数理解析研究所講究録
第 1656 巻 2009 年 1-6

Brauer-Schur functions and compound bases

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1 Introduction

The aim of this talk is to introduce a compound basis for the space of symmetric functions. This is based on a joint work with Kazuya Aokage (Okayama University) and Hiroshi Mizukawa (National Defense Academy).

Fixing an arbitrary prime number p, we construct a basis consisting of products of Schur functions and Brauer-Schur functions. The basis elements are indexed by the partitions. It is well known that the Schur functions form an orthonormal basis for our space. A natural question arises. How are these two bases connected ? In this talk I present some numerical results on the transition matrix for these bases. In particular we will see that the determinant of the transition matrix is a power of p. The explicit formulas for the determinants and for the elementary divisors involve an interesting combinatorial feature.

Our compound basis comes from the twisted homogeneous realization of the basic representation of the affine Lie algebra $A_1^{(1)}$ ([2]). Also an expression of rectangular Schur functions in terms of the compound basis is given in [3].

This is a supplement to our previous note [1].

2 Space of symmetric functions

Throughout this note V denotes the space of polynomials in infinitely many variables:

$$V = \mathbb{Q}[t_j; j \geq 1] = \bigoplus_{n=0}^{\infty} V(n).$$

Here $V(n)$ denotes the space of homogeneous polynomials of degree n, subject to deg $t_j = j$. The space V can be regarded as the ring of symmetric functions by identifying $t_j = \frac{1}{j}(x_1^j + x_2^j + \cdots)$, where x_k's are the "original" variables.

A typical basis for V is that consisting of the Schur functions. Let $P(n)$ denote the set of the partitions of n. For $\lambda \in P(n)$, the Schur function $S_\lambda(t)$ indexed by λ is defined by

$$S_\lambda(t) = \sum_{\rho \in P(n)} \chi_\rho^\lambda \frac{t_1^{m_1} t_2^{m_2} \cdots}{m_1! m_2! \cdots} \quad \in V(n).$$

Here the summation runs over all $\rho = (1^{m_1}2^{m_2}\cdots) \in P(n)$, and the integer χ_ρ^λ is the irreducible character of λ of the symmetric group $\mathfrak{S}_n$, evaluated at the conjugacy class ρ. It is known that these Schur functions are orthonormal with respect to the inner product

$$\langle F, G\rangle = F(\partial)G(t)|_{t=0},$$

where $\partial = (\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \cdots)$. By this orthogonality, $\{S_\lambda(t); \lambda \in P(n)\}$ forms an orthonormal basis for the space $V(n)$.

3 Compound bases

In the rest of the note, we always fix a prime number p. A partition $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ is said to be "p-regular" if there are no i's such that $\lambda_i = \cdots = \lambda_{i+p-1}$. The set of all p-regular partitions of n is denoted by $P^r(n)$. A partition $\lambda = (1^{m_1}\cdots n^{m_n})$ is said to be "p-class regular" if $m_{pk} = 0$ for any $k \geq 1$. The set of all p-class regular partitions of n is denoted by $P^{cr}(n)$. It is well known that these two sets have the same cardinality. In fact, there is a natural bijection

$$G : P^r(n) \longrightarrow P^{cr}(n)$$

defined as follows. Let $\lambda = (\lambda_1, \cdots \lambda_\ell)$ be p-regular. If $\lambda_i = pk$, a positive multiple of p, then replace λ_i by $(k, \cdots, k)$, a p-repetition of k. Repeat this process to get a p-class regular partition $\widetilde{\lambda}$. For example, if $p = 2$ and $\lambda = (6, 4)$, then $\widetilde{\lambda} = (3, 3, 1, 1, 1, 1)$. It is easily observed that $\ell(\widetilde{\lambda}) - \ell(\lambda)$ is divisible by $p - 1$ for any $\lambda \in P^r(n)$. This map G is called the p-Glaisher map.

For a partition $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ of n, partitions λ^r and λ^q are defined in the following way. The multiplicities $m_i(\lambda^r)$ and $m_i(\lambda^q)$ of the number i are given respectively by

$$m_i(\lambda^r) = k \quad \text{if} \quad m_i(\lambda) \equiv k \pmod p$$

and

$$m_i(\lambda^q) = \frac{m_i(\lambda) - k}{p} \quad \text{if} \quad m_i(\lambda) \equiv k \pmod p.$$

For example, if $p = 3$ and $\lambda = (5^3 4^4 2^{11} 1^2) \in P^r(55)$, then $\lambda^r = (42^2 1^2) \in P^r(10)$ and $\lambda^q = (542^3) \in P(15)$. This gives a bijection

$$\beta : P(n) \longrightarrow \bigcup_{n_0 + pn_1 = n} P^r(n_0) \times P(n_1).$$

By the theory of modular representations of the symmetric group $\mathfrak{S}_n$, the irreducible p-modular representations are indexed by the set $P^r(n)$. Let φ_ρ^λ be the Brauer character value of the irreducible representation $\lambda \in P^r(n)$, evaluated at the p-regular conjugacy class $\rho \in P^{cr}(n)$. This is an integer. One finds Brauer character tables $\Phi_n^{(p)} = (\phi_\rho^\lambda)_{\lambda,\rho}$ for some small p and n in [4]. For example, the table for $p = 2$ and $n = 5$ looks

$$\Phi_5^{(2)} = \begin{array}{c|ccc} & (1^5) & (1^2 3) & (5) \\ \hline (5) & 1 & 1 & 1 \\ (41) & 4 & 1 & -1 \\ (32) & 4 & -2 & -1 \end{array}$$

Our "Brauer-Schur function" $B_\lambda(t)$ for $\lambda \in P^r(n)$ is defined by

$$B_\lambda(t) = \sum_{\rho \in P^{cr}(n)} \varphi_\rho^\lambda \frac{t_1^{m_1} t_2^{m_2} \cdots}{m_1! m_2! \cdots}$$

Here the summation runs over all $\rho = (1^{m_1} 2^{m_2} \cdots) \in P^{cr}(n)$. We set $V^{(p)}(n) = V^{(p)} \cap V(n)$, where

$$V^{(p)} = \mathbb{Q}[t_j; j \geq 1, j \not\equiv 0 \, (\operatorname{mod} p)\}.$$

Then the Brauer-Schur functions $\{B_\lambda(t); \lambda \in P^r(n)\}$ form a basis for $V^{(p)}(n)$. In general, they are not orthogonal with respect to the inner product

$$\langle F, G \rangle = F(\partial) G(t)|_{t=0}.$$

A dual basis is obtained by using the "projective covers" of irreducible representations ([6]).

In view of the bijection β, we define, for $\lambda \in P(n)$,

$$W_\lambda(t) = B_{\lambda^r}(t) S_{\lambda^q}(t_{(p)}),$$

where $t_{(p)} = (t_p, t_{2p}, t_{3p}, \cdots)$. The functions $\{W_\lambda(t); \lambda \in P(n)\}$ are linearly independent and form a basis for the space $V(n)$. We call this the "compound basis".

4 Transition matrices

For a fixed prime number p, let $A_n^{(p)} = (a_{\lambda\mu})$ be the transition matrix between two bases, defined by

$$S_\lambda(t) = \sum_{\mu \in P(n)} a_{\lambda\mu} W_\mu(t)$$

for $\lambda \in P(n)$. We give matrices of the cases $(p, n) = (3,3), (3,4), (3,5)$ and $(3,6)$.

$$A_3^{(3)} = \begin{array}{c|ccc} & 3 & 21 & (\emptyset,1) \\ \hline (3) & 1 & 0 & 1 \\ (21) & 1 & 1 & -1 \\ (1^3) & 0 & 1 & 1 \end{array}$$

$$A_4^{(3)} = \begin{array}{c|ccccc} & 4 & 31 & 2^2 & 21^2 & (1,1) \\ \hline (4) & 1 & 0 & 0 & 0 & 1 \\ (31) & 0 & 1 & 0 & 0 & 0 \\ (2^2) & 1 & 0 & 1 & 0 & -1 \\ (21^2) & 0 & 0 & 0 & 1 & 0 \\ (1^4) & 0 & 0 & 1 & 0 & 1 \end{array}$$

$$A_5^{(3)} = \begin{array}{c|ccccccc} & 5 & 41 & 32 & 31^2 & 2^21 & (2,1) & (1^2,1) \\ \hline (5) & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ (41) & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ (32) & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ (31^2) & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ (2^21) & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ (21^3) & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ (1^5) & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

$$A_6^{(3)} = \begin{array}{c|ccccccccccc} & 6 & 51 & 42 & 3^2 & 41^2 & 321 & 2^21^2 & (3,1) & (21,1) & (\emptyset,2) & (\emptyset,1^2) \\ \hline (6) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ (51) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ (42) & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (3^2) & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ (41^2) & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ (321) & 1 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\ (2^21^2) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ (2^3) & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ (31^3) & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ (21^4) & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ (1^6) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}$$

Here the columns are labeled by the pairs (μ^r, μ^q). The partition μ^r indexing column means $(\mu^r, \emptyset)$. The minor matrix consisting of such columns that $\mu^q = \emptyset$ is nothing but the decomposition matrix $D_n^{(p)}$ of the symmetric group $\mathfrak{S}_n$ at characteristic p.

One verifies that $A_n^{(p)}$ is an integral matrix and

$$|\det A_n^{(p)}| = p^{k_n},$$

where $k_n = \sum_{\lambda \in P(n)} \ell(\lambda^q)$.

A new result which is not written in [1] is as follows. The elementary divisors of the matrix $A_n^{(p)}$ are given by

$$\{p^{\frac{\ell(\widetilde{\lambda^r})-\ell(\lambda^r)}{p-1}}; \quad \lambda \in P(n)\}.$$

Here $\widetilde{\lambda^r}$ denotes the image of the partition λ^r by p-Glaisher map G. From this result we have another formula for the determinant of $A_n^{(p)}$.

$$k_n = \sum_{\lambda \in P(n)} \frac{\ell(\widetilde{\lambda^r}) - \ell(\lambda^r)}{p-1}.$$

The elementary divisors of the matrices $A_3^{(3)}$, $A_4^{(3)}$, $A_5^{(3)}$ and $A_6^{(3)}$ are, respectively, $\{1,1,3\}$, $\{1,1,1,1,3\}$, $\{1.1.1,1,1,3,3\}$ and $\{1,1,1,1,1,1,1,3,3,3,9\}$.

Finally we mention about an orthogonality of the matrices $A_n^{(p)}$. The matrix ${}^tA_n^{(p)}A_n^{(p)}$ is block diagonal, each block labeled by the pair (n_0, n_1). Let $B_{n_0,n_1}^{(p)}$ be the block corresponding to (n_0, n_1). Then the determinant is given by

$$|\det B_{n_0,n_1}^{(p)}| = p^{\Delta_{n_0,n_1}},$$

where

$$\Delta_{n_0,n_1} = \sum_{(\lambda^r,\lambda^q) \in P^r(n_0) \times P(n_1)} \left(\frac{\ell(\tilde{\lambda^r}) - \ell(\lambda^r)}{p-1} + \ell(\lambda^q) \right).$$

The "principal block" $B_{n,0}^{(p)}$ is nothing but the Cartan matrix for $\mathfrak{S}_n$ at characteristic p ([7]).

References

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