sum_{i=0}^{n-1}\int_0^{T(\tilde{R}^i_{\mathbb{T}}(x))} X_y[\varphi_s(\tilde{R}^i_{\mathbb{T}}(x),0)]\,ds = n.$$

Therefore for each $(x,0)\in K$, we get

$$\lim_{n\to\infty}\frac{\int_0^{T_n(x)} X_x(\varphi_s(x,0))\,ds}{\int_0^{T_n(x)} X_y(\varphi_s(x,0))\,ds} = \alpha. \qquad (*)$$

Note also that

$$n\max_K T \geq T_n(x) \geq n\min_K T > 0.$$

Hence

$$\frac{n}{T_n(x)} = \frac{1}{T_n(x)}\int_0^{T_n(x)} X_y(\varphi_s(x,0))\,ds \geq \frac{1}{\max_K T} > 0. \qquad (**)$$

Let us then consider the unique invariant measure μ with support in A, it must be ergodic, see [8, Proposition 4.1.8 page 138]. We can apply Birkhoff Ergodic Theorem, see [13, Theorem 5.0.2 page 460] or [8, Theorem 4.1.2 page 136], to obtain that for μ-almost every (x,y), we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X_x(\varphi_s(x,y))\,ds = \int X_x\,d\mu$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X_x(\varphi_s(x,y))\,ds = \int X_y\,d\mu.$$

Since every orbit of A cuts $\gamma = \mathbb{T} = \mathbb{T}\times\{0\}$ in a point $(x,0)\in K$, and μ is carried by A, we obtain first from (**) that

$$\int X_y\,d\mu \geq 1/\max_K T > 0,$$

then from (*)

$$\frac{\int X_x\,d\mu}{\int X_y\,d\mu} = \alpha.$$

But α is irrational, since it is the rotation number of the lift of a homeomorphism of $\mathbb{T}$ without periodic points. $\square$

Denjoy's work in 1932 has shown that C^2 flows on $\mathbb{T}^2$ have strong restrictions on their possible dynamics, see [4]. We will state a more general version on surfaces due to A. Schwartz, see [14] or [8, Theorem 14.3.1 page 460], that we will also use in the last section to extend our observations to higher genus surfaces. The version we state is not exactly the one given by Schwartz, but it can be deduced from [14, Theorem page 453] and [14, Corollary page 457], using that every compact non-empty set invariant under a flow contains a compact minimal subset (i.e. a compact non-empty subset invariant under the flow such that every orbit contained in that subset is dense in that subset).

Theorem 6.5 (Denjoy-Schwartz). *Let M be a compact 2-dimensional surface with no boundary. Let φ_t be a C^2 flow on M. Suppose that A is a compact non-empty subset of M which is invariant under φ_t, then one of the following three statements holds:*

1) *The set A contains a fixed point of the flow φ_t.*
2) *Every recurrent orbit of φ_t contained in A is a closed orbit.*
3) *Every orbit of φ_t is dense in M, and we have $M = A = \mathbb{T}^2$.*

The reader should be warned that Denjoy has constructed counterexamples to the theorem with C^1 flows, see for example [8, §2 Chapter 12]

7 The Homogenized Hamiltonian on $\mathbb{T}^2$

The results in the previous section imply some restrictions on the subgradient $\partial\bar{H}(P)$ for a $P \in \mathbb{T}^2$. It has also consequences on the form of the Mather set $\tilde{\mathcal{M}}_P \subset \mathbb{T}^2 \times \mathbb{R}^2$ of the Hamiltonian $H_P(x,p) = H(x, P+p)$. Let us recall that $\tilde{\mathcal{M}}_P$ is invariant under the Hamiltonian flow ϕ_t^H of H. Moreover it is a graph on its projection $\mathcal{M}_P$ in $\mathbb{T}^2$. If u is a critical $C^{1,1}$ subsolution of H_P then

$$\mathcal{M}_P = \overline{\cup\{\text{supp}(\mu) \mid \mu \in \mathcal{P}(u)\}},$$

and the orbits of ϕ_t^H contained in $\tilde{\mathcal{M}}_P$ project on orbits of φ_t^u in $\mathbb{T}^2$.

Proposition 7.1. *Let $H : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a Tonelli Hamiltonian. If $P \in \mathbb{R}^2$, then one of the following things happen:*

(i) *We have $0 \in \partial\bar{H}(P)$, and therefore $\bar{H}(P) = \inf_{\mathbb{R}^2} \bar{H}$. Moreover, the Mather set $\tilde{\mathcal{M}}_P$ does satisfy one of the following possibilities:*

  (a) *$\tilde{\mathcal{M}}_P$ contains a fixed point of ϕ_t^H.*

  (b) *$\tilde{\mathcal{M}}_P$ contains a closed orbit of ϕ_t^H whose projection on $\mathbb{T}^2$ is homologous to 0.*

  (c) *$\tilde{\mathcal{M}}_P$ contains a pair closed orbits of ϕ_t^H whose projection γ_1, γ_2 on $\mathbb{T}^2$ are such that γ_2 is homologous to $-\gamma_1$.*

(ii) *There is a vector $V \in \mathbb{Z}^2 \setminus \{0\}$, and $\beta \geq \alpha > 0$ such that*

$$\partial \bar{H}(P) = \{tV \mid t \in [\alpha, \beta]\}.$$

In particular, we have $\partial \bar{H}(P) \subset \mathbb{R} \cdot \mathbb{Z}^2$. Moreover we can choose V such the Mather set $\tilde{\mathcal{M}}_P$ is a union of disjoint closed orbits of the Hamiltonian flow ϕ_t^H whose projections in $\mathbb{T}^2$ all have a homology class equal to V.

(iii) *The set $\partial \bar{H}(P)$ is reduced to one point contained in $\mathbb{R}^2 \setminus \mathbb{R} \cdot \mathbb{Z}^2$. In particular, the convex function $\bar{H}$ is differentiable at P. Moreover, in that case the Mather set $\tilde{\mathcal{M}}_P$ is a minimal uniquely ergodic subset for the Hamiltonian flow ϕ_t^H, with no closed orbit or fixed point.*

Proof. Replacing H by H_P we can assume $P = 0$. We choose a $C^{1,1}$ critical subsolution u for H. We consider the non-empty set $\mathcal{I}(u)$, which is invariant under the flow φ_t^u generated by the Lipschitz vector field χ_u. We will use the characterization of $\partial \bar{H}(0)$ given in Theorem 5.3.

If $\mathcal{I}(u)$ contains a fixed point x_0 for φ_t^u, as we have seen in §5, the Dirac mass δ_{x_0} is invariant under the flow and $\mathcal{S}(\delta_{x_0}) = 0$. Therefore we are in case (i) subcase (a).

We can from now on assume that $\mathcal{I}(u)$ does not contain any fixed point.

Suppose that the set Per of periodic points in $\mathcal{I}(u)$ is not empty. Since we are assuming that $\mathcal{I}(u)$ does not contain any fixed point, every $x \in$ Per has a minimal period $t_x > 0$. The curve $\gamma_x(t) = \varphi_t^u(x), t \in [0, t_x]$ is a simple closed curve. We can define the invariant measure μ_x

$$\int_{\mathbb{T}^2} \theta \, d\mu_x = \frac{1}{t_x} \int_0^{t_x} \theta(\varphi_s^u(x)) \, ds.$$

Again by §5, we have

$$\mathcal{S}(\mu_x) = \frac{1}{t_x}[\gamma_x].$$

If Per contains a point x such that γ_x is homologous to 0, then we are in case (i) subcase (b).

We now assume that Per is not empty and consists of points x such that the orbit $[\gamma_x]$ is never homologous to 0. Let us fix one of these points $x_0 \in$ Per, and we set $V = [\gamma_{x_0}] \in \mathbb{Z}^2 \setminus \{0\}$. We know that $\mathbb{T}^2 \setminus \gamma_{x_0}$ is homeomorphic to the annulus $\mathring{\mathbb{A}}^2$. If $x \in \text{Per} \setminus \gamma_{x_0}$, since γ_x is a simple curve in the annulus $\mathbb{T}^2 \setminus \gamma_{x_0}$ which is not homologous to 0, then necessarily $[\gamma_x] = \epsilon(x)V$ with $\epsilon(x) = \pm 1$. If for some x we have $\epsilon(x) = -1$ then we can define the invariant measure

$$\mu = \frac{1}{t_{x_0} + t_x}[t_{x_0}\mu_{x_0} + t_x\mu_x].$$

We have

$$\mathcal{S}(\mu) = \frac{1}{t_{x_0} + t_x}[t_{x_0}\mathcal{S}(\mu_{x_0}) + t_x\mathcal{S}(\mu_x)] = \frac{1}{t_{x_0} + t_x}[t_{x_0}(\frac{1}{t_{x_0}}V) + t_x(-\frac{1}{t_x}V)] = 0.$$

And we are in case (i) subcase (c).

Assume now that in fact $\epsilon(x) = 1$, for every $x \in \text{Per}$. Since every orbit of φ_t^u distinct from γ_{x_0} is contained in $\mathbb{T}^2 \setminus \gamma_{x_0}$, which is homeomorphic to the annulus $\mathring{\mathbb{A}}^2$, we can apply the Poincaré-Bendixson Theorem 6.1 to conclude that every recurrent point $x \in \mathcal{I}(u)$ is contained in Per. By Poincaré Recurrence Theorem, see [8, Page 142], any invariant measure μ with support in $\mathcal{I}(u)$ is therefore carried by Per. This has two consequences. The first one is that the Mather set $\mathcal{M}_0$ is equal to the closure of Per. The second one is

$$\mathcal{S}(\mu) = \int_{\text{Per}} \chi_u(x)d\mu.$$

Note that for $x \in \text{Per}$, we have

$$\lim_{t\to+\infty} \int_0^t \chi_u(\varphi_s^u(x))\, ds = \frac{1}{t_x}[\gamma_x].$$

Since $[\gamma_x] = V$, using Birkhoff Ergodic Theorem, see [13, Theorem 5.02 page 460] or [8, Theorem 4.1.2 page 136], we obtain

$$\mathcal{S}(\mu) = \int_{\text{Per}} \frac{1}{t_x} V\, d\mu(x) = \left[\int_{\text{Per}} \frac{1}{t_x}\, d\mu(x)\right] V.$$

Since $\partial\bar{H}(0) = \mathcal{S}(\mathcal{P}(u))$ is compact and convex, we get

$$\partial\bar{H}(0) = [\inf_{\text{Per}} \frac{1}{t_x}, \sup_{\text{Per}} \frac{1}{t_x}]V.$$

In fact, since $\mathcal{I}(u)$ does not contain fixed points, arguments similar to the Poincaré-Bendixson Theorem can show, that Per is closed in $\mathcal{I}(u)$, hence compact, and that $x \mapsto t_x > 0$ is continuous. This implies that the Mather set $\mathcal{M}_0$ is equal to Per, and also that $\inf_{\text{Per}} \frac{1}{t_x} > 0$. Therefore we are in case (ii).

It remains to consider the case when $\mathcal{I}(u)$ does not contain a fixed or a periodic point. In that case we can invoke The Poincaré-Denjoy Theorem 6.4 and Theorem 5.3 to see that we are in case (iii). □

8 Smoother critical subsolutions

What we will do in this section is to apply the Denjoy-Schwartz Theorem 6.5 to the set $\mathcal{I}(u)$. To be able to apply this theorem we will need the flow $(\varphi_t^u)_{t\in\mathbb{R}}$ defined by the vector field

$$\chi_u(x) = \frac{\partial H}{\partial p}(x, d_x u)$$

to be C^2. For this we assume, not only that H is a Tonelli Hamiltonian, but also that $\partial H/\partial p$ is of class C^2, and that the critical subsolution u is of class C^3. In which case χ_u is C^2, and therefore so is its flow φ_t^u.

Notice that for a Tonelli Hamiltonian of the form

$$H(x,p) = \frac{1}{2}\|p\|^2_{\mathrm{euc}} + V(x),$$

we have

$$\frac{\partial H}{\partial p}(x,p) = p.$$

Therefore for such a Tonelli Hamiltonian, the partial derivative $\partial H/\partial p$ is of class C^∞.

The next theorem now follows from the Denjoy-Schwartz Theorem 6.5, and Proposition 7.1.

Theorem 8.1. *Suppose $H : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a Tonelli Hamiltonian with $\partial H/\partial p$ of class C^2. Fix $P \in \mathbb{R}^2$. If $H_P(x,p) = H(x, P+p)$ admits a C^3 critical subsolution, and $\partial \bar{H}(P) \cap \mathbb{R}\cdot\mathbb{Z}^2 = \emptyset$, then $\mathcal{I}(u) = \mathbb{T}^2 = \mathcal{M}_P$, and u is a solution of the Hamilton-Jacobi Equation*

$$H(x, P + d_x u) = \bar{H}(P).$$

For the sake of completeness we would like to state the consequence of the Denjoy-Schwartz Theorem 6.5, for surfaces of higher genus. If $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian on the compact manifold M, then the homogenized Hamiltonian is a convex superlinear function on the first cohomology group $H^1(M,\mathbb{R})$. For $\omega \in H^1(M,\mathbb{R})$, the subgradient $\partial\bar{H}(\omega)$ is in that case naturally given as a subset of $H_1(M,\mathbb{R})$, the first homology group of M. Again there is a procedure to change the Hamiltonian to reduce all statements to the case when $\omega = 0 \in H^1(M,\mathbb{R})$. We will therefore state the result only in the case $\omega = 0$.

Before stating the theorem we will say that a point v in $H_1(M,\mathbb{R})$ is in a rational direction if it is of the form $v = tz$, with $t \in \mathbb{R}$, and z is in the part of $H_1(M,\mathbb{R})$ coming from the integral homology group $H_1(M,\mathbb{Z})$.

Theorem 8.2. *Assume M is a compact orientable boundaryless surface of genus $g \geq 2$. Assume $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian such that $\partial H/\partial p$ of class C^2. If H has a C^3 critical subsolution then one of the two following things happens*

(i) *We have $0 \in \partial\bar{H}(0) \subset H_1(M,\mathbb{R})$, and $\bar{H}(0) = \inf_{H^1(M,\mathbb{R})} \bar{H}$.*

(ii) *The subgradient $\partial\bar{H}(0) \subset H_1(M,\mathbb{R})$ is a polyhedron generated by at most $6g-6$ vertices which are in rational directions*

The number $6g-6$ comes from the fact that, in a surface of genus g, in any set of simple curves with cardinal $> 3g-3$ we can find a couple of curves γ_1, γ_2, with γ_2 is homologous to $\pm\gamma_1$.

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