数理解析研究所講究録
第 1654 巻 2009 年 1-18

Golden optimal processes on three dynamics :deterministic, stochastic and non-deterministic

Seiichi IWAMOTO

Department of Economic Engineering
Graduate School of Economics, Kyushu University
Fukuoka 812-8581, Japan
tel&fax. +81(92)642-2488, email: iwamoto@en.kyushu-u.ac.jp

February 23, 2009

Abstract

This paper discusses a common criterion on three different dynamics. The criterion is discounted quadratic. The dynamics are deterministic, stochastic and non-deterministic. We consider two problems from a viewpoint of Golden optimality. The first problem is to find an optimal solution – value function and optimal policy – . The second problem is to discuss whether the optimal solution is Golden or not. Is the value function Golden? Is the optimal policy Golden? We give a complete solution to the first problem through two approaches – evaluation-optimization method and dynamic programming method –. The solution of the second depends on the discount rate β $(0 < \beta < \infty)$. We show that both – deterministic and non-deterministic – dynamics allow the Golden optimal solution for $\beta = 1$. Further all the three dynamics allow the Golden optimal policy for $\beta = \dfrac{1}{\sqrt{5}}$.

Keywords: golden, optimal, policy, deterministic, stochastic, non-deterministic.
JEL classification: C61

1 Introduction

The *Golden ratio* is the symbol of beauty and practical use. It has been utilized in architecture, art, design, biolology, sicence, engineering, and others [13]. Recently it has been incorporated into *optimization problems*. There a new – Golden (and) optimal – solution is obtained. Both static problems [5–7] and dynamic problems [8, 10, 11] are studied from the Golden optimality. The static optimization is two-variable. The dynamic one is infinite variable — discrete-horizon [10,11] and continuous-time [8] —. All of them are on *deterministic* system.

This paper minimizes a discounted quadratic criterion on three – (1) deterministic, (2) stochastic and (3) non-deterministic – dynamics. We consider two problems from a viewpoint of Golden optimality. The first problem is to find an optimal solution – (i) value function, (ii) optimal policy, (iii) minimum value – . The second problem is to discuss whether the optimal solution is Golden or not. Our approaches are evaluation-optimization method and dynamic programming method. We give an optimal solution to the first. The solution of the second depends on the discount rate β $(0 < \beta < \infty)$. We show that both – (1) deterministic and (3) non-deterministic – dynamics allow the Golden optimal solution for $\beta = 1$. Further all the three dynamics allow the Golden optimal policy for $\beta = \dfrac{1}{\sqrt{5}} \approx 0.4772$

2 Golden Paths

A real number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

is called *Golden number.* It is the larger of the two solutions to quadratic equation

$$x^2 - x - 1 = 0. \tag{1}$$

Sometimes (1) is called *Fibonacci quadratic equation.* The Fibonacci quadratic equation has two real solutions: ϕ and its *conjugate* $\overline{\phi} := 1 - \phi$. We note that

$$\phi + \overline{\phi} = 1, \qquad \phi \cdot \overline{\phi} = -1.$$

Further we have

$$\phi^{-1} = \phi - 1 \approx 0.618, \quad \phi^{-2} = 2 - \phi \approx 0.382$$
$$\phi^{-1} + \phi^{-2} = 1.$$

A point $\phi^{-2}x$ splits an interval $[0,\ x]$ into two intervals $[0,\ \phi^{-2}x]$ and $[\phi^{-2}x,\ x]$. A point $\phi^{-1}x$ splits the interval into $[0,\ \phi^{-1}x]$ and $[\phi^{-1}x,\ x]$. In either case, the length constitutes the Golden ratio $\phi^{-2} : \phi^{-1} = 1 : \phi$. Thus both divisions are the *Golden section.*

Definition 2.1 *A sequence $\{x_n\}_0^\infty$ is called Golden if and only if either*

$$x_{n+1} = \phi^{-1}x_n \quad n \geq 0 \qquad or \qquad x_{n+1} = \phi^{-2}x_n \quad n \geq 0.$$

Lemma 2.1 *A Golden sequence $\{x_n\}_0^\infty$ is either*

$$x_n = c\phi^{-n} \quad n \geq 0 \ \ (Fig.\ 2) \quad or \quad x_n = c\phi^{-2n} \ \ (Fig.\ 1),$$

where c is a real constant.

Definition 2.2 *A sequence of Markov random variables* $\{X_n\}_0^\infty$ *with* $X_0 = x_0$ *is called Golden if and only if either*

$$E[X_{n+1} \mid x_n] = \phi^{-1} x_n \quad n \geq 0 \quad \text{or} \quad E[X_{n+1} \mid x_n] = \phi^{-2} x_n \quad n \geq 0.$$

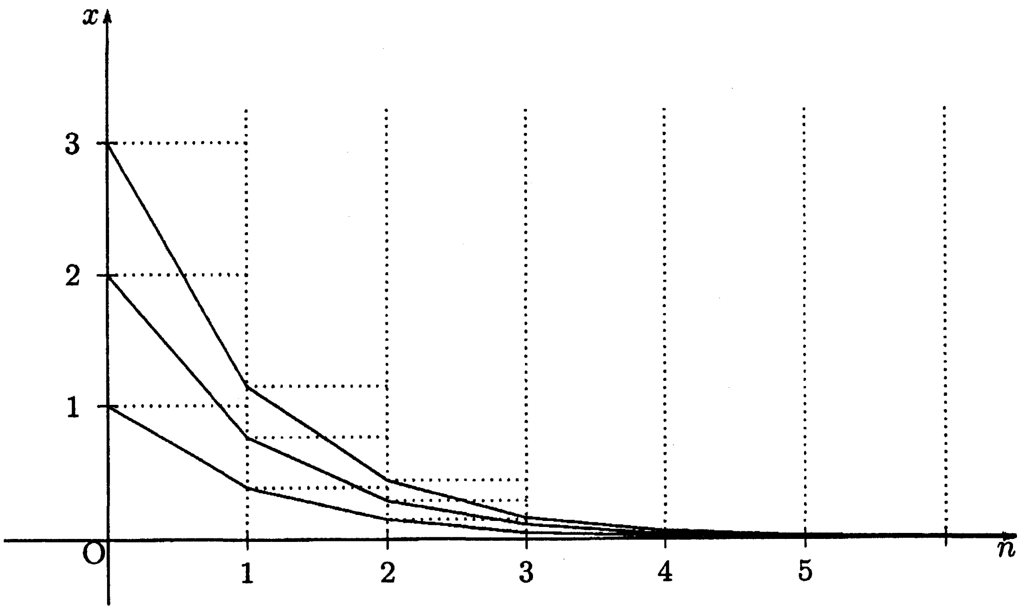

Fig. 1 Golden paths of rate ϕ^{-2} $\quad x_n = c\phi^{-2n} \quad c = 1, 2, 3$

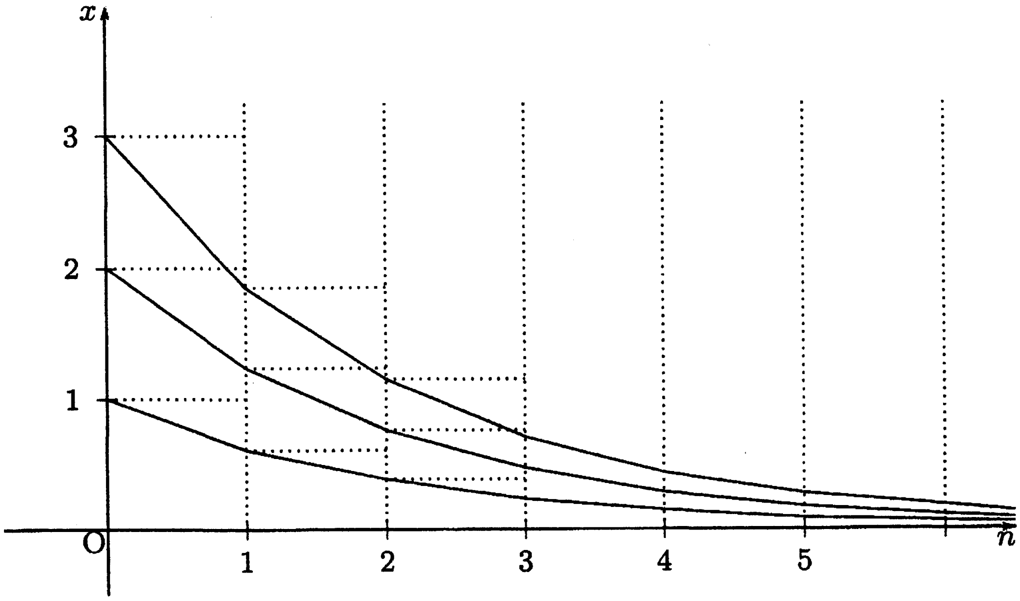

Fig. 2 Golden paths of rate ϕ^{-1} $\quad x_n = c\phi^{-n} \quad c = 1, 2, 3$

We remark that either Golden sequence is supermartingale. In either case, $E[X_{n+1} \mid x_n]$ generates a Golden section of interval $[0,\, x_n]$ for $x_n \geq 0$ and does a Golden section of interval $[x_n,\, 0]$ for $x_n \leq 0$.

Let $\{\epsilon_n\}_1^\infty$ be a sequence of independent and identical random variables with the standard normal distributuion. Then

$$E[\epsilon_n] = 0, \quad E[\epsilon_n^2] = 1.$$

For given x_0 and y_0 we define two sequences of Markov random variables $\{X_n\}$ and $\{Y_n\}$ by

$$\begin{aligned} X_{n+1} &= \phi^{-1} X_n - \epsilon_{n+1}, \quad X_0 = x_0 \\ Y_{n+1} &= \phi^{-2} Y_n - \epsilon_{n+1}, \quad Y_0 = y_0. \end{aligned}$$

Lemma 2.2 *Then $\{X_n\}$ and $\{Y_n\}$ are Golden.*

3 Three Dynamics

We consider three dynamic optimization problems with a common discounted quadratic criterion.

The first is a deterministic dynamics on which we minimizes a typical quadratic function. The problem is called linear-quadratic (LQ) [2, 3] :

$$\text{(D)} \qquad \begin{array}{lll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} \beta^n \left(x_n^2 + u_n^2\right) & \\ \text{subject to} & \text{(i)} \;\; x_{n+1} = x_n - u_n & \\ & \text{(ii)} \;\; u_n \in R^1 & n \geq 0 \\ & \text{(iii)} \;\; x_0 = c, & \end{array}$$

where $c \in R^1$. Here (i) denotes that next state x_{n+1} turns out to be $x_n - u_n$ with certainty from state x_n under decision u_n. This dynamics together with immediate cost is depicted as

$$R^1 \ni x_n \xrightarrow[\hookrightarrow \; x_n^2 + u_n^2]{\downarrow \; u_n \in R^1} \text{a unique } x_{n+1} := x_n - u_n \in R^1,$$

where $\hookrightarrow$ denotes that state x_n under decision u_n yields the stage-cost $x_n^2 + u_n^2$.

The second is a stochastic dynamics on which we minimizes the expected value of the same quadratic function as in deterministic one:

$$
\text{(S)} \quad
\begin{array}{ll}
\text{minimize} & E_{x_0}\left[\sum_{n=0}^{\infty} \beta^n (x_n^2 + u_n^2)\right] \\
\text{subject to} & \text{(i)}\ x_{n+1} = x_n - u_n - \epsilon_{n+1} \qquad n \geq 0 \\
& \text{(ii)}\ u_n \in R^1 \\
& \text{(iii)}\ x_0 = c,
\end{array}
$$

where $c \in R^1$. The problem (S) is called stochastic linear-quadratic (LQ). Here $\{\epsilon_n\}_1^\infty$ is a sequence of independently and identically distributed random variables with the standard normal distributuion. Thus (i) denotes that x_{n+1} appears on R^1 with transition probability $q(x_{n+1} \mid x_n, u_n) = \frac{1}{\sqrt{2\pi}} e^{-(x_{n+1}-x_n+u_n)^2/2}$ from x_n under u_n. This dynamics is depicted as

$$
R^1 \ni x_n \xrightarrow[\hookrightarrow\ x_n^2 + u_n^2]{\downarrow\ u_n \in R^1} x_{n+1} \quad \text{w.p.} \quad q(x_{n+1} \mid x_n, u_n) \quad \text{for any } x_{n+1} \in R^1.
$$

The third is on non-deterministic dynamics. There we minimize a total discounted weighted value of quadratic cost:

$$
\text{(N)} \quad
\begin{array}{ll}
\text{minimize} & \sum_{n=0}^{\infty} \beta^n W_{x_0}\left[x_n^2 + u_n^2\right] \\
\text{subject to} & \text{(i)}\ 0 < x_{n+1} < x_n - u_n \quad \text{with weight}\ \ 2/x_{n+1} \quad n \geq 0 \\
& \text{(ii)}\ u_n \in R^1 \\
& \text{(iii)}\ x_0 = c,
\end{array}
$$

where $c > 0$. We call the problem (N) is *nondeterministic* quadratic (Q). Here the infinite series is defined in Section 6. The successive constraint (i) denotes that x_{n+1} appears on the open interval $(0,\ x_n - u_n)$ with transition weight $2/x_{n+1}$ from x_n under u_n. This dynamics is depicted as

$$
(0, \infty) \ni x_n \xrightarrow[\hookrightarrow\ x_n^2 + u_n^2]{\downarrow\ u_n \in (-\infty, x_n)} x_{n+1} \quad \text{w.w.} \quad \frac{2}{x_{n+1}} \quad \text{for any } x_{n+1} \in (0,\ x_n - u_n).
$$

A characteristic feature of the dynamics is as follows. As next state degenerates small, its weight grows unboudedly large. The total weight from any state x_n under any decision u_n diverges to ∞ as long as $x_n - u_n > 0$:

$$
\int_0^{x_n - u_n} \frac{2}{x_{n+1}} dx_{n+1} = \infty.
$$

4 Deterministic Dynamics

Let us consider the discounted quadratic criterion on deterministic dynamics:

$$\text{(D)} \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} \beta^n (x_n^2 + u_n^2) \\ \text{subject to} & \text{(i)} \ \ x_{n+1} = x_n - u_n \qquad\qquad n \geq 0 \\ & \text{(ii)} \ \ u_n \in R^1 \\ & \text{(iii)} \ \ x_0 = c, \end{array}$$

where $c \in R^1$ is a given constant.

4.1 Evaluation-optimization

Let us solve (D) through evaluation-optimization method, which has two stages. The first stage evaluates any policy in a class of policies and the second minimizes the evaluated value over the class.

A stationary policy f^∞ is called *proportional* if the decision function is specified by $f(x) = px$, where p is a real constant. Then p is called a *proportional rate.* In this subsection, we consider the set of all proportional policies whose rate p satisfies $\beta(1-p)^2 < 1$.

Lemma 4.1 *A proportional policy $f^\infty, f(x) = px$, yields the objective value*

$$\sum_{n=0}^{\infty} \beta^n (x_n^2 + u_n^2) = \frac{r}{1-\beta q} x_0^2,$$

where $r = 1 + p^2$, $q = (1-p)^2$.

Now let us consider the ratio minimization problem

$$\text{minimize} \quad \frac{r}{1-\beta q} \qquad \text{subject to} \quad \beta q < 1.$$

This is expressed as a single-variable problem :

$$(\mathrm{C}_\beta) \qquad \begin{array}{ll} \text{minimize} & \dfrac{1+p^2}{1-\beta(1-p)^2} \\ \text{subject to} & 1 - \dfrac{1}{\sqrt{\beta}} < p < 1 + \dfrac{1}{\sqrt{\beta}}. \end{array}$$

Lemma 4.2 *The problem (C_β) has the minimum value*

$$m = \frac{2\beta - 1 + \sqrt{4\beta^2 + 1}}{2\beta} \qquad \text{at} \quad \hat{p} = \frac{\sqrt{4\beta^2+1} - 1}{2\beta}.$$

Thus we have the optimal policy $\hat{f}^{\infty}$;

$$\hat{f}(x) = \hat{p}x, \qquad \hat{p} = \frac{\sqrt{4\beta^2+1}-1}{2\beta}$$

in the proportional policy class and the value function

$$v(x) = mx^2, \qquad m = \frac{2\beta - 1 + \sqrt{4\beta^2+1}}{2\beta}.$$

We remark that

$$m = 1 + \hat{p}.$$

4.2 Dynamic programming

In this subsection, we apply dynamic programming to optimize the infinite stage problem [1,4,8,12].

Let $v(c)$ be the minimum value for $c \in R^1$. Then $v : R^1 \to R^1$ is called a *value function.* The value function v satisfies the *Bellman equation* :

$$v(x) = \min_{-\infty<u<\infty} \left[x^2 + u^2 + \beta v(x-u)\right], \quad v(0) = 0. \tag{2}$$

Lemma 4.3 *The control process* (D) *has the proportional optimal policy* $f^{\infty}, f(x) = px$, *and the quadratic value function* $v(x) = vx^2$, *where*

$$v = \frac{2\beta - 1 + \sqrt{4\beta^2+1}}{2\beta}, \qquad p = \frac{\sqrt{4\beta^2+1}-1}{2\beta}.$$

The proportional optimal policy f^{∞} splits at any time an interval $[0,\ x]$ into $[0,\ (1-p)x] = \left[0,\ \frac{x}{1+\beta v}\right]$ and $\left[\frac{x}{1+\beta v},\ x\right]$. When, in particular, $\beta = 1$, the quadratic coefficient v is reduced to the *Golden number*

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

and the proportional rate p is reduced to its *inverse number*

$$\phi^{-1} = \phi - 1 = \frac{\sqrt{5}-1}{2} \approx 0.618$$

Further the division of $[0,\ x]$ into $[0,\ \phi^{-2}x]$ and $[\phi^{-2}x,\ x]$ is Golden. That is, the ratio of length of two intervals constitutes the Golden ratio:

$$\phi^{-2} : \phi^{-1} = 1 : \phi.$$

A quadratic function $w(x) = ax^2$ is called *Golden* if $a = \phi$.

Theorem 4.1 *The control process* (D) *with unit discount rate* $\beta = 1$ *has a Golden optimal policy* $f^{\infty}, f(x) = \phi^{-1}x$, *and the Golden quadratic value function* $v(x) = \phi x^2$.

5 Stochastic Dynamics

Let us consider the stochastic dynamic process under the condition that the discount rate β shoud be $0 < \beta < 1$. Soon it will be clarified that the expected value diverges for the case $\beta \geq 1$. Our stochastic dynamic minimization problem is

$$
(\mathrm{S}) \qquad \begin{array}{lll} \text{minimize} & E_{x_0}\left[\sum_{n=0}^{\infty} \beta^n (x_n^2 + u_n^2)\right] & \\ \text{subject to} & \text{(i)} \;\; x_{n+1} = x_n - u_n - \epsilon_{n+1} & \\ & \text{(ii)} \;\; u_n \in R^1 & n \geq 0 \\ & \text{(iii)} \;\; x_0 = c, & \end{array}
$$

where an initial state $c \in R^1$ is given, and $\{\epsilon_n\}$ is a sequence of random variables that is independently and identically distributed through time and obeys the standard normal distributuion. Thus

$$E[\epsilon_n] = 0, \quad E[\epsilon_n^2] = 1.$$

We note that ϵ_n has the probability density function

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty.$$

The next state (random variable) x_{n+1} obeys the normal distribution with mean $x_n - u_n$ and unit variance, provided that a decision u_n is taken at state x_n on stage n. When a decision maker adopts a decision u on state x, the system will go to state (scalar) y with probability $q(y \mid x, u) = p(y - x + u)$:

$$q(y \mid x, u) = \frac{1}{\sqrt{2\pi}} e^{-(y-x+u)^2/2} \qquad -\infty < y < \infty.$$

We depict this dynamics as

$$(-\infty,\ \infty) \ni x \xrightarrow[\hookrightarrow\ x^2 + u^2]{\downarrow\ u \in (-\infty, \infty)} y \;\; \text{w.p.} \;\; q(y \mid x, u) \;\; \text{for any} \;\; y \in (-\infty,\ \infty).$$

5.1 Evaluation-optimization

Let us evaluate any proportional policy $f^\infty, f(x) = px$ for $0 < p < 2$. The decision maker adopts the decision $u_n = f(x_n) = px_n$ on state x_n. Hence

$$E_{x_0}\left[\sum_{n=0}^{\infty} \beta^n (x_n^2 + u_n^2)\right] = r \sum_{n=0}^{\infty} \beta^n E_{x_0}\left[x_n^2\right],$$

where $r = 1 + p^2$. The controlled dynamics $x_{n+1} = x_n - u_n - \epsilon_{n+1}$ is reduced to

$$x_{n+1} = (1-p)x_n - \epsilon_{n+1} \qquad x_0 = c. \tag{3}$$

Here we note that $|1 - p| < 1$.

Lemma 5.1 *It follows that under (3)*

$$E[x_n^2] = \frac{1}{1-q} + \left(x_0^2 - \frac{1}{1-q}\right) q^n, \tag{4}$$

where $q = (1-p)^2$.

Lemma 5.2 *A proportional policy* $f^\infty, f(x) = px$, *yields the expected value*

$$E_{x_0}\left[\sum_{n=0}^{\infty} \beta^n (x_n^2 + u_n^2)\right] = \frac{r}{1-\beta q}\left(x_0^2 + \frac{\beta}{1-\beta}\right),$$

where $r = 1 + p^2,\ q = (1-p)^2$.

Note that the term $x_0^2 + \dfrac{\beta}{1-\beta}$ is independent of p. We have reached the ratio minimization problem (C_β) in the deterministic dynamics. Lemma 4.2 gives the minimun solution of (C_β).

Thus we have the optimal policy $\hat{f}^\infty$;

$$\hat{f}(x) = \hat{p}x, \qquad \hat{p} = \frac{\sqrt{4\beta^2+1}-1}{2\beta}$$

in the proportional policy class and the value function

$$v(x) = mx^2 + \rho, \quad m = \frac{2\beta - 1 + \sqrt{4\beta^2+1}}{2\beta}, \quad \rho = \frac{2\beta - 1 + \sqrt{4\beta^2+1}}{2(1-\beta)}.$$

5.2 Dynamic programming

Let $v(c)$ be the minimum value. Then the value function $v : R^1 \to R^1$ satisfies the Bellman equation:

$$v(x) = \min_{-\infty<u<\infty} \left[x^2 + u^2 + \beta E_x\left[v(x-u-\epsilon)\right]\right]. \tag{5}$$

This is also written as the controlled integral equation

$$v(x) = \min_{-\infty<u<\infty}\left[x^2 + u^2 + \frac{\beta}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-x+u)^2} v(y)\,dy\right].$$

Lemma 5.3 *The control process* (S) *has a proportional optimal policy* $f^\infty, f(x) = px$, *and a quadratic value function* $v(x) = vx^2 + \rho$, *where*

$$v = \frac{2\beta - 1 + \sqrt{4\beta^2+1}}{2\beta}, \qquad \rho = \frac{2\beta - 1 + \sqrt{4\beta^2+1}}{2(1-\beta)} \tag{6}$$

$$p = \frac{\sqrt{4\beta^2+1}-1}{2\beta}.$$

Thus we see that the stochastic dynamic system (S) has the same optimal policy as deterministic dynamic system (D). This is what we call *certainty equivalence principle.* The value function has a difference ρ which comes from the discounted total noise under uncertainty.

6 Non-deterministic Dynamics

Now we consider the minimization problem on non-deterministic dynamics:

$$(\mathrm{N}) \quad \begin{array}{lll} \text{minimize} & \displaystyle\sum_{n=0}^{\infty} \beta^n W_{x_0}\left[x_n^2 + u_n^2\right] & \\ \text{subject to} & \text{(i)} \ \ 0 < x_{n+1} < x_n - u_n & \text{with weight} \ \ 2/x_{n+1} \\ & & n \geq 0 \\ & \text{(ii)} \ \ u_n \in R^1 & \\ & \text{(iii)} \ \ x_0 = c, & \end{array}$$

where $c > 0$ is a given constant. The constraints (i), (ii) yields the feasibily $-\infty < u_n < x_n$. Here the n-th term is defined as follows.

$$\begin{aligned} W_{x_0}\left[x_n^2 + u_n^2\right] &= \iint \cdots \int_R \gamma_0 \gamma_1 \cdots \gamma_{n-1} r_n \, dx_1 dx_2 \cdots dx_n \\ &= \iint \cdots \int_R \frac{2^n \left(x_n^2 + u_n^2\right)}{x_1 x_2 \cdots x_n} dx_1 dx_2 \cdots dx_n, \end{aligned}$$

where the *transition weight function* and *cost function* are stationary:

$$\begin{aligned} \gamma_m &= \gamma(x_m, u_m, x_{m+1}) = \frac{2}{x_{m+1}} \\ r_n &= r(x_n, u_n) = x_n^2 + u_n^2. \end{aligned}$$

The integral domain R is determined through the sequence of decision functions $f_0, f_1, \cdots, f_{n-1}$:

$$\begin{aligned} R &= \{ (x_1, x_2, \cdots, x_n) \,|\, 0 < x_1 < x_0 - u_0, \ \ \cdots, \ 0 < x_n < x_{n-1} - u_{n-1} \} \\ &\subset (0, \infty)^n, \end{aligned}$$

where $u_m = f_m(x_m)$.

When $n = 0$, we have

$$W_{x_0}[r_0] = x_0^2 + u_0^2.$$

In the following, the sequence of states

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} a \xrightarrow{b} \cdots \xrightarrow{p} s \xrightarrow{q} t \longrightarrow \cdots$$

reads

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} x_3 \xrightarrow{u_3} \cdots \xrightarrow{u_{n-2}} x_{n-1} \xrightarrow{u_{n-1}} x_n \longrightarrow \cdots$$

The first three weighted values are

$$W_x\left[y^2+v^2\right] = \int_C \frac{2(y^2+v^2)}{y}dy$$

$$W_x\left[z^2+w^2\right] = \iint_D \frac{2^2(z^2+w^2)}{yz}dydz$$

$$W_x\left[a^2+b^2\right] = \iiint_E \frac{2^3(a^2+b^2)}{yza}dydzda,$$

where

$$C = \{\, y \,|\, 0<y<x-u(x)\,\} \subset (0,\infty)$$

$$D = \{\, (y,z) \,|\, 0<y<x-u(x),\ 0<z<y-v(y)\,\} \subset (0,\infty)^2$$

$$E = \{\, (y,z,a) \,|\, 0<y<x-u(x),\ 0<z<y-v(y),\ 0<a<z-w(z)\,\} \subset (0,\infty)^3$$

We call

$$W_{x_0}\left[x_n^2+u_n^2\right] \quad \text{and} \quad \beta^n W_{x_0}\left[x_n^2+u_n^2\right]$$

n-th weighted value and *n-th discounted weighted value*, respectively. The limit of series is called a *total discounted weighted value.* Thus the objective function (of x_0) represents a total discounted weighted value by using policy $\pi = \{f_0, f_1, \dots, f_{n-1}, \dots\}$ from initial state x_0.

Then we consider the total discounted weighted value

$$J(x_0;\pi) := W_{x_0}[r_0] + \beta W_{x_0}[r_1] + \cdots + \beta^n W_{x_0}[r_n] + \cdots$$

Thus our problem is to choose a policy which minimizes the discounted total weighted value. This is expressed as

$$\mathrm{P}(x_0) \qquad \text{minimize} \quad J(x_0;\pi) \quad \text{subject to} \quad \pi \in \Pi.$$

6.1 Evaluation-optimization

First we evaluate any proportional policy $\pi = f^\infty, f(x) = px$. The decision maker adopts a decision $u = px$ on state x and the system will go to state y on open interval $(0,\ x-u) = (0,\ (1-p)x)$ with the weight $\frac{2}{y}$. Then we have inductively

$$W_x\left[x_n^2+u_n^2\right] = rq^n x^2.$$

Lemma 6.1 *A proportional policy $\pi = f^\infty, f(x) = px$, yields the objective value*

$$\sum_{n=0}^{\infty} \beta^n W_{x_0}\left[x_n^2+u_n^2\right] = \frac{r}{1-\beta q}x_0^2,$$

where $r = 1+p^2,\ q = (1-p)^2$.

Thus we have reached the same ratio minimization problem (C_β) as in deterministic dynamics. Therefore we have the optimal policy $\hat{f}^\infty$;

$$\hat{f}(x) = \hat{p}x, \qquad \hat{p} = \frac{\sqrt{4\beta^2+1}-1}{2\beta}$$

and the value function

$$v(x) = mx^2, \qquad m = \frac{2\beta - 1 + \sqrt{4\beta^2+1}}{2\beta}.$$

6.2 Dynamic programming

Let $v(x_0)$ be the minimum value. Then the value function $v : [0\,\infty) \to R^1$ satisfies the Bellman equation:

$$v(x) = \min_{-\infty<u<\infty} \left[x^2 + u^2 + \beta \int_0^{x-u} 2\frac{v(y)}{y} dy \right] \qquad v(0) = 0. \tag{7}$$

This is also written as follows:

$$v(x) = \min_{-\infty<u<\infty} \left[x^2 + u^2 + \beta W_x^u[v] \right].$$

We may assume that Eq.(7) has a quadratic form $v(x) = vx^2$, where $v \in R^1$. We solve (7) as follows. Then we have

$$\int_0^{x-u} 2\frac{v(y)}{y} dy = \int_0^{x-u} 2vy dy = v(x-u)^2.$$

Eq.(7) is reduced to a minimum equation for saclar v:

$$vx^2 = \min_{-\infty<u<\infty} \left[x^2 + u^2 + \beta v(x-u)^2 \right].$$

Thus we have reached the same situation as in deterministic dynamics as was shown in (5).

Lemma 6.2 *The control process* (N) *has the proportional optimal policy $f^\infty, f(x) = px$, and the quadratic value function $v(x) = vx^2$, where*

$$v = \frac{2\beta - 1 + \sqrt{4\beta^2+1}}{2\beta}, \qquad p = \frac{\sqrt{4\beta^2+1}-1}{2\beta}.$$

Thus as in deterministic dynamics, we have the same result on Golden optimality:

Theorem 6.1 *The control process* (N) *with unit discount rate $\beta = 1$ has a Golden optimal policy $f^\infty, f(x) = (\phi - 1)x$, and the Golden quadratic value function $v(x) = \phi x^2$.*

7 Golden Policies

Let us now discuss whether the desired optimal policy is Golden or not. Throughout three presections, we have obtained a common optimal solution. The optimal policy both for stochastic process and for non-deterministic process is identical with the optimal policy for the deterministic process. This is called *certainty equivalence principle.* The three control processes — (D), (S) and (N) — have a common proportional optimal policy

$$f^\infty; \ f(x) = px$$

and the quadratic value function

$$v(x) = \begin{cases} vx^2 & \text{for (D), (N)} \\ vx^2 + \rho & \text{for (S),} \end{cases}$$

where

$$\begin{aligned} p &= \frac{\sqrt{4\beta^2+1}-1}{2\beta} = \frac{2\beta}{1+\sqrt{4\beta^2+1}} \\ v &= \frac{2\beta-1+\sqrt{4\beta^2+1}}{2\beta}, \quad \rho = \frac{2\beta-1+\sqrt{4\beta^2+1}}{2(1-\beta)}. \end{aligned} \tag{8}$$

The rate p is determined by the coefficient v:

$$p = v - 1.$$

The proportional optimal policy f^∞ splits any interval $[0,\ x]$ into $[0,\ (1-p)x]$ and $[(1-p)x,\ x]$. We are interested in values of discount factor β which yields the two Golden sections. This asks us when $1-p$ becomes $\phi - 1$ or $2 - \phi$.

Let us now consider both p and $1-p$ as functions of β. We take

$$p(\beta) := \frac{2\beta}{1+\sqrt{4\beta^2+1}}.$$

Then

$$p'(\beta) = \frac{2}{\sqrt{4\beta^2+1}\left(1+\sqrt{4\beta^2+1}\right)} > 0.$$

Thus $p(\beta)$ is strictly increasing and

$$1-p(0) = 1, \quad 1-p(1) = 2-\phi \approx 0.382$$

This enables us to solve the equation

$$1-p(\beta) = \begin{cases} 2-\phi \\ \phi-1 \end{cases} \quad \text{i.e.} \quad p(\beta) = \begin{cases} \phi-1 \\ 2-\phi. \end{cases}$$

This is reduced to

$$\frac{2\beta}{1+\sqrt{4\beta^2+1}} = \begin{cases} \phi^{-1} \\ \phi^{-2}. \end{cases}$$

The equation has respective solutions

$$\beta = \begin{cases} 1 \\ \dfrac{\phi^2}{\phi^4 - 1} = \dfrac{1}{\sqrt{5}}. \end{cases}$$

We note that

$$\frac{\phi^2}{\phi^4-1} = \frac{1}{(\phi^2+1)(\phi-1)} = \frac{1}{2\phi-1} = \frac{2\phi-1}{5} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}} \approx 0.4772$$

7.1 Deterministic dynamics

The deterministic control process (D) has a discount factor $0 \leq \beta < \infty$.

7.1.1 Case $\beta = 1$

When $\beta = 1$, the quadratic coefficient v is reduced to the *Golden number*

$$v = \phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

and the proportional rate p is reduced to

$$p = \phi - 1 = \frac{\sqrt{5}-1}{2} \approx 0.618$$

Further the division of $[0,\ x]$ into $[0,\ \phi^{-2}x]$ and $[\phi^{-2}x,\ x]$ is Golden.

The Golden optimal policy f^∞, $f(x) = px$, yields the optimal deterministic behavior as follows. A current state x_n under the Golden optimal decision $u_n = px_n = \phi^{-1}x_n$ goes to a unique state $x_{n+1} = x_n - u_n = \phi^{-2}x_n$ into R^1. The next state is $x_{n+1} = \phi^{-2}x_n \approx 0.382x_n$ (Fig. 1). The dynamics is depicted as

$$R^1 \ni x_n \xrightarrow{\downarrow\ u_n = \phi^{-1}x_n} x_{n+1} = \phi^{-2}x_n \approx 0.382x_n \quad \text{uniquely.}$$

Thus the Golden optimal dynamics says that next state becomes $x_{n+1} = \phi^{-2}x_n \approx 0.382x_n$.

7.1.2 Case $\beta = \frac{1}{\sqrt{5}}$

We consider case $\beta = \frac{1}{\sqrt{5}} \approx 0.4772$ Then we have

$$v = 3 - \phi \approx 1.382, \qquad p = \phi^{-2} \approx 0.382$$

The division of $[0,\ x]$ into $[0,\ \phi^{-1}x]$ and $[\phi^{-1}x,\ x]$ is Golden.

The Golden optimal policy $f^{\infty}, f(x) = px$, yields the optimal deterministic behavior as follows. A current state x_n under the Golden optimal decision $u_n = \phi^{-2}x_n$ goes to a unique state $x_{n+1} = x_n - u_n = \phi^{-1}x_n \approx 0.618x_n$ (Fig. 2). The the Golden optimal dynamics

$$R^1 \ni x_n \xrightarrow{\downarrow\ u_n = \phi^{-2}x_n} x_{n+1} = \phi^{-1}x_n \approx 0.618x_n \ \text{ uniquely}$$

says that $x_{n+1} = \phi^{-1}x_n \approx 0.618x_n$.

7.2 Stochastic dynamics

The stochastic control process (S) has the discount factor restricted to $0 \le \beta < 1$. We consider the Case $\beta = \frac{1}{\sqrt{5}}$ only.

7.2.1 Case $\beta = 1$

As we have shown in (6), the total noise is $\rho = \frac{2\beta - 1 + \sqrt{4\beta^2 + 1}}{2(1-\beta)}$ for $0 \le \beta < 1$. Thus it diverges to ∞ for $\beta = 1$.

7.2.2 Case $\beta = \frac{1}{\sqrt{5}}$

We have

$$v = 3 - \phi \approx 1.382, \qquad p = \phi^{-2} \approx 0.382, \qquad \rho = \frac{\sqrt{5}}{2} \approx 1.118$$

The state sequence $\{X_n\}_0^{\infty}$ defined by

$$X_{n+1} = X_n - pX_n - \epsilon_{n+1}, \qquad X_0 = x_0$$

is stochastically *Golden* :

$$E[X_{n+1} \mid x_n] = \phi^{-1}x_n.$$

That is, the Golden optimal policy $f^{\infty}, f(x) = px$, yields the optimal stochastic behavior as follows. A current state x_n under the Golden optimal decision $u_n = px_n = \phi^{-2}x_n$ goes to x_{n+1} on R^1 with transition probability $q(x_{n+1} \mid x_n, u_n) = \frac{1}{\sqrt{2\pi}} e^{-(x_{n+1}-\phi^{-1}x_n)^2/2}$. The next state (random variable) x_{n+1} follows the normal distribution $N(\phi^{-1}x_n,\ 1)$. The mean is $\phi^{-1}x_n \approx 0.618x_n$ (see Fig. 2). The Golden optimal dynamics

$$R^1 \ni x_n \xrightarrow{\downarrow\ u_n = \phi^{-2}x_n} x_{n+1} \ \text{w.p.}\ \ q(x_{n+1} \mid x_n, u_n) \ \ \text{for any } x_{n+1} \in R^1$$

says that current state goes down to $\phi^{-1}x_n \approx 0.618x_n$ *on average.*

7.3 Non-deterministic dynamics

The non-deterministic control process (N) has a discount factor $0 \leq \beta < \infty$.

7.3.1 Case $\beta = 1$

When $\beta = 1$, it follows that

$$v = \phi \approx 1.618, \qquad p = \phi^{-1} \approx 0.618$$

Further the division of $[0,\ x]$ into $[0,\ \phi^{-2}x]$ and $[\phi^{-2}x,\ x]$ is Golden optimal.

The Golden optimal policy $f^\infty, f(x) = px$, yields the optimal non-deterministic behavior as follows. A current state x_n under the Golden optimal decision $u_n = px_n = \phi^{-1}x_n$ goes to x_{n+1} on inteval $(0,\ x_n - u_n) = (0,\ \phi^{-2}x_n)$ with transition weight $q(x_{n+1} \mid x_n, u_n) = 2/x_{n+1}$. The next state (non-deterministic variable) x_{n+1} has the unbounded weight $2/x_{n+1}$ on $(0,\ \phi^{-2}x_n) \approx (0,\ 0.382x_n)$. The Golden optimal dynamics

$$(0\,\infty) \ni x_n \xrightarrow{\downarrow\ u_n = \phi^{-1}x_n} x_{n+1} \ \text{w.w.}\ \ 2/x_{n+1} \ \ \text{for any } x_{n+1} \in (0,\ \phi^{-2}x_n)$$

says that current state goes down on a shrunken interval $(0,\ \phi^{-2}x_n) \approx (0,\ 0.382x_n)$ with the Golden rate $\phi^{-2} \approx 0.382$ (see Fig. 1).

7.3.2 Case $\beta = \frac{1}{\sqrt{5}}$

The case yields

$$v = 3 - \phi \approx 1.382, \qquad p = \phi^{-2} \approx 0.382$$

The division of $[0,\ x]$ into $[0,\ \phi^{-1}x]$ and $[\phi^{-1}x,\ x]$ is Golden optimal.

The Golden optimal policy $f^\infty, f(x) = px$, yields the optimal non-deterministic behavior as follows. A current state x_n under the Golden optimal decision $u_n = \phi^{-2}x_n$ goes to x_{n+1} on inteval $(0,\ x_n - u_n) = (0,\ \phi^{-1}x_n)$ with transition weight $q(x_{n+1} \mid x_n, u_n) = 2/x_{n+1}$. The non-deterministic x_{n+1} has the unbounded weight $2/x_{n+1}$ on $(0,\ \phi^{-1}x_n) \approx (0,\ 0.618x_n)$. The Golden optimal dynamics

$$(0\,\infty) \ni x_n \xrightarrow{\downarrow\ u_n = \phi^{-2}x_n} x_{n+1} \ \text{w.w.}\ \ 2/x_{n+1} \ \ \text{for any } x_{n+1} \in (0,\ \phi^{-1}x_n)$$

says that next state goes down on a shrunken interval $(0,\ \phi^{-1}x_n) \approx (0,\ 0.618x_n)$ with the Golden rate $\phi^{-1} \approx 0.618$ (see Fig. 2).

Finally we have the following result.

Theorem 7.1 *For the discount rate* $\beta = \dfrac{1}{\sqrt{5}}$, *three processes* (D), (S) *and* (N) *have a common Golden optimal policy* $g^\infty, g(x) = (2-\phi)x$. *Then* (D) *and* (N) *have the quadratic value function* $v(x) = (3-\phi)x^2$ *and* (S) *has the quadratic value function* $v(x) = (3-\phi)x^2 + \dfrac{\sqrt{5}}{2}$.

References

[1] R.E. Bellman, *Dynamic Programming*, Princeton Univ. Press, NJ, 1957.

[2] R.E. Bellman, *Introduction of the Mathematical Theory of Control Processes, Vol.I, Linear Equations and Quadratic Criteria; Vol.II, Nonlinear Processes*, Academic Press, NY, 1967; 1971.

[3] R.E. Bellman, *Methods of Nonlinear Analysis, Vol.I, Vol.II,* Academic Press, New York, 1969, 1972.

[4] S. Iwamoto, *Theory of Dynamic Program: Japanese*, Kyushu Univ. Press, Fukuoka, 1987.

[5] S. Iwamoto, Cross dual on the Golden optimum solutions, Proceedings of the Workshop in Mathematical Economics, RIMS Kokyu Roku No.1443, pp. 27-43, Kyoto University, 2005.

[6] S. Iwamoto, The Golden trinity — optimility, inequality, identity —, Proceedings of the Workshop in Mathematical Economics, RIMS Kokyu Roku No.1488, pp. 1-14, Kyoto University, 2006.

[7] S. Iwamoto, The Golden optimum solution in quadratic programming, Ed. W. Takahashi and T. Tanaka, Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis (Okinawa, 2005), Yokohama Publishers, 2007, pp.109–205.

[8] S. Iwamoto, Golden optimal policy in calculus of variation and dynamic programming, *Advances in Mathematical Economics* **10** (2007), pp.65–89.

[9] S. Iwamoto and M. Yasuda, Dynamic programming creates the Golden Ratio, too, *Proc. of the Sixth Intl Conference on Optimization: Techniques and Applications (ICOTA 2004)*, Ballarat, Australia, December 9-11, 2004.

[10] S. Iwamoto and M. Yasuda, Dynamic programming creates the Golden ratio, too, Proceedings of the Workshop “Decision Models and Mathematical Models under Uncertainty,” RIMS Kokyu Roku No.1477, pp. 136-140, Kyoto University, 2006.

[11] S. Iwamoto and M. Yasuda, Golden optimal path in discrete-time dynamic optimization processes, Ed. S. Elaydi, K. Nishimura and M. Shishikura, Advanced Studies in Pure Mathematics 53, 2009, ICDEA2006, pp.99–108; Proceedings of The International Conference on Differential Equations and Applications (ICDEA06), Kyoto University, Kyoto, JAPAN, July 24-28, 2006.

[12] M. Sniedovich, *Dynamic Programming*, Marcel Dekker, Inc. NY, 1992.

[13] H. Walser, *Der Goldene Schnitt*, B.G. Teubner, Leibzig, 1996.