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EXISTENCE OF HOMOCLINIC SOLUTIONS FOR A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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ABSTRACT. Let $N\geq 2$ and $\mathcal{D}\subset \mathbb{R}^{N-1}$ be a bounded domain with smooth boundary. In this paper, we consider the existence of homoclinic solutions for nonlinear elliptic problem

$$\begin{cases} \Delta u+g(x,u) & =0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial\nu} & =0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\nu(x)$ is the outward pointing normal derivative to ∂D and $g\in C^{1}(\mathbb{R}\times \mathcal{D}, \mathbb{R}^{N})$ has a spacially periodicity.

1. INTRODUCTION

Let $N\geq 2$ and $\Omega\subset \mathbb{R}^{N}$ be a cylindrical domain, i.e., $\Omega=\mathbb{R}\times \mathcal{D}$, where $\mathcal{D}\subset \mathbb{R}^{N-1}$ is a bounded open domain with a smooth boundary. In the present paper, we consider the existence of homoclinic solutions of boundary value problem

$$\text{(P)} \qquad \begin{cases} \Delta u+g(x,u) & =0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial\nu} & =0 \quad \text{on } \partial\Omega, \end{cases}$$

where $g\in C^{1}(\mathbb{R}^{N}\times \mathbb{R}, \mathbb{R})$ and $\nu=\nu(y)$ denotes the outward pointing normal derivative to $\partial \mathcal{D}$. For $x\in\Omega$, we set $x=(x_{1}, y)$, where $x_{1}\in \mathbb{R}$ and $y\in \mathcal{D}$. We impose the following conditions on g :

(g1) $\quad g(x, z)\in C^{1}(\overline{\Omega}\times \mathbb{R}, \mathbb{R})$ and is 1−periodic with respect to x_{1};

(g2) $\quad G(x, z)=\int_{0}^{z}g(x, \tau)d\tau$ is 1−periodic with respect to z.

In [2] and [3], Rabinowitz considered the existence of spacially heteroclinic solutions of problem (P) under the assumptions (g1), (g2) and an additional condition

(g3) $\quad g(x, z)$ is even with respect to $x_{1}\in \mathbb{R}$.

In [5], the existence of the heteroclinic solutions of (P) was established without the evenness condition (g3). Recently, using the results in these papers, the existence of homoclinic solutions of (P) was established in [4].

The purpose of this paper is to investigate the existence of homoclinic solutions of (P) and give sharper characterizations of the solutions. We will show

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that there is a sequence of homoclinic solutions of (P) such that each solution is given as a local minimal of corresponding functional to (P).

2. Statement of Main Result

Throughout the rest of this paper, we assume that $N \geq 2$, and conditions (g1) and (g2) hold. For $x, y \in \mathbb{R}^N$, we denote by $x \cdot y$ the inner product of x and y. For each bounded open set $U \subset \mathbb{R}^n$, we denote by $\|\cdot\|_{H^1(U)}$ and $\|\cdot\|_{L^2(U)}$ the norm of $H^1(\Omega)$ and $L^2(\Omega)$ defined by $\|u\|^2_{H^1(U)} = \int_U |\nabla u|^2 \, dx$ and $\|v\|^2 = \int_U |v|^2 \, dx$ for each $u \in H^1(U)$ and $v \in L^2(U)$, respectively. We denote by $\langle \cdot, \cdot \rangle_U$ the inner product of $H^1(U)$. Put $\Omega_i = [i, i+1] \times \mathcal{D}$ for each $i \in \mathbb{Z}$. For each function $u : H^1_{loc}(\Omega) \longrightarrow \mathbb{R}$ and $m \in \mathbb{Z}$, we denote by $u[m]$ the restriction of u on $H^1_{loc}(\Omega_m)$. Let $v \in H^1_{loc}(\Omega)$ and $j \in \mathbb{Z}$. We denote by $\tau_j v$ the function defined by

$$\tau_t v(x_1, y) = v(x_1 - t, y) \qquad \text{for all } (x_1, y) \in \mathbb{R} \times \mathcal{D}.$$

We set

$$L(u)(x) = \frac{1}{2} |\nabla u(x)|^2 - G(x, u) \qquad \text{for } u \in H^1_{loc}(\Omega) \text{ and } x \in \Omega.$$

Put

$$I_i(u) = \int_{\Omega_i} L(u) dx \qquad \text{for } i \in \mathbb{Z} \text{ and } u \in H^1(\Omega_i)$$

and

$$E = \left\{ u \in H^1(\Omega_0) : u \text{ is 1-periodic in } x_1 \right\}.$$

We put

$$c_0 = \inf_{u \in E} I_0(u) \quad \text{and} \quad M_0 = \{ u \in E : I_0(u) = c_0 \}.$$

Then the following is known.

Proposition 1 ([3]). *$M_0 \neq \emptyset$ and M_0 is an ordered set, i.e. for each $u, v \in M_0$ with $u \neq v$, $u < v$ on Ω_0 or $u > v$ on Ω_0 holds.*

Here we put

$$a_j(u) = \int_{\Omega_j} L(u) dx - c_0 \qquad \text{for } j \in \mathbb{Z} \text{ and } u \in H^1(\Omega_j),$$

and

$$J_{l,m}(u) = \sum_{j=l}^{m} a_j(u) \quad \text{for } l, m \in \mathbb{Z} \text{ with } l \leq m.$$

We also put

$$J(u) = \liminf_{l \to -\infty} J_{l,0}(u) + \liminf_{m \longrightarrow \infty} J_{1,m}(u) \qquad \text{for } u \in H^1_{loc}(\Omega),$$

$$J_{-\infty,m}(u) = \liminf_{l \to -\infty} J_{l,0}(u) + J_{1,m}(u) \qquad \text{for } u \in H^1_{loc}(\Omega) \text{ and } m \geq 1,$$

$$J_{m,\infty}(u) = J_{m,0}(u) + \liminf_{l \to \infty} J_{1,l}(u) \qquad \text{for } u \in H^1_{loc}(\Omega) \text{ and } m \leq 0.$$

For each $v, w \in M_0$ with $v < w$, we set

$$[v, w] = \left\{ u \in H^1_{loc}(\Omega) : v \leq u \leq w \right\}, \quad [v, w]_m = \left\{ u|_{\Omega_m} : u \in [v, w] \right\},$$

$$\Gamma_-(z) = \left\{u \in [v,w] : J(u) < \infty, \|u - z\|_{L^2(\Omega_j)} \longrightarrow 0, \text{ as } j \longrightarrow -\infty\right\} \text{ for } z \in \{v,w\},$$

$$\Gamma_+(z) = \left\{u \in [v,w] : J(u) < \infty, \|u - z\|_{L^2(\Omega_j)} \longrightarrow 0, \text{ as } j \longrightarrow \infty\right\} \text{ for } z \in \{v,w\},$$

and

$$\Gamma(z_1, z_2) = \Gamma_-(z_1) \cap \Gamma_+(z_2) \qquad \text{for } z_1, z_2 \in \{v,w\}.$$

Let $v, w \in M_0$ and $v < w$. We assume v, w are adjacent minimizers in $H^1_{loc}(\Omega)$, that is there are no other minimizers u_0 with $v < u_0 < w$. We call $u \in H^1_{loc}(\Omega)$ a heteroclinic solution of (P) in $[v,w]$ if $u \in \Gamma(v,w)$ and u is a solution of (P). A solution $u \in H^1_{loc}(\Omega)$ of (P) is called a homoclinic solution in $[v,w]$ if $u \in \Gamma(v,v)$ or $u \in \Gamma(w,w)$.

We put

$$c(v,w) = \inf_{u \in \Gamma(v,w)} J(u), \quad \text{for } v, w \in M_0$$

and

$$\mathcal{M}(v,w) = \{u \in \Gamma(v,w) : J(u) = c(v,w)\} \qquad \text{for } v, w \in M_0.$$

Then we have

Proposition 2 ([2]). *For each $v, w \in M_0$ which are adjacent and $v < w$, $\mathcal{M}(v,w)$ is a nonempty ordered set.*

We will consider the existence of homoclinic solution of (P) under the following conditions:

(*) $v, w \in M_0$ are adjacent elements such that $v < w$.

(**) $\mathcal{M}(v,w), \mathcal{M}(w,v)$ have adjacent elements.

(C) $\inf\{I(v) : v \in H^1(\Omega_0)\} = c_0$.

It is known that under the condition (C), we have

Proposition 3 (cf. [4, 5]). *For each $v, w \in M_0$ and $u \in \Gamma(v,w)$, $\lim_{l \to -\infty} J_{l,0}(u)$ and $\lim_{m \to \infty} J_{1,m}(u)$ exists.*

Remark 1. *From Proposition 3, it follows that for each $u \in \Gamma_-(v)$*

$$J_{-\infty,m}(u) = \lim_{l \to -\infty} J_{l,0}(u) + J_{1,m}(u) \qquad \textit{for } m \geq 1.$$

Similarly, we have for each $u \in \Gamma_+(w)$,

$$J_{m,\infty}(u) = J_{m,0}(u) + \lim_{l \to \infty} J_{1,l}(u) \qquad \textit{for } m \leq 0.$$

We can now state our main result:

Theorem 1. *Assume that* (g*1*), (g*2*), $(*), (**)$ *and* (C) *hold. Let* $v_1, v_2 \in \mathcal{M}(v,w)$ *be adjacent with* $v_1 < v_2$. *Then there exist a positive integer* n_0 *and a sequence* $\{u_n\} \subset \Gamma(v,v)$ *of homoclinic solutions of* (P) *such that*

(1) $u_n \le u_{n+1}$ *for each* $n \ge 1$;
(2) $\tau_{-n_0-n+1}v_1[0] < u_n[0] < \tau_{-n_0-n}v_2[0]$ *for each* $n \ge 1$;
(3) $\lim_{n\longrightarrow\infty} J(u_n) = c(v,w) + c(w,v)$.

Remark 2. *The analogous result holds for* $\Gamma(w,w)$.

3. Sketch of Proof of Theorem 1.

In this section, we will show the sketch of the proof of Theorem 1. Detailed proof is given in [1].

Throughout the rest of this paper, we assume that (g1), (g2), $(*), (**)$, and (C) hold. By the assumption $(**)$, we have that there are $v_1, v_2 \in \mathcal{M}(v,w)$ and $w_1, w_2 \in \mathcal{M}(w,v)$ such that v_1, v_2 are adjacent with $v_1 < v_2$ and w_1, w_2 are adjacent with $w_1 < w_2$. In the following, we fix v_1, v_2, w_1 and w_2. We put

$$\mathcal{M}_m(v,w) = \{u[m] \in C(\Omega_m) : u \in \mathcal{M}(v,w)\} \qquad \text{for } m \in \mathbb{Z}.$$

Then we have that $\tau_{-1}\mathcal{M}_m(v,w) = \mathcal{M}_{m+1}(v,w)$ for $m \in \mathbb{Z}$. Let $m \in \mathbb{Z}$. Then since $\mathcal{M}(v,w)$ is an ordered set(cf. [2]), $\mathcal{M}_m(v,w)$ is also an ordered set. Since $v_1, v_2 \in \mathcal{M}(v,w)$ are adjacent, we have that $v_1[m]$ and $v_2[m]$ are adjacent in $\mathcal{M}_m(v,w)$ and $v_1[m] < v_2[m]$. One can see

$$\begin{aligned}(3.1)\qquad (\tau_n v_1)[m] <&(\tau_n v_2)[m] < (\tau_{n-1}v_1)[m]\\ &<(\tau_{n-1}v_2)[m] < (\tau_{n-2}v_1)[m] < (\tau_{n-2}v_2)[m]\end{aligned}$$

for $m, n \in \mathbb{Z}$. Similarly, we have

$$\begin{aligned}(3.2)\qquad (\tau_n w_1)[m] <&(\tau_n w_2)[m] < (\tau_{n+1}w_1)[m]\\ &<(\tau_{n+1}w_2)[m] < (\tau_{n+2}w_1)[m] < (\tau_{n+2}w_2)[m]\end{aligned}$$

for $m, n \in \mathbb{Z}$. We put

$$W(m) = \{u \in [v,w]_0 : (\tau_{-m}v_2)[0] \le u[0] \le (\tau_{-m-1}v_1)[0]\} \text{ for each } m \in \mathbb{Z}.$$

Then we find

$$u_1 < u_2 \qquad \text{for all } u_1 \in W(m) \text{ and } u_2 \in W(m+1).$$

As a direct consequence from the regularity argument for elliptic problem, we have the following lemma. We put

$$U(m) = [W(m) + \overline{B_{r_m}(0)}] \cap \{u \in [v,w]_0 : (\tau_{-m}v_1)[0] \le u[0] \le (\tau_{-m-1}v_2)[0]\},$$

where $B_r(0)$ is an open ball in $L^2(\Omega_0)$ centered at 0 with radius $r > 0$ and r_m is a positive number, and $\overline{B_r(0)}$ stands for the closure of $B_r(0)$ with respect to the $L^2(\Omega_0)$ norm. Then $U(m)$ is a closed convex set in $H^1(\Omega_0)$.

Lemma 1. *The sequence* $\{U(m)\}_{m\in\mathbf{Z}}$ *satisfies the following conditions:*
(i) For each $m \in \mathbb{Z}$

$$(3.3)\qquad U(m) \cap U(m+1) = \emptyset.$$

(ii) If u_1, u_2 *are solutions of* (P) *such that*

$$J(u_i) < 2[c(v,w) + c(w,v)] \qquad \text{for } i = 1,2,$$

and

$$u_1[0] \in U(m) \text{ and } u_2[0] \in U(m+1) \quad \text{for some } m \in \mathbb{Z},$$

then

$$\tau_{-m}v_1[0] < u_1[0] < \tau_{-m-1}v_2[0],$$
$$\tau_{-m-1}v_1[0] < u_2[0] < \tau_{-m-2}v_2[0] \quad \text{on } \Omega_0$$

and

$$u_1[0] < u_2[0] \quad \text{on } \Omega_0.$$

In the rest of this paper, we fix $\{U(m)\}_{m\in\mathbb{Z}}$ which satisfies the properties (i) and (ii) in Lemma 1. $U(m) \subset H^1(\Omega_0)$ for each $m \in \mathbb{Z}$. From the definition, we have that

Lemma 2. *There exists* $\varepsilon_1 > 0$ *such that for each* $u \in \Gamma_-(v)$ *such that* $u[0] \in \cup_{m \geq m_{v,1}} U(m)$ *and* $J(u) \leq c(v,w) + \frac{c(w,v)}{4}$,

$$\inf_{m \geq m_{v,1}} \|v - u\|^2_{L^2(\Omega_m)} \geq \varepsilon_1.$$

To show the existence of a sequence of homoclinic solutions, we consider the shift of $U(m)$. We put

$$U_n(m) = \{\tau_n v : v \in U(m)\} \qquad \text{for each } m, n \in \mathbb{Z}.$$

Then $U_n(m) \subset H^1(\Omega_n)$ for each $m, n \in \mathbb{Z}$.

Lemma 3. *For each* $n \geq m_{v,1}$, *there exist* $\delta_{v,1}(n) > 0$ *and* $m_{v,2}(n) > m_{v,1}$ *such that*

$$J_{-\infty,m}(u) \geq c(v,w) + \delta_{v,1}(n)$$

for all $m \geq m_{v,2}(n)$, $u \in \Gamma_-(v)$ *satisfying* $J(u) < \infty$, *and* $u[m_{v,1}] \in \partial U_{m_{v,1}}(n)$.

Lemma 4. *For each* $n \geq m_{v,1}$ *and* $\varepsilon > 0$, *there exists* $m_{v,3}(n,\varepsilon) > 0 > m_{v,2}(n)$ *such that* $m_{v,3}(n,\varepsilon) > m_{v,2}(n)$ *and*

$$J_{-\infty,m}(u) \geq c(v,w) - \varepsilon$$

for all $m \geq m_{v,3}(n,\varepsilon)$ *and* $u \in \Gamma_-(v)$ *with* $u[m_{v,1}] \in U_{m_{v,1}}(n)$.

We also consider w_1, w_2 which are adjacent pair elements in $\mathcal{M}(w,v)$. We put for each $m \in \mathbb{Z}$

$$\widetilde{W}(m) = \{u \in [v,w]_0 : (\tau_m w_2)[0] \leq u[0] \leq (\tau_{m+1}w_1)[0]\} \text{ for each } m \in \mathbb{Z}.$$

and set

$$\widetilde{U}(m) = [\widetilde{W}(m) + \overline{B_{r_m}(0)}] \cap \{u \in [v,w]_0 : (\tau_m w_1)[0] \leq u[0] \leq (\tau_{m+1}w_2)[0]\}.$$

By analogous arguments as in the proof of Lemma 1, Lemma ?? and Lemma 2, we have

Lemma 5. *There exists a sequence* $\left\{\widetilde{U}(m)\right\}_{m\in\mathbf{Z}}$ *of closed convex sets in* $L^2(\Omega_0)$ *satisfying the following conditions:*
(*i*) *For each* $m \in \mathbb{Z}$

$$\widetilde{U}(m) \cap \widetilde{U}(m+1) = \emptyset. \tag{3.4}$$

(*ii*) *If* u_1, u_2 *are solutions of* (P) *such that*

$$J(u_i) < 2[c(v,w) + c(w,v)] \qquad \text{for } i = 1,2,$$

and

$$u_1[0] \in \widetilde{U}(m) \text{ and } u_2[0] \in \widetilde{U}(m+1) \quad \text{for some } m \in \mathbb{Z},$$

then

$$\tau_m w_1[0] < u_1[0] < \tau_{m+1} w_2[0],$$
$$\tau_{m+1} w_1[0] < u_2[0] < \tau_{m+2} w_2[0] \quad \text{on } \Omega_0$$

and

$$u_1[0] < u_2[0] \quad \text{on } \Omega_0.$$

Lemma 6. (1) *There exist* $m_{w,1} > 0$ *such that for each* $u \in \Gamma(v,v)$ *with* $u[0] \in \cup_{m \geq m_{w,1}} \widetilde{U}(m)$,

$$J(u) > c(v,w) + \frac{c(w,v)}{2}. \tag{3.5}$$

(2) *For each* $n \geq m_{w,1}$, *there exists* $\delta_{w,1}(n) > 0$ *and* $m_{w,2}(n) > m_{w,1}$ *such that*

$$J_{-m,\infty}(u) \geq c(w,v) + \delta_{w,1}(n)$$

for all $m \geq m_{w,2}(n)$ *and* $u \in \Gamma_+(v)$ *with* $u[-m_{w,1}] \in \partial\widetilde{U}_{-m_{w,1}}(n)$.

Lemma 7. *For each* $n \geq m_{w,1}$ *and* $\varepsilon > 0$, *there exists* $m_{w,3}(n,\varepsilon) > m_{w,2}(n)$ *such that*

$$J_{-m,\infty}(u) \geq c(w,v) - \varepsilon$$

for all $m \geq m_{w,3}(n,\varepsilon)$ *and* $u \in \Gamma_+(v)$ *with* $u[-m_{w,1}] \in \widetilde{U}_{-m_{w,1}}(n)$.

Sketch of Proof of Theorem 1. Fix a positive integer $n_0 \geq \max\{m_{v,1}, m_{w,1}\}$. Fix $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{2}\min\{\delta_{v,1}(n_0), \delta_{w,1}(n_0)\},$$

where $\delta_{v,1}$ and $\delta_{w,1}$ are positive numbers obtained in Lemma 3 and Lemma 6. We fix $m = m(n_0) > \max\{m_{v,3}(n_0,\varepsilon), m_{w,3}(n_0,\varepsilon)\}$, where $m_{v,3}(n_0,\varepsilon)$ and $m_{w,3}(n_0,\varepsilon)$ are positive integers obtained in Lemma 4 and Lemma 7. Let

$$u_0 = \min\left\{\tau_{-n_0-1+m_{v,1}} v_1, \tau_{n_0+1+2m-m_{w,1}} w_1\right\}.$$

From the definition of v_1 and w_1, we find that

$$J(u_0) \longrightarrow c(v,w) + c(w,v), \quad \text{as } m \longrightarrow \infty. \tag{3.6}$$

Then by choosing $m \geq 1$ sufficiently large, we have that

$$J(u_0) < c_2(n_0) := c(v,w) + c(w,v) + \frac{\min\{\delta_{v,1}(n_0), \delta_{w,1}(n_0)\}}{2}.$$

Let $m_1 = m_{v,1}$ and $m_2 := m_2(n_0) := 2m - m_{w,1}$. We may assume, by choosing m sufficiently large, that $u_0[m_1] = \tau_{-n_0-1}v_1[m_1]$ and $u_0[m_2] = \tau_{n_0+1}w_1[m_2]$. Then we have

$$u_0[m_1] \in U_{m_1}(n_0) \text{ and } u_0[m_2] \in \widetilde{U}_{m_2}(n_0).$$

Here we put

$$\Gamma = \left\{u \in \Gamma(v,v) : J(u) \leq c_2(n_0), u[m_1] \in U_{m_1}(n_0) \text{ and } u[m_2] \in \widetilde{U}_{m_2}(n_0)\right\}.$$

Then since $u_0 \in \Gamma$, $\Gamma \neq \emptyset$. We put $\gamma = \inf_{z\in\Gamma} J(z)$ and $u \in \Gamma$ such that $J(u) = \gamma$. The existence of u can be proved by the same argument as before. Then to prove that u is a solution of (P), it is sufficient to show that $u[m_1] \notin \partial U_{m_1}(n_0)$ and $u[m_2] \notin \partial\widetilde{U}_{m_2}(n_0)$. By Lemma 3, we have that if $u[m_1] \in \partial U_{m_1}(n_0)$, then $J_{-\infty,m}(u) \geq c(v,w) + \delta_{v,1}(n_0)$. On the other hand, noting that

$$\tau_{-2m}u[-m_{w,1}] \in \widetilde{U}_{-m_{w,1}}(n_0),$$

we have by Lemma 7 that

$$\begin{aligned} J_{m+1,\infty}(u) &= J_{-m+1,\infty}(\tau_{-2m}u) \\ &\geq c(w,v) - \varepsilon \\ &\geq c(w,v) - \frac{\min\{\delta_{v,1}(n_0), \delta_{w,1}(n_0)\}}{2}. \end{aligned} \tag{3.7}$$

Then we have that $J(u) \geq c(v,w) + c(w,v) + \delta_{v,1}(n_0)/2$. This is a contradiction. Similarly, we find that $u[m_2] \notin \partial\widetilde{U}_{m_2}(n_0)$. Therefore we obtain that there exists a solution $u_1 \in \Gamma(v,v)$ such that

$$u_1[m_1] \in U_{m_1}(n_0) \text{ and } u_1[m_2] \in \widetilde{U}_{m_2}(n_0).$$

By the same way, we have that there exists a positive integer $m_2(n_0+1) > m_2(n_0)$ and a solution $u_2 \in \Gamma(v,v)$ such that

$$u_2[m_1] \in U_{m_1}(n_0+1) \text{ and } u_2[m_2(n_0+1)] \in \widetilde{U}_{m_2}(n_0+1).$$

That is

$$\tau_{-m_1}u_1[0] \in U(n_0) \quad \text{and} \quad \tau_{-m_1}u_2[0] \in U(n_0+1).$$

By Lemma ??, we find that

$$\tau_{-n_0}v_1[0] < \tau_{-m_1}u_1[0] < \tau_{-n_0-1}v_2[0] \quad \text{on } \Omega_0,$$

$$\tau_{-n_0-1}v_1[0] < \tau_{-m_1}u_2[0] < \tau_{-n_0-2}v_2[0] \quad \text{on } \Omega_0$$

and

$$\tau_{-m_1}u_1[0] < \tau_{-m_1}u_2[0] \text{ on } \Omega_0.$$

We prove $u_1 \leq u_2$. Since $z \leq \tau_{-n_0-1}v_1[0] < \tau_{-m_1}u_2[0]$ for all $z \in W(n_0)$, we find that

$$\|\min\{\tau_{-m_1}u_1[0], \tau_{-m_1}u_2[0]\} - W(n_0)\|_{L^2(\Omega_0)} \leq \|\tau_{-m_1}u_1[0] - W(n_0)\|_{L^2(\Omega_0)}.$$

Then by the definition of $U(n_0)$, we have

$$\min\{\tau_{-m_1}u_1[0], \tau_{-m_1}u_2[0]\} \in U(n_0).$$

Similarly, we find that

$$\max\{\tau_{-m_1}u_1[0], \tau_{-m_1}u_2[0]\} \in U(n_0+1).$$

By the same argument, we have

$$\min\{\tau_{-m_2}u_1[0], \tau_{-m_2}u_2[0]\} \in \widetilde{U}(n_0), \max\{\tau_{-m_2}u_1[0], \tau_{-m_2}u_2[0]\} \in \widetilde{U}(n_0+1).$$

Here we put

$$z_1 = \min\{u_1, u_2\} \qquad \text{and} \qquad z_2 = \max\{u_1, u_2\}.$$

Then by the argument above, we have

$$z_1[m_1] \in U_{m_1}(n_0) \text{ and } z_1[m_2] \in \widetilde{U}_{m_2}(n_0)$$

and

$$z_2[m_1] \in U_{m_1}(n_0+1) \text{ and } z_2[m_2] \in \widetilde{U}_{m_2}(n_0+1).$$

Then it follow that

$$J(z_1) \geq J(u_1),\ J(z_2) \geq J(u_2) \text{ and } J(z_1) + J(z_2) = J(u_1) + J(u_2).$$

This implies that z_1 is a minimizer of Γ, i.e., z_1 is a solution of (P). Therefore we find that $u_1 \leq u_2$. By repeating the argument above, we have a sequence $\{u_n\} \subset \Gamma(v, v)$ of solutions of (P) such that

$$u_n \in U_{m_1}(n_0 + n - 1) \text{ for each } n \geq 1$$

and

$$u_1 \leq u_2 \leq u_3 \leq \cdots$$

We also have

$$\tau_{-n_0-n+1}v_1[0] < u_n[0] < \tau_{-n_0-n}v_2[0] \quad \text{for all } n \geq 1.$$

This completes the proof. □

References

1. T. Amaishi and N. Hirano, *Existence of homoclinic solutions for a nonlinear elliptic boundary value problem*, submitted.
2. P. H. Rabinowitz, *Solutions of heteroclinic type for some classes of semilinear elliptic partical differential equatuions*, J. Fac. Sci. Tokyo **1** (1994), 525–550.
3. ———, *Spatially heteroclinic solutions for a semilinear elliptic pde*, Control, Optimization, and Caclulus of Variatoin **8** (2002), 915–932.
4. ———, *Homoclinics for a semilinear elliptic pde*, Comtemporary Math. **350** (2004), 209–232.
5. ———, *A new variational characterization of spatially heteroclinic solutions of a semilinear elliptic pde*, Discrete and Continuous Dynamical Systems **10** (2004), 507–515.

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