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NONLOCAL HAMILTON-JACOBI EQUATIONS RELATED TO DISLOCATION DYNAMICS AND A FITZHUGH-NAGUMO SYSTEM

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ABSTRACT. We describe recent existence and uniqueness results obtained for nonlocal nonmonotone Eikonal equations modelling the evolution of interfaces. We focus on two model cases. The first one arises in dislocation dynamics and the second one comes from a FitzHugh-Nagumo system. The equation is nonlocal since, in both case, the velocity at a point of the boundary of the interface depends on the whole enclosed set via a convolution. In these models, the evolution is nonmonotone since we do not expect to have an inclusion principle.

1. INTRODUCTION

This text is the proceeding of conference given a the RIMS Meeting *Viscosity solutions of differential equations and related topics* in Kyoto in 2008. I would like to thank Professors H. Ishii and S. Koike for the invitation.

The aim is to describe recent results for nonlocal and nonmonotone Eikonal equations obtained in [11, 8, 10, 9] in collaboration with Guy Barles, Pierre Cardaliaguet, Régis Monneau and Aurélien Monteillet. I also refer the reader to [33] and [35]. I have chosen to sacrify some generality and to present the most significant results in two model cases: the dislocation dynamics and a FitzHugh-Nagumo system.

We are interested in the following equation

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) = c[\mathbb{1}_{\{u\geq 0\}}](x,t)|Du(x,t)| & \text{in } \mathbb{R}^N \times [0,T], \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $u_0 : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and, for all open subset $\Omega \subset \mathbb{R}^N$,

$$(1.2)\qquad c[\mathbb{1}_{\overline{\Omega}}](x,t) := \alpha(k \star \mathbb{1}_{\overline{\Omega}}(x,t)) + c_1(x,t).$$

The functions $\alpha : \mathbb{R} \to \mathbb{R}$ and $c_1 : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ are Lipschitz continuous, "$\star$" denotes some convolution between a kernel k and the indicator function of $\overline{\Omega}$. More precise assumptions will be given later. This expression of $c[\cdot]$ encompasses the two model cases.

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The paper is organized as follows. In Section 2, we briefly recall some facts about the level set approach to study front propagation problems. It is the motivation to study (1.1). Then, in Sections 3 and 4, we introduce the dynamics of dislocations and a FitzHugh-Nagumo system. Front propagation problems corresponding to these problems lead to nonlocal nonmonotone speed like (1.2). In Section 5, we introduce a notion of weak solutions for (1.1). Before stating some existence results of weak solutions, we recall some properties of the solutions of the classical Eikonal equation (Section 6). The last three sections are devoted to uniqueness results. As shown by a counter-example (Section 8), weak solutions are not unique in general. Uniqueness holds when the velocity is positive. Our results in this direction are stated in Section 9 and a sketch of proof is given in Section 10.

2. Preliminaries on the Level set approach and nonlocal nonmonotone front propagation problems

Consider the following front propagation problem: we want to find a family $(\Omega_t)_{t\geq 0}$ of open subsets of $\mathbb{R}^N$ such that every point x of the boundary $\Gamma_t := \partial\Omega_t$ (called the "front") evolves with a prescribed normal velocity given by

$$\overrightarrow{V}_{\Omega_t}(x) = h(x,t,\overline{\Omega}_t)\overrightarrow{n}_{\Omega_t}(x), \tag{2.1}$$

where $\overrightarrow{n}_{\Omega_t}(x)$ is the outer unit normal of Γ_t at x (it means that Γ_t is "oriented" by its interior Ω_t) and h is a given evolution law.

The idea of Osher & Sethian [38] for the level set approach is to introduce an auxiliary function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ whose 0-level set represents the front Γ_t. We therefore define u such that, for all $t \geq 0$,

$$u(\cdot,t) = 0 \text{ on } \Gamma_t,\ \ u(\cdot,t) > 0 \text{ in } \Omega_t \ \text{ and } \ u(\cdot,t) < 0 \text{ otherwise.} \tag{2.2}$$

Straightforward computations give:

$$\overrightarrow{n}_{\Omega_t}(x) = -\frac{Du(x,t)}{|Du(x,t)|} \quad \text{and} \quad \overrightarrow{V}_{\Omega_t}(x) = \frac{\frac{\partial u}{\partial t}(x,t)}{|Du(x,t)|}\overrightarrow{n}_{\Omega_t}(x) \quad \text{for all } x \in \Gamma_t.$$

From (2.1), we obtain the level set PDE

$$\frac{\partial u}{\partial t}(x,t) = h(x,t,\{u(\cdot,t)\geq 0\})|Du(x,t)| \quad \text{for all } x \in \Gamma_t. \tag{2.3}$$

This PDE holds *a priori* on Γ_t. The main work of Chen, Giga & Goto [18] and Evans & Spruck [22], who were the first to develop rigorously the level set approach, was to prove that (2.3) can be set and solved on $\mathbb{R}^N \times (0,T]$. This PDE is complemented with an initial data u_0 which represents the initial front (*i.e.*, (2.2) holds at $t=0$ with u_0 and a given Ω_0). One recovers Γ_t by setting

$$\Gamma_t := \{u(\cdot,t) = 0\} \quad \text{for all } t \geq 0.$$

It is worth mentioning that, even for very simple velocities, the front may develop singularities in finite time and some changes of topology may happen. Similarly, one cannot hope to find smooth solutions of (2.3). We will use the notion of viscosity solutions which are well adapted to these nonlinear problems. We refer the reader to Crandall, Ishii & Lions [20] for viscosity solutions and the book of Giga [25] for an overview of the level set approach.

Let us introduce some evolution laws h we will be interested in. The first and the simplest one is $h = c(x,t)$ (no dependence with respect to Ω_t). In this case, (2.3) becomes the classical Eikonal equation

$$\frac{\partial u}{\partial t}(x,t) = c(x,t)|Du(x,t)| \quad \text{in } \mathbb{R}^N \times [0,T] \tag{2.4}$$

(see Barles [6] and Bardi & Capuzzo Dolcetta [5] for instance). We recall some properties about this equation in Section 6 and we need to develop fine estimates for its solutions to prove uniqueness results for the more complicated velocities which follows.

We are mainly interested in nonlocal velocities which can be written

$$h(x,t,\overline{\Omega}_t) = c[\mathbb{1}_{\overline{\Omega}_t}](x,t) = \alpha(k \star \mathbb{1}_{\overline{\Omega}_t}(x,t)) + c_1(x,t). \tag{2.5}$$

They lead to the level set PDE (1.1). Notice that this PDE is nonlocal since the velocity does not depend only on local properties of Γ_t at x but on the whole set Ω_t. This brings some difficulties to study (1.1). This nonlocal dependence is enlighted by the use of the notation $c[\cdot]$.

The first typical case that we consider is the dislocation dynamics where

$$c[\mathbb{1}_{\overline{\Omega}_t}](x,t) = c_0 \star \mathbb{1}_{\overline{\Omega}_t}(x) + c_1(x,t), \tag{2.6}$$

with a space convolution: $c_0 \in L^1(\mathbb{R}^N)$ and

$$c_0 \star \mathbb{1}_{\overline{\Omega}_t}(x) = \int_{\mathbb{R}^N} c_0(x-y)\mathbb{1}_{\overline{\Omega}_t}(y)dy. \tag{2.7}$$

The second case is a velocity which governs the asymptotics of a FitzHugh-Nagumo system, namely

$$c[\mathbb{1}_{\overline{\Omega}_t}](x,t) = \alpha(v(x,t)),$$

where α is a real valued Lipschitz continuous function and v is the solution of

$$\frac{\partial v}{\partial t} - \Delta v = \mathbb{1}_{\overline{\Omega}_t} \quad \text{in } \mathbb{R}^N \times (0,T). \tag{2.8}$$

Using the representation formula for the heat equation (with a zero initial data), we have

$$v(x,t) = G * \mathbb{1}_{\overline{\Omega}_t}(x,t), \tag{2.9}$$

where "$*$" is the usual space-time convolution and G is the classical Green kernel. Therefore,

$$c[\mathbb{1}_{\{u\geq 0\}}](x,t) = \alpha(G * \mathbb{1}_{\{u\geq 0\}}(x,t)) \tag{2.10}$$

is also in the form (2.5).

Before giving some details about dislocations and FitzHugh-Nagumo systems, let us discuss the monotonicity properties of the evolutions under consideration.

Front propagation problems (local and nonlocal ones) can be classify into two categories: the monotone evolutions and the nonmonotone ones.

We say that a front propagation problem is monotone if the inclusion principle is satisfied. Otherwise it is called a nonmonotone problem. Inclusion principle can be described as follows. Start with initial sets Ω_0^1 and Ω_0^2 and let them evolves with the same velocity. They satisfy the inclusion principle if

$$\Omega_0^1 \subset \Omega_0^2 \quad \Longrightarrow \quad \Omega_t^1 \subset \Omega_t^2 \text{ for all } t \geq 0. \tag{2.11}$$

At least formally, the inclusion principle holds when

$$\Omega \subset \Omega' \subset \mathbb{R}^N \text{ and } x \in \partial\Omega \cap \partial\Omega' \quad \Longrightarrow \quad \mathcal{V}_\Omega(x) \leq \mathcal{V}_{\Omega'}(x). \tag{2.12}$$

For instance, this latter property is true for the mean curvature motion and for $h = c(x,t)$. Using the level set approach, where u^i is the solution of (2.3) corresponding to Ω^i for $i = 1, 2$, the inclusion principle implies

$$\{u_0^1 \geq 0\} \subset \{u_0^2 \geq 0\} \quad \Longrightarrow \quad \{u^1(\cdot,t) \geq 0\} \subset \{u^2(\cdot,t) \geq 0\} \text{ for all } t \geq 0.$$

Since the level set PDE (2.3) holds for all level sets (and not only the 0-level set), we get $u^1 \leq u^2$ (if $u_0^1 \leq u_0^2$). It means that one expects a comparison principle for (2.3) in the monotone case. It allows to apply Perron's method (see Ishii [31]) to build solutions for all times to (2.1).

On the contrary, for nonmonotone evolutions, (2.11)-(2.12) are violated and one cannot expect to have a comparison principle for (2.3). It is a serious obstacle to build solutions and prove uniqueness results. It happens that our typical cases (2.6) and (2.10) are nonmonotone front propagation problems. Indeed, in the case of dislocation dynamics, a physical assumption is

$$\int_{\mathbb{R}^N} c_0 = 0. \tag{2.13}$$

In consequence, (2.12) cannot be satisfied. In the FitzHugh-Nagumo model, α is merely Lispchitz continuous and this is not sufficient to ensure (2.12).

3. Dislocation dynamics

Dislocations are lines of defects which propagate in crystals. It is the main microscopic explanation of their macroscopic properties (see the books of Nabarro [36] and Hirth & Lothe [28] for the physics of dislocations and Lardner [32] for a mathematical exposition of the model). In our work, we consider a special mathematical model due to Rodney, Le Bouar & Finel [39].

Dislocation lines move preferentially in a crystallographic plane. The dynamics is given by a normal velocity proportional to the Peach-Koehler force acting on this line. This Peach-Koehler force may have two possible contributions: the first one is the self-force created by the elastic field generated by the dislocation line itself (i.e., this self-force is a nonlocal function of the shape of the dislocation line); the second one is the force created by everything exterior to the dislocation line, like the exterior stress applied on the material, or the force created by other defects. It follows that the velocity is given by (2.6) and it leads to (1.1) (*a priori* in $\mathbb{R}^2 \times [0,T]$ but we can consider any $N \geq 2$).

A mathematical study of this model was started by Monneau and his collaborators (see [3, 1, 2, 15] and the references therein). Here we focus on long-time existence and uniqueness results for (1.1). Recall that the motion is nonmonotone because of (2.13). The pioneer work in this direction is due to Alvarez, Hoch, Le Bouar & Monneau [3] where existence and uniqueness were proved for short time. The first uniqueness result was obtained by Alvarez, Cardaliaguet et Monneau [1] under the assumption that the velocity is regular enough ($C^{1,1}$) and nonnegative (*i.e.*, the front is expanding) when starting with initial sets Ω_0 having an interior ball property. In [11], we provide a new simpler proof of this fact. The techniques we introduced (lower bound gradient estimates, semiconvexity, L^1 estimates for the level sets of the solution, etc.) were re-used to obtain the results of [8, 10]. Let us finally mention the work of Cardaliaguet & Marchi [16] for dislocations with Neumann boundary conditions.

Several sets of assumptions on c_0, c_1 were used in the different works under consideration. We start with the basic ones.

(dislo-1) $c_0, c_1 \in C(\mathbb{R}^N \times [0,T])$ and there exist $\bar{c}, \bar{C} > 0$ such that, for all $x, y \in \mathbb{R}^N$, $t \in [0,T]$,

$$
\begin{aligned}
&|c_0(x,t)| + |c_1(x,t)| \leq \bar{c},\\
&|c_0(x,t) - c_0(y,t)| + |c_1(x,t) - c_1(y,t)| \leq \bar{C}|x-y|.
\end{aligned}
$$

Moreover, $c_0 \in C([0,T], L^1(\mathbb{R}^N))$.

Notice that this assumption ensures that the velocity is bounded:

$$
\begin{aligned}
c[\mathbb{1}_{\{u(\cdot,t)\geq 0\}}](x,t) &= \int_{\mathbb{R}^N} c_0(x-y)\mathbb{1}_{\{u(\cdot,t)\geq 0\}}(y)dy + c_1(x,t)\\
&\leq \sup_{0\leq t\leq T} |c_0(\cdot,t)|_{L^1(\mathbb{R}^N)} + \bar{c}.
\end{aligned}
$$

4. A FitzHugh-Nagumo type system

Consider

$$
(4.1)\quad \begin{cases}
u_t = \alpha(v)|Du| & \text{in } \mathbb{R}^N \times (0,T),\\
v_t - \Delta v = g^+(v)\mathbb{1}_{\{u\geq 0\}} + g^-(v)(1-\mathbb{1}_{\{u\geq 0\}}) & \text{in } \mathbb{R}^N \times (0,T),\\
u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 & \text{in } \mathbb{R}^N.
\end{cases}
$$

This system yields a front $\Gamma_t = \{u(\cdot,t) = 0\}$ which evolves with normal velocity $\alpha(v)$, the function v being itself the solution of a reaction-diffusion equation whose coefficients change according to the regions determined by Γ_t.

This system appears when taking the asymptotics, as $\varepsilon \to 0$, to the FitzHugh-Nagumo system

$$\tag{4.2} \begin{cases} u_t^\varepsilon - \varepsilon \Delta u^\varepsilon = \dfrac{1}{\varepsilon} f(u^\varepsilon, v^\varepsilon) & \text{in } \mathbb{R}^N \times (0,T), \\ v_t^\varepsilon - \Delta v^\varepsilon = g(u^\varepsilon, v^\varepsilon) & \text{in } \mathbb{R}^N \times (0,T), \end{cases}$$

where

$$\begin{cases} f(u,v) = u(1-u)(u-a) - v & (0 < a < 1), \\ g(u,v) = u - \gamma v & (\gamma > 0). \end{cases}$$

The functions α, g^+ and g^- in (4.1) are Lipschitz continuous and depend on f, g. Moreover g^- and g^+ are bounded and satisfy $g^- \le g^+$ in $\mathbb{R}$. Initial data u_0 and v_0 are Lipschitz continuous and v_0 is bounded and C^1.

These equations are related to wave propagation phenomena in excitable media. There exist a lot of works on this subject in biology, chemistry, physics and mathematics, see for instance [24, 37, 23, 41, 27, 17].

The issues we are interested in are the same as for dislocations. We want to define long-time solutions and prove some uniqueness properties. Giga, Goto & Ishii [26] obtained some weak solutions of (4.1). Wheras Soravia & Souganidis [40] established rigorously the convergence of (4.2) towards the limit problem (4.1) and proved the properties of α, g^+ and g^-. In particular, they found some conditions under which $\alpha > 0$. Until [10, Theorem 4.1], uniqueness was an open problem. We proved uniqueness for (4.1) when $\alpha > 0$.

To simplify, here we will choose $g^+ \equiv 1$, $g^- \equiv 0$ and $v_0 = 0$ (see [10] for the general case). To sum up, we consider (1.1) with a velocity given by (2.10), where v is the solution of (2.8) and thus may be written as (2.9). The following properties of v are straightforward.

Lemma 4.1. [10, Lemma 4.2] *For all $\chi \in L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$, the solution v of*

$$\tag{4.3} \frac{\partial v}{\partial t} - \Delta v = \chi \quad \text{in } \mathbb{R}^N \times (0,T), \quad v(x,0) = 0,$$

is continuous, $v(\cdot, t)$ is $C^{1,\beta}$ ($\beta < 1$) and, for all $x \in \mathbb{R}^N$, $0 \le s \le t \le T$,

$$|v(x,t)| \le t, \; |Dv(x,t)| \le \gamma_N \sqrt{t} \text{ and } |v(x,t) - v(x,s)| \le \gamma_N \sqrt{s}\sqrt{t-s} + t - s,$$

where γ_N is a constant which depends only on the dimension.

In the sequel, we will assume

(FN-1) $\alpha : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

From Lemma 4.1 and **(FN-1)**, we obtain some properties of the velocity $c[\chi] = \alpha(v)$. In particular, it is bounded (because v is bounded in $[0,T]$ independently of χ).

The main features of the FitzHugh-Nagumo problem are the following. On the one hand, the motion is nonmonotone since there is no monotonicity assumption on α. On the other hand, even if α is smooth, the regularity of the velocity is limited by the regularity of v which is, at the best, $C^{1,\beta}$ for all $\beta < 1$ (it comes from the regularity properties for the heat equation with L^∞ coefficients). This lack of regularity is a major difficulty and it prevents us to use the techniques of [8] which require a $C^{1,1}$ velocity (see Section 9).

5. Definition of weak solutions

In [8] and [9], we introduce a new notion of weak solution.

Definition 5.1. [8, 9] *A continuous function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is a weak solution of* (1.1) *if there exists $\chi \in L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$ such that*

(1) *u is a L^1 viscosity solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = c[\chi](x,t)|Du(x,t)| & \text{in } \mathbb{R}^N \times [0,T], \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{5.1}$$

(2) *For almost all $t \in [0,T]$,*

$$\mathbb{1}_{\{u(\cdot,t)>0\}} \leq \chi(\cdot,t) \leq \mathbb{1}_{\{u(\cdot,t)\geq 0\}} \quad \text{almost everywhere in } \mathbb{R}^N.$$

Moreover, we say that the weak solution u of (1.1) *is classical if, for almost all $t \in [0,T]$,*

$$\mathbb{1}_{\{u(\cdot,t)>0\}} = \mathbb{1}_{\{u(\cdot,t)\geq 0\}} \quad \text{almost everywhere in } \mathbb{R}^N. \tag{5.2}$$

The main difficulty to define solutions of geometrical equations like (1.1) is the fattening phenomenon which may appear (See Giga [25] and the references therein). In this case, the set $\{u(\cdot,t) = 0\}$ has positive Lebesgue measure and $t \mapsto c[\mathbb{1}_{\{u(\cdot,t)\geq 0\}}]$ is discontinuous from $[0,T]$ into $L^1(\mathbb{R}^N)$. When there is no fattening, χ is uniquely determined by

$$\chi(\cdot,t) = \mathbb{1}_{\{u(\cdot,t)>0\}} = \mathbb{1}_{\{u(\cdot,t)\geq 0\}}.$$

This definition makes interest for equations which are well-posed when the non-local term is frozen. More precisely, the point is to be able to solve (5.1) in the sense of L^1 viscosity solutions for a fixed χ. Notice that L^1 viscosity solutions appear naturally since, in the dislocation case for instance, the convolution regularizes the velocity in space but not in time, namely $(x,t) \mapsto c[\chi](x,t)$ is merely measurable. The generalization of the notion of viscosity solutions for equations with measurable in time coefficients is due to Ishii [30]. For further references see [8, Appendix A] where the results we need are collected.

This notion of solutions is very weak. In general, there is no uniqueness (see Section 8) but it provides general existence results. When the velocity is positive, we prove that the solutions are in fact classical and we obtain some uniqueness results (see Section 9).

6. Preliminaries on the classical eikonal equation and lower bound gradient estimate

Consider (2.4) with an initial data u_0. Classical assumptions on the speed c are:

(eikonal) $c \in C(\mathbb{R}^N \times [0,T])$ and there exist $\overline{c}, \overline{C} > 0$ such that, for all $x, y \in \mathbb{R}^N, t \in [0,T]$,

$$0 \leq c(x,t) \leq \overline{c},$$
$$|c(x,t) - c(y,t)| \leq \overline{C}|x-y|.$$

Assume moreover that

(lower-bound) (Lower bound gradient estimate on the initial front) $u_0 : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and there exists $\eta_0 > 0$ such that

$$-|u_0| - |Du_0| + \eta_0 \leq 0 \quad \text{in } \mathbb{R}^N \text{ in the viscosity sense.} \tag{6.1}$$

Some comments about this latter hypothesis are given below. The first part of the following theorem is classical, see Crandall & Lions [21] and Ishii [29]. The second part comes from Ley [34] and still holds in the context of L^1 viscosity solutions.

Theorem 6.1. [34]

(i) *(Lipschitz regularity) Under the assumption* **(eikonal)**, (2.4) *has a unique viscosity solution u. If u_0 is Lipschitz continuous, then u is Lipschitz continuous and, for all $x \in \mathbb{R}^N$, $t \in [0,T]$,*

$$|Du(x,t)| \leq e^{\overline{C}T}|Du_0|_\infty, \qquad |u_t(x,t)| \leq \overline{c}e^{\overline{C}T}|Du_0|_\infty .$$

(ii) *(Preservation of the lower bound gradient estimate) Assume that* **(eikonal)** *and* **(lower-bound)** *hold true. Then there exists $\eta = \eta(T, \overline{C}, \overline{c}, \eta_0) > 0$ such that*

$$-|u(x,t)| - |Du(x,t)| + \eta \leq 0 \text{ in } \mathbb{R}^N \times [0,T] \text{ in the viscosity sense.} \tag{6.2}$$

In the context of the level set approach, (6.1) and (6.2) imply a lower bound gradient estimate on the front Γ_t. Indeed, suppose that u_0, u are smooth. If x is on the front, then $u(x,t) = 0$ and (6.2) implies $|Du(x,t)| \geq \eta > 0$. It follows from the implicit function theorem that the front is a smooth hypersurface. But u_0, u are not smooth in general and (6.1), (6.2) has to be understood in a generalized sense (see [34] for details). Nevertheless the lower bound gradient estimate holds almost everywhere in a neighborhood of the front. This is enough to prove some L^1 type estimates for level sets like $\{-\delta \leq u(\cdot,t) \leq \delta\}$ (with $\delta \approx 0$) which are crucial.

At this step, let us make a very important remark. Since the velocity is bounded (*cf.* Sections 3 et 4), let us say by a constant $\overline{V}$, we have a finite speed if propagation. With the notations of (2.2), if

$$\Gamma_0 \cup \Omega_0 = \{u_0 \geq 0\} \subset \overline{B}(0, R_0), \tag{6.3}$$

then

$$\text{(6.4)} \qquad \Gamma_t \cup \Omega_t = \{u(\cdot,t) \geq 0\} \ \subset \ \overline{B}(0, R_0 + \overline{V}T) \ \text{ for all } t \geq 0.$$

It means that, starting with compact fronts, everything takes place in a big fixed ball $\overline{B}(0, R_0 + \overline{V}T)$. Thanks to the expression (1.2) for the velocity together with the assumptions **(dislo-1)** and **(FN-1)**, we deduce that the velocity $c[\chi]$ satisfies **(eikonal)** with constants which are independent of $\chi \in L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$ as soon as χ is compactly supported in $\overline{B}(0, R_0+\overline{V}T)$. It follows that the greater part of the results for the the classical eikonal equation applies to our model problems.

7. Existence of weak solutions and classical solutions

We have

Theorem 7.1. [8, 9] *Under the assumptions* **(dislo-1)** *(dislocations case) or* **(FN-1)** *(FitzHugh-Nagumo case), for all Lipschitz continuous u_0 such that* (6.3) *holds, Equation* (1.1) *admits at least a weak solution in* $\mathbb{R}^N \times [0,T]$.

As said above, one does not have any comparison principle which allows to build visocsity solutions by Perron's method. We need to use other strategies. In the case of dislocations, existence is proved in [8, Theorem 1.2] by approximation: the velocity $c[\mathbb{1}_{\{u\geq 0\}}]$ is regularized by replacing the indicator function by a continuous function. We can apply Schauder's theorem to the perturbated equation and extract a convergent subsequence by Ascoli's theorem. To conclude, it remains to prove that the limit is a solution. This is not obvious because we are not in the classical framework of viscosity solutions. At this point, we need to use a new stability result for measurable in time equations which was proved by Barles [7]. For the FitzHugh-Nagumo system, existence was proved in [26] for a different notion of weak solutions. In [9], we introduce a general framework yielding weak solutions (in the sense of Definition 5.1) for both model problems (and even more general cases). Our proof is based on Kakutani's fixed point theorem (see [4]) which was already the main ingredient of the proof in [26]. We end by recalling that, since the velocity $c[\chi]$ satisfies **(eikonal)** with constants which are independent of χ, we can apply Theorem 6.1 (i) and (6.4) in our proof.

Let us state some additional assumptions in order to obtain classical solutions.

(dislo-2) For all $x \in \mathbb{R}^N$, $t \in [0,T]$, $0 \leq -|c_0(\cdot,t)|_{L^1(\mathbb{R}^N)} + c_1(x,t)$.

(FN-2) $0 \leq \alpha$.

A consequence of these new assumptions is that $c[\chi](x,t)$ is nonnegative for all $\chi \in L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$, $x \in \mathbb{R}^N$ and $t \in [0,T]$.

Theorem 7.2. [8, 9] *Under the assumptions* **(dislo-1-2)** *(dislocations case) or* **(FN-1-2)** *(FitzHugh-Nagumo case), for all Lipschitz continuous u_0 such* *that* (6.3) *and* **(lower-bound)** *hold, the weak solutions of* (1.1) *are classical ones.*

The proof is straightforward using the preservation of the lower bound gradient estimate of Theorem 6.1 since this latter property implies that the front has zero Lebesgue measure and therefore (5.2) holds.

8. A COUNTER-EXAMPLE TO UNIQUENESS

The following example comes from [8, Example 3.1] and is inspired by [12]. It takes profit of the fact that the velocity vanishes.

We set $N = 1$ and consider the following PDE of type (1.1),

$$\begin{cases} \dfrac{\partial U}{\partial t} = (1 \star \mathbb{1}_{\{U(\cdot,t)\geq 0\}}(x) + c_1(t))|DU| & \text{in } \mathbb{R} \times (0,2], \\ U(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}, \end{cases} \tag{8.1}$$

where we choose $c_1(x,t) := c_1(t) = 2(t-1)(2-t)$ and $u_0(x) = 1 - |x|$. Notice that $1 \star \mathbb{1}_A = \mathcal{L}^1(A)$ for all measurable subset $A \subset \mathbb{R}$.

We start by studying auxiliary problems in the time intervals $[0,1]$ and $[1,2]$ which will be useful to build a family of weak solutions for (8.1) in $[0,2]$.

1. Construction of a solution for $0 \leq t \leq 1$. The function $x_1(t) = (t-1)^2$ is solution of $\dot{x}_1(t) = c_1(t) + 2x_1(t)$ on $(0,1)$ with $x(0) = 1$ (note that $\dot{x}_1 \leq 0$ in $[0,1]$). Consider

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dot{x}_1(t) \left| \dfrac{\partial u}{\partial x} \right| & \text{in } \mathbb{R} \times (0,1], \\ u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}. \end{cases} \tag{8.2}$$

From Theorem 6.1, there exists a unique continuous viscosity solution u of (8.2). By Lax-Oleinik formula, $u(x,t) = u_0(|x| - x_1(t) + 1)$. Hence, for $0 \leq t \leq 1$, we have

$$\{u(\cdot,t) > 0\} = (-x_1(t), x_1(t)) \quad \text{et} \quad \{u(\cdot,t) \geq 0\} = [-x_1(t), x_1(t)]. \tag{8.3}$$

In Step 3, we will establish that u is a weak solution of (8.1) on $[0,1]$.

2. Construction of solutions for $1 \leq t \leq 2$. For all measurable functions $0 \leq \gamma(t) \leq 1$, let y_γ be the unique solution of $\dot{y}_\gamma(t) = c_1(t) + 2\gamma(t)y_\gamma(t)$ on $(1,2)$ with $y_\gamma(1) = 0$. By comparison, one has $0 \leq y_0(t) \leq y_\gamma(t) \leq y_1(t)$ for $1 \leq t \leq 2$, where y_0, y_1 are the solutions of the previous equation with $\gamma(t) \equiv 0$ and 1. Note that $\dot{y}_\gamma \geq 0$ in $[1,2]$. Then, consider

$$\begin{cases} \dfrac{\partial u_\gamma}{\partial t} = \dot{y}_\gamma(t) \left| \dfrac{\partial u_\gamma}{\partial x} \right| & \text{in } \mathbb{R} \times (1,2], \\ u_\gamma(\cdot, 1) = u(\cdot, 1) \quad \text{in } \mathbb{R}, \end{cases}$$

where u is the solution of (8.2). Again, this problem has a unique continuous viscosity solution u_γ which is zero if $|x| \leq y_\gamma(t)$ and $u_\gamma(x,t) = u(|x| - y_\gamma(t), 1)$

otherwise (note that, since $u(\cdot,1) \le 0$, by the maximum principle, $u_\gamma \le 0$ in $\mathbb{R}\times[1,2]$). It follows

$$(8.4)\ \{u_\gamma(\cdot,t)>0\} = \emptyset \ \text{ and } \ \{u_\gamma(\cdot,t)\ge 0\} = \{u_\gamma(\cdot,t)=0\} = [-y_\gamma(t), y_\gamma(t)].$$

3. There are several weak solutions to (8.1). For $0 \le \gamma(t) \le 1$, set

$$\begin{array}{lll} c_\gamma(t) = c_1(t) + 2x_1(t), & U_\gamma(x,t) = u(x,t) & \text{if } (x,t) \in \mathbb{R}\times[0,1], \\ c_\gamma(t) = c_1(t) + 2\gamma(t)y_\gamma(t), & U_\gamma(x,t) = u_\gamma(x,t) & \text{if } (x,t) \in \mathbb{R}\times[1,2]. \end{array}$$

Then, from Steps 1 and 2, U_γ is the continuous viscosity solution of

$$\begin{cases} \dfrac{\partial U_\gamma}{\partial t} = c_\gamma(t)\left|\dfrac{\partial U_\gamma}{\partial x}\right| & \text{in } \mathbb{R}\times(0,2], \\ U_\gamma(\cdot,0) = u_0 \quad \text{in } \mathbb{R}. \end{cases}$$

Taking $\chi_\gamma(\cdot,t) = \gamma(t)\mathbb{1}_{[-y_\gamma(t),y_\gamma(t)]}$ for $1 \le t \le 2$, from (8.3) et (8.4), we obtain

$$\mathbb{1}_{\{U_\gamma(\cdot,t)>0\}} \le \chi_\gamma(\cdot,t) \le \mathbb{1}_{\{U_\gamma(\cdot,t)\ge 0\}}$$

(see Figure 1). This implies that all the functions U_γ, for measurable $0 \le \gamma(t) \le 1$, are weak solutions of (8.1).

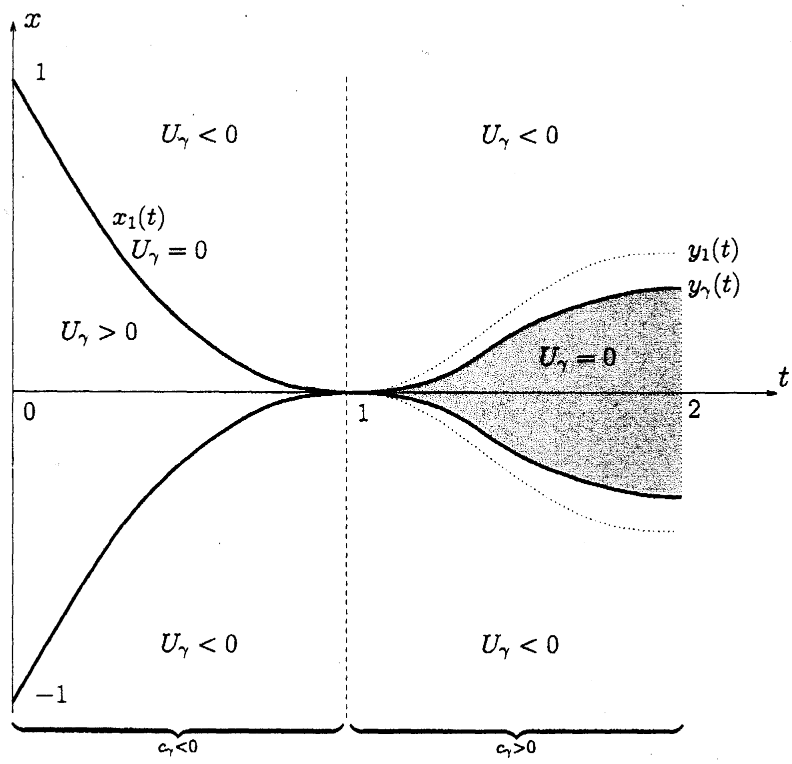

FIGURE 1

9. Uniqueness results

We obtained several uniqueness results under different assumptions in the case of dislocations. I focus on the most recent one here (established in [10], see Theorem 9.1). This result requires the weakest assumption on the regularity of the velocity which needs to be merely Lipschitz continuous. I describe simultaneously the case of dislocations dynamics and the FitzHugh-Nagumo system. A sketch of proof is given in Section 10.

When the velocity is only Lipschitz continuous, we assume that it is positive. In order to get this property, we need to reinforce **(dislo-2)** and **(FN-2)**:

(dislo-3) There exists $\underline{c} > 0$ such that, for all $x \in \mathbb{R}^N$, $t \in [0,T]$, $0 < \underline{c} \leq -|c_0(\cdot,t)|_{L^1(\mathbb{R}^N)} + c_1(x,t)$.

(FN-3) There exists $\underline{c} > 0$ such that $0 < \underline{c} \leq \alpha$.

We have

Theorem 9.1. [10, Theorems 3.1 and 4.1] *Assume* **(dislo-1-3)** *(dislocations case) or* **(FN-1-3)** *(FitzHugh-Nagumo case) and suppose that* u_0 *satisfies* **(lower-bound)**, (6.3) *and that* $\Gamma_0 := \{u_0 = 0\}$ *is* C^2. *Then there exists a unique (classical) viscosity solution for* (1.1).

Let us now explain what are the results we obtained previously for dislocations when the velocity is more regular, namely $C^{1,1}$ or semiconvex. We recall that $f : \mathbb{R}^N \to \mathbb{R}$ is semiconvex if, for all $x, y \in \mathbb{R}^N$,

$$f(x+y) + f(x-y) - 2f(x) \geq -L|y|^2. \tag{9.1}$$

We refer the reader to the book of Cannarsa & Sinestrari [14] for details about semiconcavity (a function f is semiconcave if $-f$ is semiconvex. If f is both semiconvex and semiconcave then it is $C^{1,1}$).

Theorem 9.2. *(Dislocations case) Suppose that* **(dislo-1)** *holds, that* u_0 *satisfies* (6.3) *and* **(lower-bound)** *and that*

$c_0(\cdot,t)$ *and* $c_1(\cdot,t)$ *are semiconvex uniformly with respect to* $t \in [0,T]$.

(1) [1, Theorem 4.3] *and* [11, Theorem 4.2] *If* **(dislo-2)** *holds and* u_0 *is semiconvex, then there exists unique (classical) viscosity solution for* (1.1).
(2) [8, Theorem 1.3] *If* **(dislo-3)** *holds, then there exists unique (classical) viscosity solution for* (1.1).

10. Sketch of the proof of the uniqueness Theorem 9.1

I give a sketch of the proof of Theorem 9.1 and I will point out how getting the results of Theorem 9.2 for more regular velocities. For the whole proof, see [10, Proofs of Theorems 3.1 and 4.1].

Under the assumptions of Theorem 9.1, consider two classical solutions u^1 and u^2 of (1.1) with the same initial data u_0 (the existence is given by Theorems 7.1 and 7.2).

1. Preliminary estimates. As explained at the end of Section 6, the subsets $\{u^i(\cdot,t) \geq 0\}$, $i = 1, 2$ are contained in a ball $\overline{B}(0, R_0+\overline{V}T)$ and $c[\mathbb{1}_{\{u^i(\cdot,t)\geq 0\}}]$ satisfies **(eikonal)** with fixed constants. Therefore, the conclusions of Theorem 6.1 hold true for the u^i's. In particular, for $\overline{\delta} > 0$ small enough, we have the lower bound gradient estimate

$$(10.1) \quad |Du^i| \geq \frac{\eta}{2} \quad \text{for almost all } (x,t) \text{ such that } x \in \{-\overline{\delta} \leq u^i(\cdot,t) \leq \overline{\delta}\}.$$

For $0 \leq \tau \leq T$, define

$$\delta_\tau = \sup_{\mathbb{R}^N \times [0,\tau]} |u^1 - u^2|.$$

Since $\delta_0 = 0$ and using the continuity of u^i, we can choose $\tau > 0$ small enough in order to have $\delta_\tau < \overline{\delta}$.

Since the u^i's satisfy (1.1), by a classical comparison result for eikonal equations with different speeds (see [11, Lemma 2.2]), we have

$$(10.2) \quad \delta_\tau \leq |Du_0|_\infty e^{\overline{C}\tau} \int_0^\tau |c[\mathbb{1}_{\{u^1(\cdot,t)\geq 0\}}] - c[\mathbb{1}_{\{u^2(\cdot,t)\geq 0\}}](\cdot,t)|_\infty dt.$$

Now, the purpose is to bound the previous integral by a quantity like

$$(10.3) \qquad o_\tau(1)\delta_\tau.$$

It follows $\delta_\tau = 0$ for small τ. By a step-by-step argument, we can conclude $\delta_T = 0$. At this step, we have to distinguish the dislocations case and the FitzHugh-Nagumo one. The difference lies in the convolution kernel which appears in the velocity. For dislocations, this kernel is bounded whereas it is not bounded in the FitzHugh-Nagumo case (the heat kernel is not bounded with respect to time). In this latter case, we need fine perimeter estimates in order to get (10.3).

2. Dislocations case. We continue the computation (10.2) by using (2.6).

$$\begin{aligned} \delta_\tau &\leq |Du_0|_\infty e^{\overline{C}\tau} \int_0^\tau |c_0(\cdot,t) \star (\mathbb{1}_{\{u^1(\cdot,t)\geq 0\}} - \mathbb{1}_{\{u^2(\cdot,t)\geq 0\}})|_\infty dt \\ &\leq \overline{c}|Du_0|_\infty e^{\overline{C}\tau} \int_0^\tau \int_{\mathbb{R}^N} \left(\mathbb{1}_{\{-\delta_\tau \leq u^1 \leq 0\}} + \mathbb{1}_{\{-\delta_\tau \leq u^2 \leq 0\}}\right) dxdt, \end{aligned}$$

since c_0 is bounded (see **(dislo-1)**) and

$$(10.4) \quad |\mathbb{1}_{\{u^1\geq 0\}} - \mathbb{1}_{\{u^2\geq 0\}}| \leq \mathbb{1}_{\{-\delta_\tau \leq u^1 \leq 0\}} + \mathbb{1}_{\{-\delta_\tau \leq u^2 \leq 0\}} \quad \text{in } \mathbb{R}^N \times [0,\tau].$$

It remains to deal with

$$\int_0^\tau \int_{\mathbb{R}^N} \mathbb{1}_{\{-\delta_\tau \leq u^i \leq 0\}} dxdt.$$

Depending on the type of assumptions, there are several ways to proceed to obtain one of the results of Theorems 9.1 or 9.2. Let us start by a heuristic computation which enlights the interest of the lower bound gradient estimate and the perimeter estimates in the proof of Theorem 9.2. By the coarea formula, using (10.1), we have

$$\begin{aligned}\int_{\mathbb{R}^N} \mathbb{1}_{\{-\delta_\tau\leq u^i(\cdot,t)\leq 0\}}dx &= \int_{-\delta_\tau}^{0}\int_{\{u^i(\cdot,t)=s\}} |Du|^{-1}d\mathcal{H}^{N-1}ds\\ &\leq \frac{2\delta_\tau}{\eta}\sup_{-\delta_\tau\leq s\leq 0}\mathrm{Per}(\{u(\cdot,t)=s\}).\end{aligned}$$

It means that, if we may obtain a bound for the perimeter of $\{u(\cdot,t)=s\}$ with $s\approx 0$ for all $t\in[0,\tau]$, then we are done. This bound was obtained in [1] as a consequence of the propagation of the interior ball property for the front when the velocity is $C^{1,1}$. In [11], we follow an equivalent strategy (semiconvexity associated with a lower bound gradient estimate is equivalent to the interior ball property, see [11, Lemma A.1]). This latter strategy has the advantage to require only volume estimates of $\mathcal{L}^N(\{-\delta_\tau\leq u^i\leq 0\})$ ([11, Section 3]) and not perimeter estimates which are more delicate to prove. Moreover, we could improve these L^1 estimates for less regular velocities. In [8], we prove that, if **(dislo-3)** holds and the velocity is semiconvex, then an interior ball property is created during the evolution and the desired perimeter estimates follow.

Let us come back to the proof of Theorem 9.1. Let $\varphi:\mathbb{R}\to\mathbb{R}^+$ be a continuous function such that $\delta_\tau\varphi'=\mathbb{1}_{[-\delta_\tau,0]}$ (it suffices to take φ which is zero on $(-\infty,-\delta_\tau]$, 1 on $\mathbb{R}^+$ and linear with a slope $1/\delta_\tau$ on $[-\delta_\tau,0]$). It follows (see [10, Proposition 5.5]), using the lower bound gradient and the equation, that

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}^N}\mathbb{1}_{\{-\delta_\tau\leq u^i\leq 0\}}dxdt\\ =\;&\int_0^\tau\int_{\mathbb{R}^N}\delta_\tau\varphi'(u^i(x,t))dxdt\\ \leq\;&\int_0^\tau\int_{\mathbb{R}^N}\delta_\tau\varphi'(u^i(x,t))\,\frac{c[\mathbb{1}_{\{u^i\geq 0\}}](x,t)}{\underline{c}}\,\frac{|Du^i|}{\eta}\,dxdt\\ =\;&\frac{\delta_\tau}{\underline{c}\eta}\int_0^\tau\int_{\mathbb{R}^N}\varphi'(u^i(x,t))\,\frac{\partial u^i}{\partial t}\,dxdt\\ =\;&\frac{\delta_\tau}{\underline{c}\eta}\int_0^\tau\int_{\mathbb{R}^N}\frac{\partial}{\partial t}\varphi(u^i(x,t))dxdt\\ \leq\;&\frac{\delta_\tau}{\underline{c}\eta}\left(\mathcal{L}^N(\{u^i(\cdot,\tau)\geq-\delta_\tau\})-\mathcal{L}^N(\{u_0\geq 0\})\right).\end{aligned}$$

The dominated convergence theorem implies that $o_\tau(1)\delta_\tau$ is an upper-bound. It completes the proof in the case of dislocations.

3. FitzHugh-Nagumo system. In this case, we estimate (10.2) as follows.

$$\begin{aligned}|c[\mathbb{1}_{\{u^1(\cdot,t)\geq 0\}}]-c[\mathbb{1}_{\{u^2(\cdot,t)\geq 0\}}](\cdot,t)|_\infty &= |(\alpha(v_1)-\alpha(v_2))(\cdot,t)|_\infty\\ &\leq \overline{C}|(v_1-v_2)(\cdot,t)|_\infty,\end{aligned}$$

where v_i is the solution of (4.3) with $\chi = \mathbb{1}_{\{u^i \geq 0\}}$. We continue the computation (10.2) by taking profit of the formula for v_i given by Lemma 4.1 and (10.4):

$$(10.5)\, \delta_\tau \leq |Du_0|_\infty e^{\overline{C}\tau} \int_0^\tau \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s) \left(\mathbb{1}_{\{-\delta_\tau \leq u^1 \leq 0\}} + \mathbb{1}_{\{-\delta_\tau \leq u^2 \leq 0\}}\right) dydsdt.$$

At this step, we cannot conclude as in the dislocation case since G is not bounded. Moreover, the merely Lipschitz continuity of the velocity does not imply some interior ball properties. We overcome this difficulty by establishing some interior cone properties which are more involved.

At first we have

$$\{-\delta_\tau \leq u^i \leq 0\} \;\subset\; E_i(t) := \left(\{u^i(\cdot,t) \geq 0\} + \frac{2\delta_\tau B(0,1)}{\eta}\right) \setminus \{u^i(\cdot,t) \geq 0\}.$$

The above inclusion means that one can keep under control the size of $\{-\delta_\tau \leq u^i \leq 0\}$ by broaden a bit the 0-level set. Notice this is hopeless in general; the lower bound gradient estimate is crucial.

Next step is devoted to show that the subsets $\{u^i(\cdot,t) \geq 0\}$ satisfy a uniform interior cone property. Namely, for each $x \in \partial\{u^i(\cdot,t) \geq 0\}$, there exists a cone $C_x^{\rho,\theta}$ with a degree of opening θ and a height ρ whose vertex is x and such that $C_x^{\rho,\theta} \subset \{u^i(\cdot,t) \geq 0\}$ (see Figure 2). The proof of this

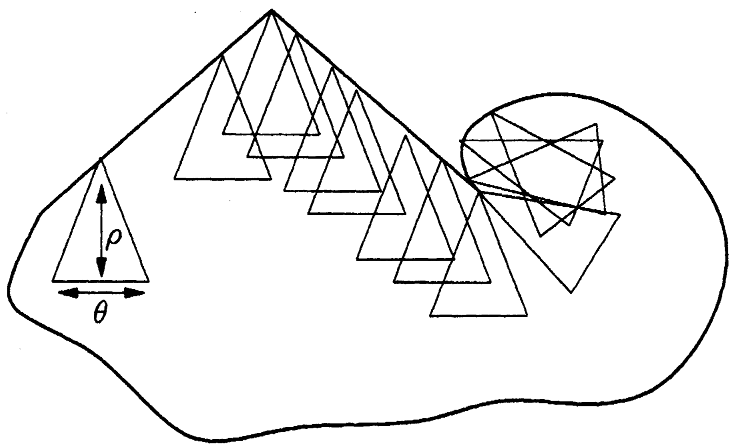

FIGURE 2

result is based on the positiveness of the velocity **(FN-3)** and the nonsmooth Pontryagine maximum principle (see Clarke [19]). Such tools were already used for proving the creation of the interior ball property in [8] (see also [13] and [1]).

Then, we prove that a bounded subset satisfying the uniform interior cone property has a finite perimeter.

Theorem 10.1. [10, Theorem 5.8] *Let K be a compact subset of $\mathbb{R}^N$ satisfying the uniform interior cone property with parameters θ and ρ. Then, there exists $\Lambda = \Lambda(N, \rho, \theta)$ such that, for all $R > 0$,*

$$\mathcal{H}^{N-1}(\partial K \cap \overline{B}(0,R)) \leq \Lambda\, \mathcal{L}^N(K \cap \bar{B}(0, R + \rho/4)). \tag{10.6}$$

The proof of this result is involved and used Besicovitch's covering theorem.

From the two previous results, we get [10, Lemma 4.4]:

$$\int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)\mathbb{1}_{E_i(t)} dyds \leq \tilde{\Lambda}\, \frac{2\delta_\tau}{\eta},$$

where $\tilde{\Lambda}$ depends on the given data and Λ (see (10.6)). Plugging this estimate in (10.5), we obtain an upper-bound like (10.3). The proof of the theorem is complete.

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