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On C_2-cofiniteness of $\mathbb{Z}_2$-orbifold models of vertex operator algebras[1]

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1 Introduction

The notion of C_2-cofiniteness of vertex algebras has recently been become very important in the representation theory of vertex operator algebra. The C_2-cofiniteness property is a finite codimensionality of a particuler subspace of vertex operator algebra and follows a lot of other finiteness properties (see [M] for example).

The final aim of the work is to show that the following conjecture " any orbifold model of a simple, C_2-cofinite vertex operator algebra is C_2-cofinite". For this purpose, as a first step, we experimentally consider the case of commutative vertex algberas. In commutative case, it seems not to be natural to assume that a vertex algebras is simple. Then we have an example of C_2-cofinite commutative vertex algebra whose $\mathbb{Z}_2$-orbifold model is not C_2-cofinite. We give a criterion for the C_2-cofiniteness of $\mathbb{Z}_2$-orbifold models of C_2-cofinite, finitely generated commutative vertex algebra.

2 Vertex algebras and some notions

A vertex algebra is a triple $(V, Y(\,\cdot\,,z), \mathbf{1})$ of a vector space over $\mathbb{C}$, a linear map $Y(\,\cdot\,,z) : V \mapsto \operatorname{End} V[[z,z^{-1}]]$ and a distinguished vector $\mathbf{1}$ called a vacuum vector, where $\operatorname{End} V[[z,z^{-1}]]$ is a formal integral power series of z with $\operatorname{End} V$ as coefficients. For $a \in V$, we write $Y(a,z) = \sum_{m\in\mathbb{Z}} a_{(m)} z^{-n-1}$ where the coefficients $a_(m) \in \operatorname{End} V$. We may regards the map $V \times V \ni (a,b) \mapsto a_{(m)}b \in V$ with $a,b \in V$ and $m \in \mathbb{Z}$ as a bilinear multiplication on V. Then the following is satisfied:

(1) For any $a,b \in V$, $a_{(n)}b = 0$ for sufficiently large integer n.

(2) (Borcherds identity) For any $a,b \in V$,

$$\begin{aligned}
&\sum_{i=0}^{\infty} \binom{q}{i} (a_{(p+i)}b)_{(q+r-i)}c \\
&\quad = \sum_{i=0}^{\infty} (-1)^i \binom{p}{i} (a_{(p+q-i)}b_{(r+i)}c - (-1)^p b_{(p+r-i)}a_{(q+i)})c.
\end{aligned} \tag{2.1}$$

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(3) $\mathbf{1}_{(n)} = \delta_{n,-1}\mathrm{id}_V$ for $n \in \mathbb{Z}$.

We canonically have a linear map $D : V \ni a \mapsto a_{(-2)}\mathbf{1} \in V$. The linear map D satisfies the following identities;

$$(Da)_{(m)} = -ma_{(m-1)} \quad \text{for } a \in V, m \in \mathbb{Z}, \tag{2.2}$$

$$D(a_{(m)}b) = (Da)_{(m)}b + a_{(m)}D(b) \quad \text{for } a, b \in V, m \in \mathbb{Z}. \tag{2.3}$$

The second identity means that D is a derivation of V.

A vertex algebra V is said to be **commutative** if $a_{(n)}b = 0$ for any $n \in \mathbb{Z}_{\geq 0}$. In this case, we have $[a_{(m)}, b_{(n)}] = 0$ in $\mathrm{End}\, V$ for any $a, b \in V$ and $m, n \in \mathbb{Z}$.

Let S be a finite set of V. If V is spanned by a set of the form

$$\{a^1_{(-n_1)} \cdots a^r_{(-n_r)}\mathbf{1} | a^i \in S, n_i \in \mathbb{Z}_{>0}\}$$

then it is called that V is **strongly generated** by S (see [Ar] for more properties). By (2.2), we have $a_{(-m-1)} = \frac{1}{m!}(D^m a)_{(-1)}$ for $a \in V$ and $m \in \mathbb{Z}_{\geq 0}$. Therefore V is strongly generated by S if and only if V is generated by S when V is regarded as a noncommutative, nonassociative differential algebra with -1-product as multiplication and with derivation D.

We consider a subspace $C_2(V)$ defined by

$$C_2(V) = \mathrm{span}\{\, a_{(-2)}b \,|\, a, b \in V\,\}.$$

A vertex algebra V is called **C_2-cofinite** if $V/C_2(V)$ is finite dimensional. We set

$$C_2(U, W) = \mathrm{span}\{\, a_{(-2)}b \,|\, a \in U, b \in W\,\}$$

for a subset U, W of V.

An *automorphism* of a vertex algebra $(V, Y(\,\cdot\,, z), \mathbf{1})$ is a linear isomorphism g satisfying $g(a_{(m)}b) = g(a)_{(m)}g(b)$ for $a, b \in V$ and $g(\mathbf{1}) = \mathbf{1}$. For a finite automorphism group G, $V^G = \{a \in V \,|\, g(a) = a\}$ has naturally a vertex algebra structure. This vertex algebra is called an orbifold model of V.

3 Commutative vertex algebras

Borcherds introduced a notion of a vertex algebra in [B]. In this paper he showed that a commutative vertex algebra is nothing but a unital commutative associative algebra with a derivation. We recall the correspondence in this section.

Let A be a commutative associative algebra with unit 1, and D its arbitrary derivation. We denote the triple by $(A, D, 1)$ and call it a unital differential commutative algebra.

For a unital differential commutative algebra $(A, D, 1)$, we set $\mathbf{1} = 1$ and define $Y(a, z) = \sum_{i=0}^{\infty} \rho(D^i a)\frac{z^i}{i!}$ for $a \in A$, where ρ is the (left) regular representation of A. Then $(A, Y(\,\cdot\,, z), \mathbf{1})$ becomes a vertex algebra. Since $a_{(m)} = 0$ for

$m \in \mathbb{Z}_{\geq 0}$, A is commutative, and we also have $a_{(-m)} = \rho(D^{m-1}a)/(m-1)!$ for $m \in \mathbb{Z}_{>0}$. On the other hand, for a commutative vertex algebra $A(Y(\cdot, z), \mathbf{1})$, A has a unital commutative associative algebra structure with multiplication $ab = a_{(-1)}b$ and unit $\mathbf{1}$. Then as mentioned above, $D \in \operatorname{End} A$ defined by $D(a) = a_{(-2)}\mathbf{1}$ for $a \in A$ is a derivation. Thus we have a unital differential commutative algebra $(A, D, \mathbf{1})$.

Let $(A, D, 1)$ be a unital differential commutative algebra. A D-invariant ideal I of A is an ideal of A as commutative algebra satisfying $D(I) \subset I$. For any ideal I of A, we have D-invariant ideal $\sum_{i=0}^{\infty} D^i(I)$. We see that an ideal I of A as commutative algebra is D-invariant if and only if an ideal of A as a vertex algebra. For $a_1, \ldots, a_r \in A$, we set $(a_1, a_2, \ldots, a_r; D)$ a D-invariant ideal generated by $a_1, \ldots, a_r$.

If A is a commutative vertex algebra, then $C_2(A)$ is a D-invariant ideal generated by $D(V)$. In fact for any subspace $U \subset A$, $C_2(U, A)$ is a D-invariant ideal of A generated by $D(U)$.

4 Polynomial ring

Let $\Lambda = \{1, \ldots, k\}$ and set

$$A = \mathbb{C}[x_j^{(i)} \,|i \in \Lambda, j \in \mathbb{Z}_{>0}] \tag{4.1}$$

be the ring of all polynomials in formal variable $x_j^{(i)}$ with $i \in \Lambda$ and $j \in \mathbb{Z}_{>0}$. Let D be a derivation mapping $x_j^{(i)}$ to $x_{j+1}^{(i)}$ for any $i \in \Lambda, j \in \mathbb{Z}_{>0}$. Then A is a unital differential commutative algebra. As a vertex algebra it is strongly generated by $S = \{x_1^{(i)} \,|i \in \Lambda\}$, and we have

$$C_2(A) = \left(x_j^{(i)} \middle| \, i \in \Lambda, j \geq 2\right).$$

Hence $A/C_2(A) \cong \mathbb{C}[S]$.

We define an automorphism g of A by $g(x_j^{(i)}) = -x_j^{(i)}$ for $i \in \Lambda, j \in \mathbb{Z}_{>0}$. Set $A^{\pm} = \{a \in A \,|\, g(a) = \pm a\}$ respectively. Next we consider the subset $C_2(A^+, A)$. We can first show that the following lemma:

Lemma 4.1. $x_{j_1}^{(i_1)} \cdots x_{j_r}^{(i_r)} \in C_2(A^+, A)$ *if* $r \geq 3$, $i_p = i_q$ *for some* $1 \leq p \neq q \leq r$ *and* $j_s \geq 2$ *for some* $1 \leq s \leq r$.

Proof. We may assume that $i_1 = i_2$. First we note that for $a, b \in A^-$, $D(a)b + aD(b) = D(ab) \in D(A^+)$. Thus for any $c \in A$, $D(a)bc \equiv -aD(b)c$ modulo $C_2(A^+, A)$. Hence we see that j_s with $2 \leq s \leq r$ can be reduced to 1 by adding $j_s - 1$ to j_1 and multiplying $(-1)^{j_s - 1}$. For example, we have the congruence relations $x_3x_2x_5u \equiv -x_4x_1x_5u \equiv x_5x_1x_4u \equiv \cdots \equiv -x_8x_1x_1u$. Therefore, $x_{j_1}^{(i_1)} \cdots x_{j_r}^{(i_r)}$ is congruent to a nonzero scalar multiple of the monomila $x_p^{(i_1)}x_1^{(i_1)} \cdots x_1^{(i_r)}$, where $p = \sum j_s - r + 1$ or $x_p^{(i_1)}x_2^{(i_1)} \cdots x_1^{(i_r)}$, where $p = \sum j_s - r$.

On the other hand for $m, n \in \mathbb{Z}_{>0}$, if $m-n$ is odd then

$$x_m^{(i_1)} x_n^{(i_1)} \equiv \pm \frac{1}{2} D \left(\left(x^{(i_1)}_{\frac{m+n-1}{2}} \right)^2 \right) \equiv 0 \mod C_2(A^+).$$

Hence both of $x_p^{(i_1)} x_1^{(i_1)} \cdots x_1^{(i_r)}$ and $x_p^{(i_1)} x_2^{(i_1)} \cdots x_1^{(i_r)}$ are in $C_2(A^+, A)$. □

In the proof we show that $x_m^{(i)} x_n^{(i)} \in \mathbb{C}_2(A^+)$ if $m-n$ is odd. We also see that if $m-n$ is even then $x_m^{(i)} x_n^{(i)}$ is congruent to a nonzero multiple of the square of $x_p^{(i)}$ with $p = (m+n)/2$. Actually, we have

$$\begin{aligned} &A/C_2(A^+, A) \\ &\cong \mathbb{C}[S] \oplus \bigoplus_{r=2}^{\infty} \bigoplus_{i=1}^{k} \mathbb{C} \left(x_r^{(i)} \right)^2 \oplus \bigoplus_{t=3}^{k} \bigoplus_{1 \le i_1 < \cdots < i_t \le k} \bigoplus_{p=2}^{\infty} \mathbb{C} x_p^{(i_1)} x_1^{(i_2)} \cdots x_1^{(i_t)} \end{aligned} \tag{4.2}$$

as vector spaces[3]. We see that both A and A^+ are not C_2-cofinite.

To construct a C_2-cofinite commutative vertex algebra strongly generated by a finite set, we take a D-invariant ideal I of A and consider the quotient algebra $V = A/I$. Since $C_2(V) = (C_2(A) + I)/I$, $V/C_2(V) = A/(C_2(A) + I)$ which is isomorphic to the quotient of $A/C_2(A)$ by $(I + C_2(A))/C_2(A)$. Therefore V is C_2-cofinite if and only if $(I + C_2(A))/C_2(A)$ is finite codimensional in the polynomial ring $\mathbb{C}[S]$.

We assume that $g(I) = I$, that is, $I = I^+ \oplus I^-$ with $I^\pm = I \cap A^\pm$ respectively. Then g acts on V as an automorphism. We set $V^\pm$ the ± 1-eigenspace for g respectively. We shall see in the next section that in general V^+ is not C_2-cofinite if V is C_2-cofinite.

5 A condition for the C_2-cofiniteness of V^+

Let I be a D-invariant ideal of the polynomial ring A. Suppose that $V = A/I$ is C_2-cofinite. In this section we seek a sufficient and necessary condition for I such that V^+ is C_2-cofinite. Before doing this, we give an example that V^+ is not C_2-cofinite. We consider the case $\Lambda = \{1\}$ and omit the upper index (1) form the generators $x_j^{(1)}$ for simplicity.

Example 5.1. Let $I = (x_1^2 - x_2^2; D)$. Since $D(x_1^2 - x_2^2) \in C_2(A^+)$, we have

$$I + C_2(A^+, A) = A(x_1^2 - x_2^2) + C_2(A^+, A).$$

By Lemma 4.1, we see that the quotient space of the right hand side in the above formula by $C_2(A^+, A)$ becomes

$$\left(\mathbb{C}(x_1^2 - x_2^2) + \mathbb{C}x_1^3 + C_2(A^+, A) \right) / C_2(A^+, A).$$

[3]The algebra structure of $A/C_2(A^+, A)$ can be seen easily. But we do not state here

On the other hand $V/C_2(V^+, V)$ is isomorphic to the quotient of $A/C_2(A^+, A)$ by $(I + C_2(A^+, A))/C_2(A^+, A)$. Thus we see that $V^+/C_2(V^+)$ is contains the direct sum $\bigoplus_{r=2}^{\infty} \mathbb{C}\,(x_r)^2$ of infinitely many one dimensional vector spaces. Therefore $V^+/C_2(V^+)$ is infinite dimensional and V^+ is not C_2-cofinite. In this case we have no polynomial in I with the monomial x_1.

Example 5.2. We consider the case $I = (x_1 - x_2; D)$. In this case, it is easy to see that $V = A/I$ is isomorphic to $\mathbb{C}[x]$, where x corresponds to the image of x_1 in V, and D is given by $D = x\frac{d}{dx}$. Thus $C_2(V) = (x)$ and hence V is C_2-cofinite. We find that $C_2(V^+)$ is an ideal generated by x^2 in $\mathbb{C}[x^2]$. Thus V^+ is also C_2-cofinite. In this case $x_1 - x_2 \in I$ has the monomial x_1.

In the above two examples, x_1 is a generator of A as a vertex algebra, and the difference of them is that I contains a polynomial with a nonzero scalar multiple of the generator x_1 as one of monomials or not. We can generalize this to the case A is generated by more than one generator as a vertex algebra.

Theorem 5.3. *Let A be as in (4.1), I a D-invariant ideal, and g an automorphism such that $g(x_j^{(i)}) = -x_j^{(i)}$ for $i \in \Lambda$, $j \in \mathbb{Z}_{\geq 0}$. For a positive integer r, let $A_{\geq r}$ be the ideal consisting of all polynomials whose degrees are greater than or equal to r. Suppose that $V = A/I$ is C_2-cofinite and that $g(I) = I$. Set $V^+ = \{u \in V | g(u) = u\}$ the $\mathbb{Z}_2$-orbifold model of V with respect to g. Then $V/C_2(V^+, V)$ is finite dimensional if and only if $(I + A_{\geq 1})/(I + A_{\geq 2})$ is finite dimensional.*

We first note that

$$\begin{aligned} V/C_2(V^+, V) &\cong (A/I)/\left((C_2(A^+, A) + I)/I\right) \cong A/(C_2(A^+, A) + I) \\ &\cong (A/C_2(A^+, A))/((C_2(A^+, A) + I)/C_2(A^+, A)). \end{aligned}$$

By using Lemma 4.1 and the explicit description of $A/C_2(A^+, A)$ in (4.2), we can show the theorem. The main idea is that the condition $(I+A_{\geq 1})/(I+A_{\geq 2})$ of the ideal I implies that for large enough $p \in \mathbb{Z}$, degree one monomial $x_p^{(i)}$ is equivalent to a polynomial consisting of monomials whose degrees are greater than one modulo I. This fact shows that any monomials in which the sum of lower indices j of generators $x_j^{(i)}$ are sufficiently large are congruent to polynomials whose degrees are greater than $k + 1$ modulo I, where k is the cardinality of Λ. Such polynomials are in $C_2(A^+, A) + I$ by Lemma 4.1. This is the rough sketch of a proof.

6 Conclusion

In this report, we consider only $\mathbb{Z}_2$-orbifold models of finitely, strongly generated commutative vertex algebra and give a sufficient and necessary condition for its C_2-cofiniteness under the assumption that based vertex algebra is C_2-cofinite. We expect that the theorem can be extend to the case any cyclic group whose order is not only two but arbitrary positive integer.

As for the noncommutative case, the idea using to prove Theorem 5.3 can not be applied to show the same statement directly although give some new idea. One of the tool to avoid the noncommutativity is an abelianization of vertex algebra by means of Li's standard filtration (see [Li4] and [Ar]). This abelianization is very useful to show the C_2-cofiniteness of vertex algebra itself. But it does not still give enough property for proving C_2-cofiniteness of orbifold models. We need further study on information which a noncommutativity of a vertex algebra has to get hints for the problem.

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