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On Lie algebras and triple systems -B_3-type case-

Noriaki Kamiya[1]
University of Aizu, Aizuwakamatsu, 965-8580, Japan
e-mail: kamiya@u-aizu.ac.jp

1 Introduction

In future, it is expected that triple systems will be useful for the characterization of noncommutative structures in mathematics and physics as well as that of (classical) Yang-Baxter equations ([9],[17],[20]).

Our aim is to use triple systems to investigate a characterization of differential geometry and mathematical physics from the viewpoint of nonassociative algebras that contain a class of Lie algebras or Jordan algebras ([7], [8], [14], [16], [17], [20]). Thus, in particular, for B_3-type Lie algebras, we will provide some examples of triple systems and their correspondence with extended Dynkin diagrams in this article.

A $(2\nu+1)$ graded Lie algebra is a Lie algebra of the form $g=\oplus_{k=-\nu}^{\nu}g_k$ such that $[g_k, g_l]\subset g_{k+l}$. It is well-known that 3-graded Lie algebras are essentially in bijection with certain theoretic objects called Jordan pairs. Kantor remarked that more general graded Lie algebras correspond to generalized Jordan triple systems. In particular, the graded Lie algebra

$$g_{-2}\oplus g_{-1}\oplus g_0\oplus g_1\oplus g_2$$

has the structure of a triple product on the subspace g_{-1}, and is known as a generalized Jordan triple system (GJTS) of second order or a (-1,1)-Freudenthal-Kantor triple system (F-K.t.s.)([7], [12]). Also $g_{-1}\oplus g_1$ has the structure of a Lie triple system (in particular, a system over a real number is known to correspond with a symmetric Riemannian space). We will discuss the corresponding geometrical object by means of these triple systems. To the notation and terminology used for the geometry, we can be found in ([4], [5], [19]). We will often use the symbols g and L to denote a Lie algebra or Lie superalgebra as is conventionally used ([2],[3],[6],[23]).

Speaking from the viewpoint of an algebraic study, our purpose is to propose a unified structural theory for triple systems in nonassociative algebras. In previous works ([11],[12],[13]), we have studied the Peirce decomposition

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of the GJTS U of second order by employing a tripotent element e of U (for a tripotent element, $\{eee\} = e$).
The Peirce decomposition of U is described as follows:

$$U = U_{00} \oplus U_{\frac{1}{2}\frac{1}{2}} \oplus U_{11} \oplus U_{\frac{3}{2}\frac{3}{2}} \oplus U_{-\frac{1}{2}0} \oplus U_{01} \oplus U_{\frac{1}{2}2} \oplus U_{13},$$

where $L(a) = \{eea\} = \lambda a$ and $R(a) = \{aee\} = \mu a$ if $a \in U_{\lambda\mu}$.

These viewpoints have formed the basis of our study on triple systems.

We are concerned with triple systems which have finite dimensionality over a field Φ of characteristic $\neq 2$ *or* 3, unless otherwise specified.

This note is an announcement of new results, and the details will be published elsewhere.

2 Definitions and Preamble

To make this paper as self-contained as possible, we first recall the definition of a generalized Jordan triple system of second order (hereafter, referred to as the GJTS of 2nd order), and the construction of Lie algebras associated with GJTS of 2nd order.

A vector space V over a field Φ, endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto \{xyz\}$, is said to be a *GJTS of 2nd order* if the following two conditions are satisfied:

$$\begin{array}{ll} (J1) & \{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}(GJTS) \\ (K1) & K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0(\text{2nd order}), \end{array}$$

where $L(a,b)c = \{abc\}$ and $K(a,b)c = \{acb\} - \{bca\}$.

Remark. If $K(a,b) \equiv 0$ (identically zero), then this triple system is a Jordan triple system (JTS), i.e., it satisfies the relations $\{acb\} = \{cba\}$ and $GJTS$..

We can also generalize the concept of the GJTS of 2nd order as follows (for examples, see [7],[8],[10],[14] and the references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, if the triple product satisfies

$$\begin{array}{l} (ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \\ K(K(a,b)c,d) - L(d,c)K(a,b) + \varepsilon K(a,b)L(c,d) = 0, \end{array}$$

where $L(x,y)z = (xyz)$ and $K(a,b)c = (acb) - \delta(bca)$, then it is said to be a (ε,δ)-*Freudenthal-Kantor triple system* (hereafter abbreviated as (ε,δ)-F-K.t.s).

Furthermore, if the (ε,δ)-F-K.t.s) satisfies

$$dim_{\Phi}\{K(a,b)\}_{span} = 1,$$

then it is said to be *balanced.*

Remark. We set $S(x,y) := L(x,y) + \varepsilon L(y,x)$, and $A(x,y) := L(x,y) - \varepsilon L(y,x)$, then this $S(x,y)$ (resp. $A(x,y)$) is a derivation (resp. anti-derivation) of $U(\varepsilon,\delta)$.

We generally denote the triple products by $\{xyz\}$, (xyz), $[xyz]$, and $< xyz >$. Bilinear forms are denoted by $< x|y >$, (x,y), and $B(x,y)$.

Remark. Note that the concept of a GJTS of 2nd order coincides with that of $(-1,1)$-F-K.t.s. Thus we can construct simple Lie algebras or superalgebras by means of the standard embedding method (for example, [2], [3], [7]–[11], [13], [14], [15], [21]).

Proposition 1([8],[15]). *Let $U(\varepsilon,\delta)$ be an (ε,δ)-F-K.t.s. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J < xyz > = < JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta),[xyz])$ is a Lie triple system (the case of $\delta = 1$) or an anti-Lie triple system (the case of $\delta = -1$) with respect to the product*

$$[xyz] := < xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > .$$

Corollary. *Let $U(\varepsilon,\delta)$ be an (ε,δ)-F-K.t.s. Then the vector space $T(\varepsilon,\delta) = U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a Lie triple system (the case of $\delta = 1$) or an anti-Lie triple system (the case of δ = -1) with respect to the triple product defined by*

$$[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}.$$

Thus we can obtain the standard embedding Lie algebra (the case of $\delta = 1$) or Lie superalgebra (the case of $\delta = -1$), $L(\varepsilon,\delta) = D(T(\varepsilon,\delta),T(\varepsilon,\delta)) \oplus T(\varepsilon,\delta)$, associated with $T(\varepsilon,\delta)$, where $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$. That is, these vector spaces $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ and $T(\varepsilon,\delta)$ imply

$$D(T(\varepsilon,\delta),T(\varepsilon,\delta)) := \begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix}_{span}, \text{ and}$$

$$T(\varepsilon,\delta) := \{\begin{pmatrix} x \\ y \end{pmatrix} |x,y \in U(\varepsilon,\delta)\}_{span}.$$

In fact, we have

$$L_0 = \{\begin{pmatrix} L(a,b) & 0 \\ 0 & \varepsilon L(b,a) \end{pmatrix}\}_{span} = \{L(a,b)\}_{span},$$

$L_{-2} = \{\begin{pmatrix} 0 & \delta K(c,d) \\ 0 & 0 \end{pmatrix}\}_{span} = \{K(c,d)\}_{span}$and $L_0 = DerU \oplus Anti-DerU.$

Remark. For the standard embedding algebras obtained from these triple systems, note that $L(\varepsilon,\delta) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_{-1} \oplus L_{-2}$ (or $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$) is a 5-graded Lie algebra or Lie superalgebra, such that $L_{-1} = g_{-1} = U(\varepsilon,\delta) and\ DerT(U) := D(T(\varepsilon,\delta), T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_{-2}$ with $[L_i, L_j] \subseteq L_{i+j}$.

By straightforward calculations, for the correspondence of the (1,1) balanced F.K.t.s with the (-1,1) balanced F.K.t.s, we obtain the following.

Proposition 2. *Let* $(U, < xyz >)$ *be a* $(1,1)$ *F-K.t.s. If there is an endomorphism* J *of* U *such that* $J < xyz > = < JxJyJz >$ *and* $J^2 = -Id$, *then* $(U, \{xyz\})$ *is a GJTS of 2nd order (that is, (-1,1)-F-K.t.s.) with respect to the new product defined by* $\{xyz\} := < xJyz >$.

We now give an explicit example of a JTS and a Lie triple system.

Example. Let U be a vector space with a symmetric bilinear form $< , >$. Then the triple system $(U, [xyz])$ is a Lie triple system with respect to the product

$$[xyz] = < y,z > x - < z,x > y.$$

That is, this triple system is induced from the JTS

$$\{xyz\} = \frac{1}{2}(< x,y > z + < y,z > x - < z,x > y),$$

by means of

$$[xyz] = \{xyz\} - \{yxz\}.$$

3 Construction of B_3-type Lie algebras from several triple systems

In this section, we will discuss the construction of simple B_3-type Lie algebras associated with several triple systems (the details will be described in a future paper).

a) the case of a JTS,

b) the case of a balanced GJTS,

c) the case of a GJTS of 2nd order,

d) the case of a derivation induced from a JTS.

To consider these cases, we will start with an extended Dynkin diagram for the B_3-type Lie algebra.

$$\begin{array}{ccccc} 1 & & 2 & & 2 \\ \circ & \cdots & \circ & => & \circ \\ & & | & & \\ & & \circ & & \\ & & -\rho & & \end{array}$$

where $-\rho = \alpha_1 + 2\alpha_2 + 2\alpha_3$.

For the root system, it is well known that

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$.

3.1 The case of a JTS

First we study the case of $g_{-1} = U = Mat(1, 5; \Phi)$. (Hereafter, we assume $\Phi = C$.)

In this case, g_{-1} is a JTS respect to the product

$$\{xyz\} = x^t yz + y^t zx - z^t xy,$$

where ${}^t x$ denotes the transpose matrix of x.

By straightforward calculations, the standard embedding Lie algebra $L(U) = g$ can be shown to be 3-graded B_3-type Lie algebra with $g_{-1} \oplus g_0 \oplus g_1$. Thus, we have

$$\begin{aligned} g_0 &= DerU \oplus Anti - DerU \\ &= B_2 \oplus \Phi H, \ where\ H := \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} \end{aligned}$$

$$Der(g_{-1} \oplus g_1) \cong \{\circ \cdots \circ \Longrightarrow \circ\} = B_3, \quad (\odot\ omitted).$$

$$\begin{array}{ccccc} 1 & & 2 & & 2 \\ \odot & \cdots & \circ & => & \circ \\ & & | & & \\ & & \circ & & \\ & & -\rho & & \end{array}$$

Furtheremore, we obtain

$$\begin{aligned} DerU &= \{L(x,y) - L(y,x)\}_{span} = B_2, \\ Anti - DerU &= \{L(x,y) + L(y,x)\}_{span} = \Phi H, \end{aligned}$$

$$g_0 = \{\begin{pmatrix} L(x,y) & 0 \\ 0 & -L(y,x) \end{pmatrix}\}_{span} = \{S(x,y) + A(x,y)\}_{span},$$

where

$$S(x,y) = L(x,y) - L(x,y), A(x,y) = L(x,y) + L(y,x).$$

Here, g_{-1} corresponds to the root system

$$\{\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+2\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3\}$$

3.2 The case of a balanced GJTS

Second we study the case of $g_{-1} = U = Mat(2,3;\Phi)$.

In this case, g_{-1} is a balanced GJTS of 2nd order w.r.t. the product

$$\{xyz\} := z^tyx + x^tyz - zJ_3\,{}^txyJ_3,$$

where $J_3 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$.

By straightforward calculations, it can be shown that $L(U) = g$ is a 5-graded B_3-type Lie algebra with $g_{-2} \oplus \cdots \oplus g_2$ and $dim\ g_{-2} = 1$. Thus, we have

$$g_0 = DerU \oplus Anti - DerU = A_1 \oplus A_1 \oplus \Phi H,\ where\ H := \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}$$
$$Der(g_1 \oplus g_1) = g_{-2} \oplus g_0 \oplus g_2 = A_1 \oplus A_1 \oplus A_1(\odot\ omitted) \cong DerT(U)$$

$$\begin{array}{ccccc} 1 & & 2 & & 2 \\ \circ & \cdots & \odot & => & \circ \\ & & | & & \\ & & \circ & & \\ & & -\rho & & \end{array}$$

Furthermore we obtain

$$g_{-2} = \{K(x,y)\}_{span} = \Phi Id \cdots \text{ which is one dimensional,}$$

i.e., balanced. This g_{-1} corresponds to the root system

$$\{\alpha_2,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_2+\alpha_3,\ \alpha_2+2\alpha_3,\ \alpha_1+\alpha_2+2\alpha_3,\}$$

g_{-2} corresponds to the highest root

$$\{\alpha_1+2\alpha_2+2\alpha_3\},$$

and $g/(g_{-2} \oplus g_0 \oplus g_2) \cong T(= g_{-1} \oplus g_1)$ is the tangent space of a quaternion symmetric space of dimension 12, since T is a Lie triple system associated with g_{-1}.

3.3 The case of a GJTS of 2nd order

Third we study the case of $g_{-1} = U = Mat(1,3;\Phi)$.

In this case, g_{-1} is a GJTS of 2nd order with respect to the product

$$\{xyz\} = x^t yz + z^t yx - y^t xz.$$

By straightforward calculations, it can be shown that $L(U)$ is a 5-graded B_3-type Lie algebra with $g_{-2} \oplus \cdots \oplus g_2$ and $dim\ g_{-2} = 3$,

$$\begin{aligned} g_0 &= DerU \oplus Anti - DerU = A_2 \oplus \Phi H, \ where\ H := \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} \\ Der(g_{-1} \oplus g_1) &= g_{-2} \oplus g_0 \oplus g_2 = A_3(\odot\ omitted) \ \cong DerT(U). \end{aligned}$$

$$\begin{array}{ccccc} 1 & & 2 & & 2 \\ \circ & \cdots & \circ & => & \odot \\ & & | & & \\ & & \circ & & \\ & & -\rho & & \end{array}$$

Furthermore, we obtain

$$g_{-2} = \{K(x,y)\}_{span} = Alt(3,3;\Phi).$$

That is, the triple system $g_{-1}(resp.\ g_{-2})$ corresponds to th the root system

$\{\alpha_3,\ \alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3\}(resp.\ \{\alpha_2+2\alpha_3,\ \alpha_1+\alpha_2+2\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3\})$,

implying that

$\underbrace{\circ\cdots\cdots\circ}\cdots \Longrightarrow \odot$ ($\odot$ omitted) and

$$g_0 = A_2 \oplus \Phi H.$$

Remark. Following [18], for the case of a GJT of 2nd order, note that $g_{-2}(\cong \mathbf{k})$ has the structure of the JTS associated with a GJTS of 2nd order.

3.4 The case of a derivation induced from a JTS

Finally, we study the case of $g_{-1} = U = Mat(1,7;\Phi)$.

In this case g_{-1} is a JTS with respect to the product

$$\{xyz\} = x^t yz + y^t zx - z^t xy.$$

For this case, we obtain

$$\begin{aligned} DerU = & \quad \{L(x,y) - L(y,x)\}_{span} = Alt(7,7;\Phi) \quad \cong B_3, \\ Anti - DerU \ \cong & \ \ \Phi H \text{ (which is one dimensional)}. \end{aligned}$$

The standard embedding Lie algebra is a 3-graded B_4-type Lie algebra with $g_{-1} \oplus g_0 \oplus g_1$.

Furthermore, we have

$$\odot \cdots \underbrace{\circ \cdots \circ \Longrightarrow \circ} \ (\odot \text{ omitted})$$

$$g_0 = B_3 \oplus \Phi H.$$

This case is obtained from $DerU$ such that $U = Mat(1,7;\Phi)$ with the JTS structure.

Remark. In the above constructions, note that there exist four different constructions for the B_3-type Lie algebras. It appears that these results may be applicable to mathematical physics, for example, quark theory and gravity theory.

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