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On Measuring Utility from Demand: without Proofs*

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Abstract

This paper provides a method which calculates a preference relation from a demand function. Our method works well with only weak axiom. We think that our method is more intuitive than in previous works on the integrability theory. Also, this method is so simple that it is easy to execute calculation. Moreover, if the demand function obeys strong axiom, then our method can calculate a smooth utility function, which assures the applicability on the method of comparative statics. In addition to these results, we guarantee the recoverability, that is, uniqueness of the preference relation corresponding with a demand function.

Keywords: integrablity; weak axiom; comparative statics; recoverability.

JEL Classification Numbers: D11

1 Introduction

The purpose of this paper is to provide an easy computational method which calculates a preference relation corresponding with a demand function with weak or strong axiom. The result of this paper is related to the integrability theory, especially so-called indirect approach of the integrability theory.

In economics, it is thought that the behavior of theoretical model should resemble observed data. Models starting from a demand function are better

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in this sense than from an utility function since a demand function can be compared with observed data. But models starting from a demand function are often confronted with difficulty on welfare analysis. In partial equilibrium theory, there exists a criterion of welfare measured from a demand function, called the total surplus.[1] In general equilibrium theory, however, there is no such a criterion. Thus there should be some method which calculates a preference relation from a demand function.

The aim of this paper is to provide such a method. This method works well even if the demand function obeys only weak axiom. We think that our method is more intuitive than in previous works such as Richter(1966, 1979), Hurwicz and Uzawa(1969), Debreu(1972, 1976) and so on. Also, our method is so simple that it is easy to execute actual calculation. Moreover, if the demand function obeys strong axiom, then our method can calculate a smooth utility function. This result is important since the information of a smooth utility function is essential to applying the method of comparative statics. In addition to these results, we guarantee the "recoverability", that is, uniqueness of the preference relation corresponding with a demand function. This result is independent of Mas-Colell(1977) since our result can apply some demand functions without strong axiom.

In section 2, we demonstrate our three main results. Our starting point is a demand function and goal is a preference relation. However, to compute a preference relation from a demand function directly is too difficult. Hence we put a buffer between a demand function and a preference relation, that is, an inverse demand function.

Theorem 1 provides an easy and simple method which calculates a preference relation from an inverse demand function. This method can easily be interpreted. Choose any x, v from the consumption set. Let $y(t)$ denote some function which expresses the indifference curve passing through x in the plane spanned by x and v, and t^* is a real number such that $y(t^*)$ is proportional to v. Then we can decide that x is preferred to v if and only if $y(t^*)$ is larger than v. Hence our task is only to find such y. In Theorem 1, we provide a method which calculates y by solving a simple ordinary differential equation(ODE) which is built from an inverse demand function, and find a condition of an inverse demand function under which the calculated preference relation properly corresponds with this inverse demand function, named condition (A). Also, we find another condition of an inverse demand function under which the calculated preference relation is transitive and has

[1]With some additional assumptions, the total surplus in a partial equilibrium model is identical with the sum of utility in the corresponding general equilibrium model. See Hicks(1939).

a smooth utility function, named condition (B).

Next three propositions connect demand functions with inverse demand functions. Proposition 1 has at least two implications. At first, Proposition 1 provides an applicable sufficient condition of a demand function on the existence of a smooth inverse demand function.[2] Secondly, Proposition 1 provides a method which calculates such an inverse demand function from a demand function. In addition to these implications, Proposition 1 make clear that condition (A) of an inverse demand function corresponds with weak axiom of a demand function. While Proposition 1 treats only single-valued demand functions, Proposition 2 treats multi-valued demand functions. Also, Proposition 3 makes clear that condition (B) of an inverse demand function correspond with strong axiom of a demand function. Combining Theorem 1 with these propositions, we can say that our computational method works well under weak axiom and can calculate a smooth utility function under strong axiom.

At last, Proposition 4 treats the problem of the recoverability. This property is important to perform welfare analysis. If the recoverability does not hold, there may be another preference relation corresponding with the same demand function. Fortunately, the recoverability holds and thus our preference relation can be trusted.

In section 3, we try to explain our computation method for three purposes, and refer to several important remarks. At first, we explain how to calculate it analytically. Secondly, we explain how to compute it approximately. At last, we explain the method of comparative statics, that is, how to compute the derivatives of utility approximately. In remarks, we treat the relationship between Euler difference approximation method and revealed preference relation and the case in which demand may take the corner solution. Section 4 is the concluding remarks.

2 Main Results

2.1 The Model

Fix a positive integer n and suppose $n \geq 2$. We write $\Omega = \mathbb{R}^n_{++}$ to make clear that $\mathbb{R}^n_{++}$ is the consumption set. Let $x \gg y$ mean $x - y \in \mathbb{R}^n_{++}$, $x \gneq y$ mean $x - y \in \mathbb{R}^n_+ \setminus \{0\}$ and $x \geq y$ mean $x - y \in \mathbb{R}^n_+$.

[2]We stress that this is a new result: the author cannot find any other result in this context to assure the existence of a smooth inverse demand function without the assumption on the existence of a smooth utility function.

We call a binary relation $\succsim$ on Ω a *preference relation* if it is complete, that is,

$$\forall x, y \in \Omega, \text{ we have } (x,y) \in \succsim \text{ or } (y,x) \in \succsim.$$

We also write $x \succsim y$ as $(x,y) \in \succsim$, $x \succ y$ as $(x,y) \in \succsim$ and $(y,x) \notin \succsim$, and $x \sim y$ as $(x,y) \in \succsim$ and $(y,x) \in \succsim$.

Let $\succsim$ be a preference relation. $\succsim$ is said to be

$$\begin{array}{rll} \textit{continuous} & \text{if} & \succsim \text{ is closed in } \Omega^2, \\ \textit{strongly monotone} & \text{if} & x \succ y \text{ whenever } x \gneq y, \\ \textit{transitive} & \text{if} & x \succsim y \text{ and } y \succsim z \text{ imply } x \succsim z, \\ \textit{p-transitive} & \text{if} & x \succsim y \text{ and } y \succsim z \text{ imply } x \succsim z \\ & & \text{whenever } \dim(\text{span}\{x,y,z\}) \leq 2. \\ \textit{represented by } u & \text{if} & u : \Omega \to \mathbb{R} \text{ and } u(x) \geq u(y) \Leftrightarrow x \succsim y. \end{array}$$

We call a multi-valued function $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \twoheadrightarrow \Omega$ a *demand function* if it satisfies homogeneity of degree zero and Walras' law.[3] A demand function f is said to satisfy[4]

$$\begin{array}{rll} \textit{weak axiom} & \text{if} & \forall x \in f(p,m),\ \forall y \in f(q,w), \\ & & q \cdot x \leq w \text{ and } p \cdot y \leq m \text{ imply } x \in f(q,w), \\ \textit{strong axiom} & \text{if} & \forall x_1, ..., x_k \text{ s.t. } x_i \in f(p_i, m_i) \text{ for any } i = 1, ..., k, \\ & & \text{if } p_i \cdot x_{i+1} \leq m_i \text{ for any } i = 1, ..., k-1 \\ & & \text{and } p_k \cdot x_1 \leq m_k, \text{ then } x_1 \in f(p_k, m_k). \end{array}$$

If f is a single-valued function and C^1-class, then we define the *Slutsky matrix* of f, denoted by S_f, such that

$$S_f(p,m) = D_p f(p,m) + D_m f(p,m) \times f(p,m)^T,$$

where A^T denote the transposed matrix of A.

Suppose $\succsim$ is a preference relation. Then we define,

$$f^{\succsim}(p,m) = \{x \in \Omega | p \cdot x \leq m \text{ and } \forall y \in \Omega,\ [p \cdot y \leq m \Rightarrow (x,y) \in \succsim]\}.$$

Note that $f^{\succsim}$ is a demand function if $\succsim$ is strongly monotone.

[3]Since the consumption set Ω is not closed, we do not assume f is nonempty-valued.

[4]Sometimes our strong axiom is called *congruence axiom*. See Richter(1966), for example.

Suppose $g : \Omega \longrightarrow \mathbb{R}^n_{++}$ is C^1-class. We say g satisfies[5]

condition (A) if $\forall x \in \Omega$, $\forall w \in \mathbb{R}^n$ such that $g(x) \cdot w = 0$,
$w^T Dg(x) w \leq 0$,
condition (B) if $g^i(\partial_j g^k - \partial_k g^j) + g^j(\partial_k g^i - \partial_i g^k) + g^k(\partial_i g^j - \partial_j g^i) = 0$.

Choose any demand function f. Then we call a function $g : \Omega \longrightarrow \mathbb{R}^n_{++}$ an *inverse demand function of f* if,

$$\forall x \in \Omega, x \in f(g(x), g(x) \cdot x).$$

If g is an inverse demand function of $f^{\succsim}$, then g is also called an *inverse demand function of $\succsim$*.

Remarks on Definitions: In above definitions, p-transitivity of preferences and contitions (A) and (B) of inverse demand functions are less familier and thus they should be interpreted.

P-transitive preferences often appear from early times in the context of a non-integrable demand function. For example, Samuelson(1950) referred to such a preference in his "three-sided tower" argument. Likewise, this property appears very naturally in our Theorem 1.

P-transitivity has an interesting similarity to weak axiom. In fact, weak axiom is equivalent to strong axiom *on any plane.* Rose(1958) showed that weak axiom implies strong axiom when $n = 2$. Similarly, we can prove the following fact even if $n > 2$: for any demand function f, f satisfies weak axiom if and only if for any $x_1, ..., x_k$ such that $\dim(\mathrm{span}\{x_1, ..., x_k\}) \leq 2$ and $x_i \in f(p_i, m_i)$ for any $i = 1, ..., k$, if $p_i \cdot x_{i+1} \leq m_i$ for any $i = 1, ..., k-1$ and $p_k \cdot x_1 \leq m_k$, then $x_1 \in f(p_k, m_k)$. In general, strong axiom corresponds with transitivity. Thus strong axiom *on any plane*(and also weak axiom) corresponds with transitivity *on any plane*, that is, p-transitivity.

Conditions (A) and (B) of inverse demand functions are closely related to the notion of *Antonelli matrix.* For any smooth $g : \Omega \longrightarrow \mathbb{R}^n_{++}$, the Antonelli matrix of g, denoted by $A_g(x)$, is defined as follows: $A^{i,j}_g(x) = \partial_j g^i(x) - g^j(x)\partial_n g^i(x)$ for any $i, j \in \{1, ..., n-1\}$. Then for any g normalized by $g^n \equiv 1$, we can easily verify that condition (A) is equivalent to the negative semi-definiteness of $A_g(x)$ and condition (B) is equivalent to the symmetry of $A_g(x)$. See Katzner(1970), Richter(1979) or Hurwicz and Richter(1979) for the more detailed arguments.

[5] Unless otherwise stated, we treats x^i as the i-th coordinate of vector x.

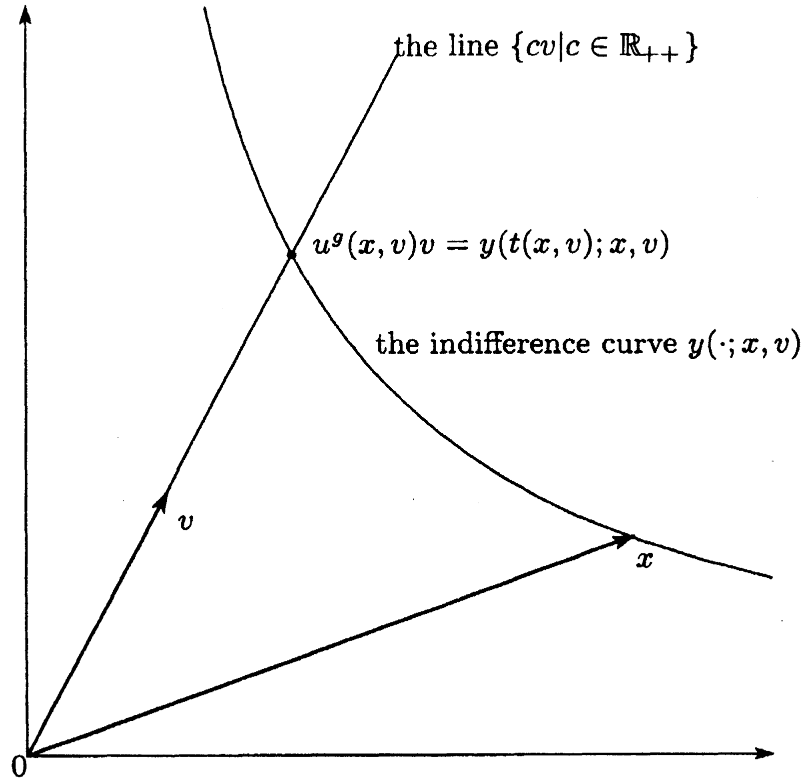

Figure 1: An illustration of our method

2.2 From an Inverse Demand Function to a Preference Relation

Choose any $g : \Omega \to \mathbb{R}^n_{++}$. We define $u^g : \Omega^2 \to \mathbb{R}$ and $\succsim^g$ in the following manner.

Consider the following initial value problem;

$$\begin{aligned} \dot{y} &= (g(y) \cdot x)v - (g(y) \cdot v)x, \\ y(0) &= x. \end{aligned} \tag{1}$$

Suppose $y(t;x,v)$ is the nonextendable solution of (1). If x is proportional to v, then we define $t(x,v) = 0$. Otherwise, we define $t(x,v)$ as the unique t such that $y(t;x,v)$ is proportional to v.[6] Finally, we define

$$u^g(x,v) = \frac{y^1(t(x,v);x,v)}{v^1},$$

and $\succsim^g = (u^g)^{-1}([1,+\infty[)$.

[6]The existence and uniqueness of such t is argued in the proof of Theorem 1.

We interpret our definitions of u^g and $\succsim^g$ as follows. Choose any commodity basket $v \in \Omega$. Fix any $x \in \Omega$. Then for any t,

$$\dot{y}(t; x, v) \cdot g(y(t; x, v)) = 0,$$

and thus the trajectory of $y(t; x, v)$ is orthogonal to price vector $g(y(t; x, v))$ at each t. Hence the trajectory of $y(t; x, v)$ is tangent to budget hyperplane at each point and thus it means the indifference curve passing through x. Since $y(t(x, v); x, v) = u^g(x, v)v$, both x and $u^g(x, v)v$ are in the trajectory of $y(\cdot; x, v)$ and thus we can guess x is indifferent to $u^g(x, v)v$. Therefore, $u^g(x, v)$ represents the amount of commodity basket v whom the consumer thinks to be indifferent to x. Figure 1 is an illustration of this method.

Theorem 1 Let $\ell \geq 2$ and suppose $g : \Omega \to \mathbb{R}^n_{++}$ is C^ℓ-class. Then,

(I) *Well-definedness.* Both u^g and $\succsim^g$ are well-defined. $\succsim^g$ is a continuous, strongly monotone and p-transitive preference relation. u^g is continuous on Ω^2 and C^ℓ-class on $\{(x, v) \in \Omega^2 | \forall c \in \mathbb{R}, x \neq cv\}$.

(II) *Inverse demand function.* The following three statements are equivalent:

a) g is an inverse demand function of $\succsim^g$.

b) g satisfies condition (A).

c) $(1 - \lambda)x + \lambda y \succsim^g x$ for any $x, y \in \Omega$ such that $y \succsim^g x$ and any $\lambda \in [0, 1]$.[7]

(III) *Transitivity and smoothness.* Fix any $v \in \Omega$ and define $u^g_v : x \mapsto u^g(x, v)$. Then the following four statements are equivalent:

a) g satisfies condition (B).

b) $\succsim^g$ is transitive.

c) u^g_v is C^ℓ-class and there exists $\lambda : \Omega \to \mathbb{R}$ such that $Du^g_v = \lambda g$.

d) u^g_v is C^ℓ-class and represents $\succsim^g$.

We should show that condition (A) is independent of condition (B). Consider the following functions.

$$g_1(x^1, x^2) = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix},$$

[7]We call this property the *convexity toward the origin* on indifference curve of $\succsim^g$.

$$g_2(x^1, x^2, x^3) = \begin{pmatrix} 9 & 12 & 16 \\ 16 & 9 & 12 \\ 12 & 16 & 9 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

Clearly g_1 does not satisfy condition (A) but satisfies condition (B). Also, we can show that g_2 does not satisfies condition (B). Meanwhile, Gale(1960) showed that g_2 is an inverse demand function of some demand function f which satisfies weak axiom. Hence our next proposition(Proposition 1) guarantees g_2 satisfies condition (A). Therefore, condition (A) is independent of condition (B). By our Theorem 1, this fact also implies there exists a preference relation which does not satisfy transitivity but satisfies p-transitivity.

Note that under condition (B), condition (A) is equivalent to the quasi-concavity of u^g_v. See Otani(1983) for the more detailed arguments.

We should make mention of an alternative definition of $t(x,v)$. Define $w = (v \cdot x)v - (v \cdot v)x$. If x is not proportional to v, then $w \cdot z = 0 \Leftrightarrow \exists c \in \mathbb{R}, z = cv$ for any $z \in \mathrm{span}\{x, v\}$, $w \cdot \dot{y}(t; x, v) > 0$ for any t and $w \cdot x < 0$. Hence we have,

$$t(x,v) = \min\{t \geq 0 | y(t; x, v) \cdot w \geq 0\}.$$

This feature is useful when we approximate $u^g(x,v)$ numerically. Let (y_t) be any difference approximation of $y(\cdot; x, v)$ and t^* be the first t such that $y_t \cdot w \geq 0$. Then y_{t^*} approximates $y(t(x,v); x, v)$.

It may be hard to treat equation (1) directly. Next corollary relaxes this difficulty.

Corollary 1 Suppose $g : \Omega \to \mathbb{R}^n_{++}$ is C^2-class, $\sigma : \Omega^3 \to \mathbb{R}$ is C^1-class with respect to the first n-variables and $\sigma(y, x, v) \neq 0$ whenever x is not proportional to v. Consider the following initial value problem;

$$\begin{aligned} \dot{y} &= \sigma(y, x, v)[(g(y) \cdot x)v - (g(y) \cdot v)x], \\ y(0) &= x. \end{aligned} \tag{2}$$

Let $y^*(\cdot; x, v)$ be the nonextendable solution of (2). For any $x, v \in \Omega$ such that x is not proportional to v, there exists the unique number $t^*(x,v)$ such that $y^*(t^*(x,v); x, v)$ is proportional to v. Define $t^*(x,v) = 0$ if x is proportional to v. Then for any $(x,v) \in \Omega^2$,

$$y^*(t^*(x,v); x, v) = u^g(x,v)v.$$

This corollary has two meanings.

At first, to calculate u^g, we can use equation (2) instead of equation (1). Under suitable σ, equation (2) may be easier to solve than equation (1). Hence Corollary 1 is useful for actual calculation.

Secondly, consider the case in which σ does not depend on x, v. Then the right-hand side of equation (2) can be rewritten as the following expression:

$$(\sigma(y)g(y) \cdot x)v - (\sigma(y)g(y) \cdot v)x.$$

Notice that σg means an alternative normalized form of g. Thus Corollary 1 means u^g and $\succsim^g$ are invariant from any exchange of normalization on inverse demand function.

In section 3, we introduce three simple examples to demonstrate how to use this theorem and corollary.

2.3 From a Demand Function to an Inverse Demand Function

In Theorem 1, we assumed the domain of g is equal to Ω. Therefore, we implicitly assumed the corresponding demand function f is surjective, that is, $f(\mathbb{R}^n_{++} \times \mathbb{R}_{++}) = \Omega$. The implicit assumption of a demand function is not only this surjective condition. The rank condition of S_f is also a necessary condition of the existence of g. That is, for any smooth demand function f having a smooth inverse demand function g, the $n-1$-th principal submatrix of $S_f(g(x), g(x) \cdot x)$ is regular and thus the rank of S_f is equal to $n-1$.[8]

Next proposition provides a sufficient condition of the existence of a smooth inverse demand function.

Proposition 1 Let $\ell \geq 2$, A be an open set in $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$, $f : A \to \Omega$ be a C^ℓ-class surjective single-valued demand function such that the rank of $S_f(p, m)$ is equal to $n-1$ for any $(p, m) \in A$. If f satisfies weak axiom, then there exists the unique inverse demand function $g : \Omega \to \mathbb{R}^n_{++}$ such that $g^n \equiv 1$. Moreover, g is C^ℓ-class, g satisfies condition (A) and $f = f^{\succsim^g}$.

Proposition 1 has several meanings. At first, it provides an appropriate sufficient condition of the existence on smooth g. Secondly, Proposition 1 tells us the way to calculate smooth g for given f. That is, $g(x)$ is the unique

[8] See Samuelson(1950).

solution of the following equations;

$$\begin{aligned} f(p, p \cdot x) &= x, \\ p^n = 1. & \end{aligned}$$

Hence we can get $g(x)$ by solving above equations blindly.

It is often hard to solve this equation analytically. Thus sometimes we have to use the approximation of $g(x)$, instead of $g(x)$ itself. This fact is related to the third meaning of Proposition 1: Proposition 1 allows us to omit verifying g satisfies condition (A). In fact, to verify g satisfies condition (A) directly, we must know the value of $g(x)$ and $Dg(x)$. But $g(x)$ may be not able to calculate precisely, and thus its verification may be impossible. Proposition 1 makes its verification possible.

Finally, Proposition 1 assures $f = f^{\succsim^g}$. If this property does not hold, then our calculated preference relation does not correspond with f precisely and thus our method does not work well. Proposition 1 guarantees this property and thus our method works well under weak axiom.

There is a remark on Proposition 1. Kihlstrom, Mas-Colell and Sonnenschein(1976) showed that the negative semi-definiteness of Slutsky matrix(NSD) is equivalent to WWA(weak weak axiom). When the demand function is single-valued and smooth, condition (A) is equivalent to NSD and thus it is also equivalent to WWA. However, WWA seems not to assure $f = f^{\succsim^g}$. This fact suggest the possibility on the existence of a demand function f which has a smooth inverse demand function g satisfies condition (A) but $f \neq f^{\succsim^g}$. Troublingly, the author cannot find such an example, and this problem is still an open question.[9]

Although we could not find any appropriate sufficient condition of the existence on a smooth inverse demand function when f is not single-valued, the following proposition still holds.

Proposition 2 Choose any demand function $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \twoheadrightarrow \Omega$, and suppose $g : \Omega \to \mathbb{R}^n_{++}$ is the unique inverse demand function of f such that $g^n \equiv 1$. If g is C^2-class and f satisfies weak axiom, then g satisfies condition (A) and $f = f^{\succsim^g}$.

For condition (B), next proposition holds.

[9]Kihlstrom, Mas-Colell and Sonnnenschein(1976) also provided an example of demand function which does not satisfy weak axiom but satisfies WWA. However, the range of this demand function is not the positive orthant, and thus this is not a desired example.

Proposition 3 Let $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \twoheadrightarrow \Omega$ is a demand function satisfying weak axiom, $g : \Omega \longrightarrow \mathbb{R}^n_{++}$ is the unique inverse demand function of f such that $g^n \equiv 1$, and g is C^2-class. Then the following three statements are equivalent.

1) f satisfies strong axiom.

2) There exists a transitive preference $\succsim$ such that $f = f^{\succsim}$.

3) g satisfies condition (B).

If f is a C^1-class single-valued function on an open set $A \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, then the following statement is also equivalent to above statements.

4) $S_f(p, m)$ is symmetric for any $(p, m) \in A$.

By Proposition 3, we can see that condition (B) is related to strong axiom. Combining Theorem 1 with these propositions, we can say that our computational method can calculate a smooth utility function when the demand function satisfies strong axiom.

Note that the equivalence between 1) and 2) differs from classical results on revealed preference theory, since we allow $f(p, m) = \emptyset$ for some (p, m). It is very famous that strong axiom of a single-valued choice function is equivalent to the rationalizability of this function.[10] This result can easily be extended to the case of non-empty multi-valued choice function. But nonempty-valuedness cannot be dropped. In fact, any choice function which takes empty value on some finite set cannot rationalize even if it satisfies strong axiom.

2.4 The Recoverability Property

Next, we treat the recoverability problem. Of course, with no additional assumption we can never prove such property.[11] But we can prove it, if we restrict the range of preference by adding the weakened continuity condition.

[10] See, for example, Mas-Colell, Whinston and Green(1995).

[11] Let $\succsim_l$ be a lexicographic order on Ω. For any C^2-class $g : \Omega \longrightarrow \mathbb{R}^n_{++}$ such that $w^T Dg(x) w < 0$ for any $w \in \mathbb{R}^n$ such that $w \neq 0$ and $g(x) \cdot w = 0$, define

$$\succsim' = \succ^g \cup \{(x, y) \in \Omega^2 | x \sim^g y \text{ and } x \succsim_l y\}.$$

Then we can easily prove that $\succsim'$ is p-transitive, $\succsim' \neq \succsim^g$ and $f^{\succsim'} = f^{\succsim^g}$.

Proposition 4 Suppose $g : \Omega \to \mathbb{R}^n_{++}$ is C^2-class and satisfies condition (A). If $\succsim$ is a p-transitive preference relation such that $f^{\succsim} = f^{\succsim^g}$, then $x \succ^g y$ implies $x \succ y$. If, in addition, the set $U(x,y) = \{z \in \Omega \cap \text{span}\{x,y\} | z \succsim x\}$ is closed in $\Omega \cap \text{span}\{x,y\}$ for any $x, y \in \Omega$, then $\succsim = \succsim^g$.

3 The How-to Manual and Several Remarks

We now explain the "cookbook" procedure for using the method in the last section. We separate it into three parts. Each part treats the computation procedure for specific purpose.

Procedure for Computing Utility Analytically

At first, we treats the method to compute utility analytically. Fix f and suppose f satisfies several good properties. By Theorem 1, we can compute utility analytically if

1)we can find a smooth inverse demand g, and
2)the ODE $\dot{y} = (g(y) \cdot x)v - (g(y) \cdot v)x$ can be solved analytically.

However, each condition 1) and 2) is rarely satisfied, and we cannot compute utility in many cases. Fortunately, condition 2) can be relaxed by using Corollary 1. Corollary 1 tells us that the right-hand side of above ODE can replace $\sigma(y,x,v)[g(y) \cdot x)v - (g(y) \cdot v)x]$ for suitable σ. So condition 2) can be relaxed;

2')we can find some suitable σ such that the ODE $\dot{y} = \sigma(y,x,v)[(g(y) \cdot x)v - (g(y) \cdot v)x]$ can be solved analytically.

Hereinafter, we introduce three examples and show how to use Theorem 1 and Corollary 1.

Example 1 (Cobb-Douglas type) Let $\alpha \in]0,1[$. We start from the following demand function;

$$x^1 = \alpha\frac{m}{p^1},\ x^2 = (1-\alpha)\frac{m}{p^2}.$$

At first, we calculate an inverse demand function. To put $p^2 = 1$, we get $x^2 = (1-\alpha)m$ and thus,

$$p^1 = \frac{\alpha x^2}{(1-\alpha)x^1}.$$

Hence we get an inverse demand function g;

$$\begin{aligned} g^1(x) &= \frac{\alpha x^2}{(1-\alpha)x^1}, \\ g^2(x) &= 1. \end{aligned}$$

Secondly, we construct the indifference curve y. Consider the following initial value problem;

$$\begin{cases} \dot{y}^1 = \sigma(y,x)g^2(y)(x^2 - x^1), \\ \dot{y}^2 = \sigma(y,x)g^1(y)(x^1 - x^2), \\ y^1(0) = x^1,\ y^2(0) = x^2. \end{cases}$$

Here we choose σ such that above problem can easily be solved. Let $\sigma(y,x) = (1-\alpha)y^1(x^2 - x^1)^{-1}$ if $x^1 \neq x^2$ and $\sigma(y,x) = 0$ otherwise. Then, if $x^1 \neq x^2$, we have

$$\begin{aligned} \dot{y}^1 &= (1-\alpha)y^1, \\ \dot{y}^2 &= -\alpha y^2. \end{aligned}$$

To solve this equation, we have

$$y^1(t) = \exp(\log x^1 + (1-\alpha)t),\ y^2(t) = \exp(\log x^2 - \alpha t).$$

Finally, we calculate the intersection of the trajectory of y with the diagonal. To solve $y^1(t) = y^2(t)$, we have

$$t = (\log x^2 - \log x^1),$$

and,

$$u^g_{(1,1)}(x) = y^1(t) = \exp(\alpha \log x^1 + (1-\alpha)\log x^2) = (x^1)^\alpha (x^2)^{1-\alpha},$$

which completes these calculations.[12] ■

Example 2 (CES type) Let $\alpha \in]0,1[, \rho < 1, \rho \neq 0$ and consider the following demand function;

$$x^1 = \frac{\alpha^{\frac{1}{1-\rho}}(p^1)^{\frac{-1}{1-\rho}}m}{\alpha^{\frac{1}{1-\rho}}(p^1)^{\frac{-\rho}{1-\rho}} + (1-\alpha)^{\frac{1}{1-\rho}}(p^2)^{\frac{-\rho}{1-\rho}}},\ x^2 = \frac{(1-\alpha)^{\frac{1}{1-\rho}}(p^2)^{\frac{-1}{1-\rho}}m}{\alpha^{\frac{1}{1-\rho}}(p^1)^{\frac{-\rho}{1-\rho}} + (1-\alpha)^{\frac{1}{1-\rho}}(p^2)^{\frac{-\rho}{1-\rho}}}.$$

[12] We omit the case $x^1 = x^2$, since clearly $u^g_{(1,1)}(x) = x^1 = (x^1)^\alpha(x^2)^{1-\alpha}$. To avoid complication, we continue to omit this case in succeeding examples.

To put $p^2 = 1$, we have

$$(p^1)^{\frac{-\rho}{1-\rho}} = \frac{(1-\alpha)^{\frac{1}{1-\rho}}}{\alpha^{\frac{1}{1-\rho}}x^2}[m - x^2].$$

Hence,

$$(p^1)^{\frac{-1}{1-\rho}} = \frac{(1-\alpha)^{\frac{1}{1-\rho}}x^1}{\alpha^{\frac{1}{1-\rho}}x^2},$$

and thus,

$$p^1 = \frac{\alpha(x^2)^{1-\rho}}{(1-\alpha)(x^1)^{1-\rho}}.$$

Therefore, we find an inverse demand function g such that

$$\begin{aligned} g^1(x) &= \frac{\alpha(x^2)^{1-\rho}}{(1-\alpha)(x^1)^{1-\rho}}, \\ g^2(x) &= 1. \end{aligned}$$

Next, consider the following initial value problem;

$$\begin{cases} \dot{y}^1 = \sigma(y,x)g^2(y)(x^2 - x^1), \\ \dot{y}^2 = \sigma(y,x)g^1(y)(x^1 - x^2), \\ y^1(0) = x^1,\ y^2(0) = x^2. \end{cases}$$

Here we choose σ such that above problem can easily be solved. Let $\sigma(y,x) = (x^2 - x^1)^{-1}(1-\alpha)(y^1)^{1-\rho}$. Then we have

$$\begin{aligned} \dot{y}^1 &= (1-\alpha)(y^1)^{1-\rho}, \\ \dot{y}^2 &= -\alpha(y^2)^{1-\rho}. \end{aligned}$$

To solve this equation, we get

$$\begin{aligned} y^1(t) &= [(x^1)^\rho + \rho(1-\alpha)t]^{\frac{1}{\rho}}, \\ y^2(t) &= [(x^2)^\rho - \rho\alpha t]^{\frac{1}{\rho}}. \end{aligned}$$

To solve $y^1(t) = y^2(t)$, we have

$$t = \frac{(x^2)^\rho - (x^1)^\rho}{\rho},$$

and thus,

$$u^g_{(1,1)}(x) = y^1(t) = [\alpha(x^1)^\rho + (1-\alpha)(x^2)^\rho]^{\frac{1}{\rho}},$$

which completes these calculations. ■

Example 3 (Linear utility type) Consider the following demand function;

$$f(p,m) = \begin{cases} \{x \in P | p \cdot x = m\} & \text{(if } p = c(\alpha, 1-\alpha) \text{ for some } c \in \mathbb{R}_{++}.) \\ \emptyset & \text{(otherwise.)} \end{cases}$$

Then we can easily get an inverse demand function g such that $g(y) \equiv (\alpha, 1-\alpha)$. Next, consider an initial value problem below;

$$\begin{cases} \dot{y}^1 = \sigma(y,x) g_2(y)(x^2 - x^1), \\ \dot{y}^2 = \sigma(y,x) g_1(y)(x^1 - x^2), \\ y^1(0) = x^1, \; y^2(0) = x^2. \end{cases}$$

Set $\sigma(y,x) \equiv 1$. Then we have $y^1(t) = x^1 + (1-\alpha)(x^2 - x^1)t$ and $y^2(t) = x^2 - \alpha(x^2 - x^1)t$. To solve $y^1(t) = y^2(t)$, we have $t = 1$ and thus,

$$u^g_{(1,1)}(x) = y^1(t) = \alpha x^1 + (1-\alpha)x^2,$$

which completes these calculations. ■

Procedure for Computing Utility Approximately

In many cases, suitable σ cannot be found and thus above computation method cannot be done. However, if we refrain from straining to get the rigorous value of u^g, we can ignore condition 2') and widely expand the applicability of this method.

Fix any $x, v \in \Omega$ and suppose these are linearly independent. To compute $u^g(x,v)$, we need to know the intersection of $L = \{cv | c \in \mathbb{R}\}$ and the trajectory of $y(\cdot; x, v)$. Suppose $w = (v \cdot x)v - \|v\|^2 x$. Then we can easily verify $w \cdot z = 0 \Leftrightarrow z \in L$ for any $z \in \text{span}\{x, v\}$, and $\dot{y}(t; x, v) \cdot w > 0$ for any t. Therefore, $y(t; x, v) \cdot w < 0$ for $t < t(x,v)$, $y(t; x, v) \cdot w > 0$ for $t > t(x,v)$ and $y(t; x, v) \cdot w = 0$ for $t = t(x,v)$.

Hence we can get $u^g(x,v)$ approximately by following steps;

i)compute $y(\cdot; x, v)$ by some method of difference approximation(For example, Euler method, Runge-Kutta method, etc.) and get $(y_t)_t$, where $y_0 = x$,
ii)compute t^* such that $y_{t^*} \cdot w \geq 0$ and $y_t \cdot w < 0$ for any t such that $0 < t < t^*$,
iii)compute $s^* \in [0,1]$ such that $[(1-s^*)y_{t^*} + s^* y_{t^*-1}] \cdot w = 0$ and let $u^g(x,v) \approx \dfrac{(1-s^*)y^1_{t^*} + s^* y^1_{t^*-1}}{v^1}$.

Also, we may be able to relax condition 1). We use the information of g at only step i) of above method, that is, step for computing the difference approximation $(y_t)_t$. So we need to find the value of $g(y)$ for only finitely many y.

By Proposition 1, if f satisfies several conditions, smooth g such that $g^n \equiv 1$ must be exists, and for any p with $p^n = 1$, $p = g(y)$ if and only if $f(p, p \cdot y) = y$. Therefore, we can find $g(y)$ by solving $f(p, p \cdot y) = y$.

Moreover, if we admit the approximation on g, condition 1) can be further relaxed. Our modified method is here;

i-0)let $y_0 = x$,
i-1)compute $g(y)$ for several y by some approximation method and y_1 for given $g(y)$,[13]
ii-1)calculate $y_1 \cdot w$. If $y_1 \cdot w < 0$, we go to i-2). Otherwise, we define $t^* = 1$ and go to iii),

$\vdots$

i-t)compute $g(y)$ for several y by some approximation method and y_t for given $g(y)$,
ii-t)calculate $y_t \cdot w$. If $y_t \cdot w < 0$, we go to i-t+1). Otherwise, we define $t^* = t$ and go to iii),

$\vdots$

iii)compute $s^* \in [0, 1]$ such that $[(1-s^*)y_{t^*} + s^* y_{t^*-1}] \cdot w = 0$ and let $u^g(x, v) \approx \dfrac{(1-s^*)y^1_{t^*} + s^* y^1_{t^*-1}}{v^1}$.

Procedures for Comparative Statics

In economics, people sometimes need an information of the derivatives of utility for using the method of comparative statics. However, if u^g cannot be solved analytically, these derivatives cannot be calculated. Meanwhile, if g is C^3-class and satisfies condition (B), then u^g_v is also C^3-class and thus we can get an estimate of its derivative by computing the following *central difference estimation*:

$$\frac{u^g_v(x + h(0, ..., 1, ..., 0)) - u^g_v(x - h(0, ..., 1, ..., 0))}{2h},$$

where h is sufficiently small real number.[14]

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[13]If we use simple Euler method, we need to compute $g(y_0)$ only. But other method may require the information of $g(y)$ for other y. For example, fourth-order Runge-Kutta method requires the information of $g(y)$ for three different y other than y_0.

[14]We must be careful not to choose too small h. If h is very small, the numerator of this fraction is also very small and thus your computer may count its value zero!

The Relationship between the Euler Method and Revealed Preference

Euler method is a famous difference approximation method of differential equation. Consider the following initial value problem:

$$\begin{aligned}\dot{y} &= f(y),\\ y(0) &= x.\end{aligned}$$

Define the sequence (y_t) as follows. At first, choose any $h > 0$. Next, define $y_0 = x$. Thirdly, we define y_t recursively by the following difference equation:

$$y_t = y_{t-1} + hf(y_{t-1}).$$

(y_t) is called the (explicit) *Euler difference approximation* of y. In fact, $y_t \approx y(ht)$ when f satisfies several conditions and $h > 0$ is sufficiently small.

Euler method has a very special feature in our computing method. Again consider the initial value problem (1). We can easily check that $\dot{y} \perp g(y)$, and thus, if (y_t) is a Euler difference approximation of (1), y_t must be revealed preferred to y_{t+1} for any t.

Remember our computation method. The error of its method comes from two causes. First cause is the rounding error, that is, the error comes from the fact computer cannot deal with real number directly. Second cause is the truncation error, that is, the error comes from the difference between $y(t)$ and its Taylor approximation. Above fact tells us the truncation error must be negative, whenever we use Euler method to approximate. So the approximation value of $u^g(x, v)$ must be less than the true value of $u^g(x, v)$ whenever the absolute value of rounding error is sufficiently small. Especially, we can believe $u^g(x, v) > 1$ and thus $x \succ^g v$ when its approximation is greater than 1.

For Demand Including Corner Solutions

Until now, we treat the consumption set $\Omega = \mathbb{R}^n_{++}$, that is, we ignore the existence of corner solutions. In some important cases, however, corner solutions must arise. Especially, if $\succsim$ is so-called *quasi-linear* preference, $f^{\succsim}$ must include corner solutions. So our previous method cannot treat such a case and thus we should expand our method.

However, it is very difficult. $\mathbb{R}^n_+$ is not a smooth manifold and thus f is usually not smooth when f has a corner solution. So the differential approach cannot applied directly. But our method can avoid this difficulty.

Example 4 (quasi-linear type) Let the consumption set Ω be $\mathbb{R}^2_+$ and consider the demand f corresponding with the utility $u(x^1, x^2) = \sqrt{x^1} + x^2$.

By easy computation, we have

$$f(p,m) = \begin{cases} \left(\frac{(p^2)^2}{(2p^1)^2}, \frac{1}{4p^1p^2}(4p^1m - (p^2)^2)\right) & (\text{if } m \geq \frac{(p^2)^2}{4p^1}), \\ \left(\frac{m}{p^1}, 0\right) & \text{otherwise.} \end{cases}$$

Also, by easy computation, we get

$$g(x) = \left(\frac{1}{2\sqrt{x^1}}, 1\right)$$

is an inverse demand of f, that is, $x = f(g(x), g(x) \cdot x)$ for any $x \in \Omega$ such that $x^1 \neq 0$.

Suppose $x \in \mathbb{R}^2_{++}$ and $x^1 \neq x^2$, and consider the following initial value problem;

$$\dot{y} = (x^2 - x^1)^{-1}[(g(y) \cdot x)(1,1) - (g(y) \cdot (1,1))x],$$
$$y(0) = x.$$

Then we can get

$$y^1(t; x, (1,1)) = x^1 + t,$$
$$y^2(t; x, (1,1)) = x^2 + \sqrt{x^1} - \sqrt{x^1 + t}.$$

Note that the domain of y can be regarded as $[-x^1, +\infty[$. So we can extend $\succsim^g$ as follows: let t^* be the unique t such that $y^2(t; x, (1,1)) = 0$ and define $z_+(x) = y(t^*; x, (1,1))$ and $z_*(x) = y(-x^1; x, (1,1))$. For any $z = (z^1, 0)$ such that $z^1 > 0$, there exists $x \in \mathbb{R}^2_{++}$ such that $z = z_+(x)$. Then we define $z \succsim^g y$ iff $x \succsim^g y$ and $y \succsim^g z$ iff $y \succsim^g x$. Similarly, for any $z = (0, z^2)$ such that $z^2 > 0$, there exists $x \in \mathbb{R}^2_{++}$ such that $z = z_*(x)$. Then we define $z \succsim^g y$ iff $x \succsim^g y$ and $y \succsim^g z$ iff $y \succsim^g x$. Finally, we define $x \succsim^g 0$ for any $x \in \mathbb{R}^2_+$. It is easy to show that $\succsim^g$ is a well-defined preference and $\sqrt{x^1} + x^2$ represents $\succsim^g$ on Ω.

Now, we extend the method of above example. Fix some $g : A \longrightarrow \mathbb{R}^n_{++}$ and suppose $\mathbb{R}^n_{++} \subset A \subset \mathbb{R}^n_+$. We suppose also g is C^2-class on $\mathbb{R}^n_{++}$, continuous on A and satisfies condition (A) and (B). Then we can define $u^g : (\mathbb{R}^n_{++})^2 \longrightarrow \mathbb{R}$ by the way of theorem 1. We will extend u^g in the following manner.

At first, fix $x \in \mathbb{R}^n_{++}$ and $v \in (\mathbb{R}^n_+ \setminus \mathbb{R}^n_{++} \cup \{0\})$. Let $z = x + v$. Then $z \in \mathbb{R}^n_{++}$ and $\{x, z\}$ is linearly independent. Now we consider the following initial value problem;

$$\dot{y} = (g(y) \cdot x)z - (g(y) \cdot z)x,$$

$$y(0) = x.$$

Let $y(\cdot; x, z)$ be the nonextendable solution of above problem, and the domain of y is $]a, b[$. Recall that $\dot{y} \cdot w > 0$ where $w = (z \cdot x)z - (z \cdot z)x$. So there exists

$$c^* = \lim_{t \uparrow b}(y(t; x, z) \cdot w) \in]0, +\infty].$$

If $c^* = +\infty$, we let $u^g(x, v) = +\infty$. Otherwise, we can easily verify that there exists $y^* = \lim_{t \uparrow b} y(t; x, z)$ and $c > 0$ such that $y^* = cv$. Then we define $u^g(x, v) = c$. Also, we define $u^g(v, x) = \inf\{c > 0 | u^g(cx, v) \geq 1\}$.[15]

Next, fix $x, v \in \mathbb{R}^n_+ \setminus (\mathbb{R}^n_{++} \cup \{0\})$. Let e denote the vector $(1, 1, ..., 1)$ and $c = u^g(x, e)$. If $c \neq 0$, we define $u^g(x, v) = u^g(ce, v)$. Suppose not. If $u^g(v, e) = 0$, we define $u^g(x, v) = 1$. Otherwise, we define $u^g(x, v) = +\infty$.

Thirdly, fix any $x \in \mathbb{R}^n \setminus \{0\}$. If $u^g(x, e) \neq 0$, then we define $u^g(x, 0) = +\infty$ and $u^g(0, x) = 0$. If not, we define $u^g(x, 0) = u^g(0, x) = 1$. Finally, we define $u^g(0, 0) = 1$. Then we get a definition of u^g on $(\mathbb{R}^n_+)^2$ into $\mathbb{R} \cup \{+\infty\}$. We define $\succsim^g = u^g([1, +\infty])$. Then the next theorem holds.

Theorem 2 $\succsim^g$ is transitive and monotone preference and $x \in f^{\succsim^g}(g(x), g(x) \cdot x)$ for any $x \in A$.[16]

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[15]There exists an alternative definition of $u^g(v, x)$; that is, $u^g(v, x) = \sup\{c \geq 0 | c = 0 \text{ or } u^g(cx, v) \leq 1\}$. Such extension of $\succsim^g$ may differ from our extension. See the next footnote for example.

[16]In fact, $\succsim^g$ may be discontinuous even if $n = 2$. Let

$$h_1(c) = \begin{cases} e^{-\frac{1}{c^2}} & (\text{if } c > 0), \\ 0 & (\text{otherwise}), \end{cases}$$

and

$$h(c) = 1 - \frac{1}{h_1(1)} h_1(1 - c) + \tan(\frac{\pi}{2} h_1(c - 2)).$$

Define $c : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}$ such that for any $x, y > 0$, $c(x, y)$ is the unique positive real number c such that

$$(x^{\frac{1}{1+\frac{1}{c}}} + y^{\frac{1}{1+\frac{1}{c}}})^{1+\frac{1}{c}} = h(c),$$

and define $g : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}^2_{++}$ such that,

$$g(x, y) = (x^{-\frac{1}{1+c(x,y)}}, y^{-\frac{1}{1+c(x,y)}}).$$

Then g is C^∞-class and satisfies condition (A) and (B), and c represents $\succsim^g$ on $\mathbb{R}^2_{++}$. However, $\succsim^g$ cannot have any transitive and continuous extension on $\mathbb{R}^2_+$, since $(0, 1)$ is the limit point of both $c^{-1}(1)$ and $c^{-1}(2)$. Especially, our extension method leads to a transitive and discontinuous extension of $\succsim^g$.

This example also shows that the recoverability of such problem cannot be achieved. In fact, there is infinitely many extension of $\succsim^g$ such that g is an inverse demand of it.

4 Concluding Remarks

Comparison with Related Literatures

In this paper, we based on the indirect approach of integrability theory and thus assumed the existence of smooth inverse demand. This assumption means, roughly speaking, a restriction of demand to what satisfies the rank condition of Slutsky matrix.

The method of Hurwicz and Uzawa(1971) differs from our method in this sense. They do not assume the existence of smooth inverse demand and obtain the *Expenditure function* directly from a PDE (partial differential equation) which is usually called *Shepherd's lemma*, and compute utility for its Expenditure function. To compare with our method, Hurwicz-Uzawa method can neglect the rank condition of Slutsky matrix, and thus there exists a demand to which our method cannot be applied but Hurwicz-Uzawa method can be applied.[17]

But our method also has several virtues which Hurwicz-Uzawa method does not have. Firstly, our method uses the ODE (ordinary differential equation). To treat ODE is much easier than PDE in general, our method is much easier to apply than Hurwicz-Uzawa method. Secondly, our method can treat the case demand has the corner solution. Hurwicz-Uzawa method, however, cannot treat most of such function since these demands usually have kinked Engel curves. For example, their method cannot treat our Example 4. Thirdly, our method can be applied even if the Slutsky matrix of given demand is not symmetric. Fourthly, our method can skip to check the symmetry of Slutsky matrix and condition (E) of Hurwicz and Uzawa(1971), even when f could satisfy these conditions. At last, our method can be applied even if f is the multi-valued demand.

Afriat(1967)'s approach radically differs from both our and Hurwicz-Uzawa method. His approach begins with not demand function but demand *data*, and calculates a utility corresponding with these demand data. To compare with our method, this approach can omit to search a demand hypothesis well matched to actual data.

However, Afriat's approach has a serious problem to use it for welfare analysis. Suppose there is only two demand data such that $((1,2),(\frac{1}{2},\frac{1}{4}))$ and $((2,1),(\frac{1}{4},\frac{1}{2}))$.[18] We can easily show that there is so many transitive and continuous preferences which supports these demand data. Especially, we cannot know whether $(1,2)$ is preferred than $(2,1)$ or not. In other words, Afriat's approach does not have the recoverability property, and thus the

[17] See section 9 of Hurwicz and Uzawa(1971).

[18] The data (x,p) corresponds to $x \in f(p,1)$.

difficulty of welfare analysis remains whenever people uses its approach.

Future Tasks

Two tasks remain.

Firstly, we discussed the computation method of preference from the demand including corner solutions, and derive $\succsim^g$ for initial g. But we do not find the appropriate sufficient condition of the existence of a continuous inverse demand on the range of demand which is smooth on interior of the domain. When demand does not have any corner solution, the answer of this question is in theorem 2. However, if demand has some corner value, the answer is still open.

Secondly, we assumed the existence of smooth inverse demand throughout this paper. Probably it is possible to extend our method for more general case. Suppose $n = 2$. If $\succsim$ is a transitive, continuous and strong monotone preference such that $f^{\succsim}$ is surjective, then any indifference curve of $\succsim$ is a *convex function* and thus it has a both left and right derivatives everywhere. So if we can recover its indifference curve from only the information on right derivatives, the applicability of our method will widely expand since this extension of our method no longer needs the smoothness of inverse demand. But we do not know how to construct such a method right now.

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