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Non-tempered automorphic representations of inner forms of $Sp(4)$

九州大学 GCOE 研究員　安田 貴徳 (Takanori Yasuda)
Faculty of Mathematics,
Kyushu University

1 Introduction

For a reductive group G defined over a number field k, an unitary representations of $G(\mathbb{A}_k)$ on the space of L^2-automorphic forms $L^2(G(k)\backslash G(\mathbb{A}_k))$ is defined by the right regular action. As for the irreducible decomposition of its discrete spectrum $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}_k))$, Arthur gave a conjecture [2]. It says that $L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A}_k))$ should decompose into $G(\mathbb{A}_k)$-invariant subspaces parametrized by elliptic A-parameters. For an elliptic A-parameter ψ, the set Π_ψ of irreducible automorphic representations of $G(\mathbb{A}_k)$ appearing the associated subspaces is called A-packet for ψ. For any place v of k, a finite set Π_{ψ_v} of irreducible admissible representations of $G(k_v)$, which is called a local A-packet, should exist so that Π_ψ is a subset of

$$\{\bigotimes_v{}'\pi_v \mid \pi_v \in \Pi_{\psi_v} \text{ and } \pi_v \text{ is unramified for almost all } v\}.$$

Arthur also conjectured the multiplicity of $\pi \in \Pi_\psi$ in the associated subspace for ψ. To describe the multiplicity, we need the information about global and local S-group for ψ, and pairings between S-groups and A-packets.

In this note, we treat the case that G is a non-split inner form of $Sp(4)$. ($Sp(4)$ is the isometry group of 4-dimensional symplectic space.) I give an evaluation of the multiplicities of non-tempered irreducible automorphic representations which appear in the residual spectrum, or are CAP representations (Theorem 4.1, Theorem 5.1 and Proposition 6.2). Here a cuspidal representation π is said to be of CAP if for any cusp form ϕ in π which is $\mathbf{K}$-finite where $\mathbf{K}$ is a maximal compact subgroup of $G(\mathbb{A}_k)$, there exists an element ϕ' of an irreducible component of the residual spectrum such that ϕ and ϕ' share the same absolute values of Hecke eigenvalues at almost all places of k. According to Arthur's conjecture, any irreducible non-tempered automorphic representation of $G(\mathbb{A}_k)$ appears in A-packet for some A-parameter ψ of DAP type. Here an A-parameter $\psi : \mathcal{L}_k \times SL(2,\mathbb{C}) \to {}^LG$ where $\mathcal{L}_k$ is the hypothetical Langlands group of k and ${}^LG$ is the L-group of G is said to be of DAP type if ψ is elliptic and the restriction to $SL(2,\mathbb{C})$ of ψ is non-trivial. This implies that irreducible non-tempered automorphic representations should be exhausted by the irreducible components of the residual spectrum and CAP representations.

From the evaluation of the multiplicity for irreducible non-tempered automorphic representation of $G(\mathbb{A}_k)$, we can guess a explicit description of multiplicity of these representations (Expectation 8.1). Our interest is whether this description coincides with the Arthur's conjectural multiplicity. More precisely, the problem is whether there are pairings between S-groups and A-packets such that the description coincides with the Arthur's multiplicity defined by these pairings. Our main result is that such pairings exist (Section

8). Remark that the local pairings defined in this result satisfy the conjecture of Hiraga and Saito.

2 Inner forms of $Sp(2)$

Let k be a number field and $\mathbb{A}$ its adele ring. $|\ |_{\mathbb{A}}$ denotes the idele norm of $\mathbb{A}^\times$. For any place v of k, we write k_v for the completion of k at v and $|\ |_v$ for the v-adic norm. Let μ be a non-trivial character of $\mathbb{A}$ which is trivial on k.

Let D be a quaternion division algebra over k. We write ν, τ and ι for the reduced norm, the reduced trace and the main involution of D, respectively. We write S_D for the set of places v of k at which D is ramified, which has finite and even elements. Let $W = D^{\oplus 2}$ be the free left module over D with rank two, and we equip it with a hermitian form $\langle\ ,\ \rangle$ given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1\,{}^{\iota}y_2 + y_1\,{}^{\iota}x_2 \qquad (x_1, x_2, y_1, y_2 \in D).$$

Let G be the unitary group of this form, so that

$$G = \left\{ g \in GL(2, D) \,\middle|\, g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^{*}g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Here we write ${}^{*}(a_{i,j}) = ({}^{\iota}a_{j,i})$ for $(a_{i,j}) \in \mathbb{M}(2, D)$. It can be regarded as a reductive group defined over k. It is non-quasisplit and an inner form of $Sp(2)$ with respect to a quadratic extension k' of k such that all $v \in S_D$ do not split fully in k'/k. Fix a k-parabolic subgroup P and its Levi factor M as

$$P = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in G \right\}, \quad M = \left\{ \begin{pmatrix} x & 0 \\ 0 & ({}^{\iota}x)^{-1} \end{pmatrix} \,\middle|\, x \in D^\times \right\},$$

P is the unique proper parabolic subgroup of G up to $G(k)$-conjugate and corresponds to the Siegel parabolic subgroup via an inner twist. We write again ν for the character of M corresponding to the reduced norm. U denotes the unipotent radical of P, so that

$$U = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \,\middle|\, \tau(y) = 0 \right\}.$$

$G(k)\backslash G(\mathbb{A})$ becomes a locally compact Hausdorff space and has a non-zero $G(\mathbb{A})$-invariant measure up to scalars. Fix such a measure dg. Then the space $L^2(G(k)\backslash G(\mathbb{A}))$ of square-integrable functions on $G(k)\backslash G(\mathbb{A})$ is defined and the representation ρ of $G(\mathbb{A})$ on $L^2(G(k)\backslash G(\mathbb{A}))$ is defined by

$$\rho(g)f(x) = f(xg) \qquad (x, g \in G(\mathbb{A})).$$

This representation has an orthogonal decomposition;

$$L^2(G(k)\backslash G(\mathbb{A})) = L^2_{\mathrm{disc}}(G) \oplus L^2_{\mathrm{cont}}(G),$$

where $L^2_{\mathrm{disc}}(G)$ is the maximal completely reducible closed subspace of $L^2(G(k)\backslash G(\mathbb{A}))$ and $L^2_{\mathrm{cont}}(G)$ is its orthogonal complement. For $\phi \in L^2(G(k)\backslash G(\mathbb{A}))$ its constant term ϕ_P along $P = MU$ is defined by

$$\phi_P(g) = \int_{U(k)\backslash U(\mathbb{A})} \phi(ug)du \quad (g \in G(\mathbb{A}))$$

where du is a Haar measure of $U(k)\backslash U(\mathbb{A})$. $L_0^2(G)$ denotes the space of cuspidal elements of $L^2(G(k)\backslash G(\mathbb{A}))$, that is, elements whose constant terms along P vanish. It is known that $L_0^2(G)$ is a $G(\mathbb{A})$-invariant closed subspace contained in $L^2_{\text{disc}}(G)$ [7]. We write $L^2_{\text{res}}(G)$ for its orthogonal complement in $L^2_{\text{disc}}(G)$, which is called the residual spectrum. In this note, we call an irreducible component of $L^2_{\text{disc}}(G)$ an irreducible automorphic representation of $G(\mathbb{A})$. Any irreducible automorphic representation π of $G(\mathbb{A})$ has a decomposition into a restricted tensor product $\pi \simeq \bigotimes_v' \pi_v$. From the Langlands' spectral theory of Eisenstein series, the residual spectrum of G coincides with the space of residues of Eisenstein series associated to the cuspidal representations of $M(\mathbb{A})$.

3 Decomposition of Discrete spectrum

Assume the existense of the hypothetical Langlands group $\mathcal{L}_k$ of k. The L-group ${}^LG$ of G is $\widehat{G} \times W_k = SO(5,\mathbb{C}) \times W_k$ where W_k is the Weil group of k. By an *A-parameter* is meant a continuous homomorphism $\phi : \mathcal{L}_k \times SL(2,\mathbb{C}) \to {}^LG$ such that

(i) writing $p_k : \mathcal{L}_k \to W_k$ for the conjectural homomorphism and $p_2 : {}^LG \to W_k$ the projection to the second component, $p_2 \circ \phi = p_k$,

(ii) its restriction to $\mathcal{L}_k$ is a Langlands parameter with bounded image [4], and

(iii) its restriction to $SL(2,\mathbb{C})$ is analytic.

Two A-parameter are equivalent if they are $\widehat{G}$-conjugate. The set of equivalence classes of A-parameters is denoted by $\Psi(G)$. We write C_ψ for the centralizer of the image of ψ in $\widehat{G}$. An A-parameter ψ is said to be *elliptic* if C_ψ is contained in the center $Z(\widehat{G})$ of $\widehat{G}$. The subset of elliptic elements of $\Psi(G)$ is denoted by $\Psi_0(G)$. An A-parameter ψ is of *DAP type* (DAP is the abbreviation of "Discrete Associated to Parabolic") if

(i) ψ is elliptic, and

(ii) $\psi|_{SL(2,\mathbb{C})}$ is not trivial.

$\Psi_{\text{DAP}}(G)$ denotes the subset of $\Psi_0(G)$ consisting of the elements of DAP type. From the property (iii) of the definition of A-parameter, elements of $\Psi(G)$ is classified by the irreducible decomposition of their restriction to $SL(2,\mathbb{C})$. As for homomorphisms $SL(2,\mathbb{C}) \to \widehat{G}$ we have the following result.

Proposition 3.1 ([5]). *1. (Jacobson-Morozov)*
$\{\textit{homomorphism } SL(2,\mathbb{C}) \to SO(5,\mathbb{C})\}/_\sim \approx \{\textit{nilpotent orbits in } \mathfrak{so}(5,\mathbb{C})\}/_\sim,$

2. $\{\textit{nilpotent orbits in } \mathfrak{so}(5,\mathbb{C})\}/_\sim \approx \{\textit{partition}\,[n_1^{k_1},\dots,n_l^{k_l}]\,\textit{of}\ 5 \mid n_i : \textit{even} \Rightarrow k_i : \textit{even}\}$
$= \{[1^5],[2^2,1],[3,1^2],[5]\}.$

Here $\sim$ means $SO(5,\mathbb{C})$-conjugacy.

By this proposition we have a decomposition

$$\Psi_0(G) = \Psi_0(G)_{[1^5]} \sqcup \Psi_0(G)_{[2^2,1]} \sqcup \Psi_0(G)_{[3,1^2]} \sqcup \Psi_0(G)_{[5]}. \tag{3.1}$$

In addition, we have

$$\Psi_{\mathrm{DAP}}(G) = \Psi_0(G)_{[2^2,1]} \sqcup \Psi_0(G)_{[3,1^2]} \sqcup \Psi_0(G)_{[5]}.$$

Arthur's conjecture [2] implies a coarse decomposition

$$L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A})) = \bigoplus_{\psi\in\Psi_0(G)} L^2(G)_\psi. \tag{3.2}$$

The set of irreducible automorphic representations appearing in $L^2(G)_\psi$ is denoted by Π^G_ψ. By (3.1) and (3.2) we have the decomposition,

$$L^2_{\mathrm{disc}}(G(k)\backslash G(\mathbb{A})) = L^2_{[1^5]}(G) \oplus L^2_{[2^2,1]}(G) \oplus L^2_{[3,1^2]}(G) \oplus L^2_{[5]}(G).$$

Arthur's conjecture also implies the space spanned by non-tempered cuspidal representations of $G(\mathbb{A})$ coincides with

$$\bigoplus_{\psi\in\Psi_{\mathrm{DAP}}(G)} L^2(G)_\psi = L^2_{[2^2,1]}(G) \oplus L^2_{[3,1^2]}(G) \oplus L^2_{[5]}(G).$$

We will consider the multiplicity for irreducible non-tempered automorphic representation. Since $\Psi_0(G)_{[5]}$ consists of one element $\psi_0 = \mathbf{1} \otimes \mathrm{Sym}^4$ where Sym^4 is the 4-th symmetric power of $SL(2,\mathbb{C})$, and $L^2_{[5]}(G) = L^2(G)_{\psi_0}$ should be $\mathbb{C}\cdot\mathbf{1}$, we will treat mainly the case of $L^2_{[2^2,1]}(G)$ and $L^2_{[3,1^2]}(G)$. We will say that $\psi \in \Psi_0(G)_{[2^2,1]}$ and irreducible components of $L^2_{[2^2,1]}(G)$ are of Saito-Kurokawa type, and $\psi \in \Psi_0(G)_{[3,1^2]}$ and irreducible components of $L^2_{[3,1^2]}(G)$ are of Soudry type.

4 Residual spectrum of G

Theorem 4.1 ([10]). *Let k be a totally real number field. The irreducible components of the residual spectrum of G consist of the following representations.*

(1) The trivial representation $\mathbf{1}_G$,

(2) The unique irreducible quotient $J^G_P(\sigma)$ of $\mathrm{Ind}^{G(\mathbb{A})}_{P(\mathbb{A})}(\sigma|\nu|^{1/2}_{\mathbb{A}})$. Here σ runs over the set of infinite dimensional irreducible self-dual cuspidal representations of $M(\mathbb{A})$ whose standard L-functions $L(\sigma,s)$ do not vanish at $s=1/2$, and

(3) The theta lift $R(V)$ from the trivial representation of $G(V_\mathbb{A})$ under the Weil representation $\omega_{V,\mu}$. Here V runs over the set of local isometry classes of (-1)-hermitian right D-spaces of dimension one, and $G(V)$ is the unitary group of V.

In the case (1) and (2), the multiplicity of each representation is one. In the case (3), the multiplicity of each representation is $2^{|S_D|-2}$.

Remark 4.2. *In case of general k, it can be shown that the residual spectrum is exhausted by these automorphic representations and the multiplicity of representation in the case (3) is greater than or equal to $2^{|S_D|-2}$.*

All irreducible representations appearing in the residual spectrum are non-tempered by the Langlands classification. Therefore these representations belongs to some A-packets of DAP type. From the description of local components of these automorphic representations these associated A-parameters should be the following.

(1) $\mathbf{1}_G$ correspond to

$$\phi = \psi_0 = \mathbf{1}_5 \otimes \mathrm{Sym}^4 \times p_k.$$

(2) $J_P^G(\sigma)$ correspond to

$$\phi = \left((\phi_\sigma \otimes \mathrm{St}) \oplus (\mathbf{1}_{\mathcal{L}_k} \otimes \mathbf{1}_{SL(2,\mathbb{C})})\right) \times p_k.$$

Here St is the standard representation of $SL(2,\mathbb{C})$ and ϕ_σ is the Langlands parameter associated to σ, whose image is contained by $SL(2,\mathbb{C})$.

(3) $R(V)$ correspond to

$$\phi = \left((\mathrm{Ind}_{W_{k'}}^{W_k} \mathbf{1}_{W_{k'}} \otimes \mathbf{1}_{SL(2,\mathbb{C})}) \oplus (\omega_{k'/k} \otimes \mathrm{Sym}^2)\right) \circ p_k \times p_k.$$

Here $\omega_{k'/k}$ is the quadratic character of W_k associated to k'/k. Remark the image of $\mathrm{Ind}_{W_{k'}}^{W_k} \mathbf{1}_{W_{k'}}$ is contained by $O(2,\mathbb{C})$.

As we have already explained $\mathbf{1}_G$ spans $L^2_{[5]}(G)$. The A-parameters of (2) are of Saito-Kurokawa type and those of (3) are of Soudry type.

5 CAP representations of Saito-Kurokawa type

Let B be a quaternion algebra over k. ι_B denotes the main involution of B and S_B is defined similarly for S_D. Take $\eta \in D$ such that ${}^\iota\eta = -\eta$ and $\eta^2 = p \in k^\times$. We write $K = k(\eta)$, which is a quadratic extension of k. Suppose that K can be embedded in B. A K/k-skew-hermitian form on B is defined by

$$h(x,y) := \eta \cdot ({}^{\iota_B}x \cdot y)_K \quad (x, y \in B),$$

where $(\ )_K$ is the projection to K-part of B. A (D,ι)-skew-hermitian right space $(V_{B,\eta}, h_{B,\eta})$ of rank 2 is defined as

$$(V_{B,\eta}, h_{B,\eta}) = (B \otimes_K D, h \otimes \mathbf{1}).$$

$G(V_{B,\eta})$ denotes the unitary group of $(V_{B,\eta}, h_{B,\eta})$. This is an inner form of $O(4)$. We write $G_0(V_{B,\eta})$ for the k-group of elements of $G(V_{B,\eta})$ whose reduced norm is one. Writing $\widetilde{B}$ for the quaternion algebra over k such that $B \cdot \widetilde{B} = D$ in the Brauer group of k,

$$G_0(V_{B,\eta}) \simeq \left\{(b, \widetilde{b}) \in B^\times \times \widetilde{B}^\times \,\middle|\, \nu_B(b) = \nu_{\widetilde{B}}(\widetilde{b})^{-1}\right\} \Big/ \{(z, z^{-1}) \mid z \in \mathbb{G}_m\}. \qquad (5.1)$$

Therefore any irreducible cuspidal representation of $G_0(V_{B,\eta}, \mathbb{A})$ is written in the form $\sigma_B \otimes \sigma_{\widetilde{B}}$ where σ_B and $\sigma_{\widetilde{B}}$ are irreducible cuspidal representations of $B_\mathbb{A}^\times$ and $\widetilde{B}_\mathbb{A}^\times$, respectively. Since $(G(V_{B,\eta}), G)$ is a dual reductive pair we can consider the Weil representation $\omega_{V_{B,\eta},\mu}$ of $G(V_{B,\eta}, \mathbb{A}) \times G(\mathbb{A})$. Let $(\sigma, V_\sigma)(\simeq \bigotimes_v \sigma_v)$ be an irreducible cuspidal representation of

$B_{\mathbb{A}}^{\times}$ with trivial central character. Any element of V_σ can be regarded as an automorphic form on $G_0(V_{B,\eta}, \mathbb{A})$ by (5.1). For $\phi \in V_\sigma$ and $f \in \mathcal{S}(V_{B_{\mathbb{A}},\eta})$, define

$$\theta(f,\phi)(g) = \int_{h \in G_0(V_{B,\eta},k)\backslash G_0(V_{B,\eta},\mathbb{A})} \theta(f,h,g)\phi(h)dh \quad (g \in G(\mathbb{A}))$$
$$\theta(f,h,g) = \sum_{x \in V_{B,\eta}} \omega_{V_{B,\eta},\psi}(h,g)f(x) \quad (h \in G(V_{B,\eta},\mathbb{A})).$$

Put $\Theta(\sigma, B, \eta) = \{\theta(f,\phi) \mid f \in \mathcal{S}(V_{B_{\mathbb{A}},\eta}), \phi \in V_\sigma\}$. This is $G(\mathbb{A}_f) \times (\mathfrak{g}_\infty, \mathbf{K}_\infty)$-module by the right regular action. Here $\mathbb{A}_f, \mathbb{A}_\infty$ are the finite and infinite parts of $\mathbb{A}$, and $\mathfrak{g}_\infty$ is the complexification of the Lie algebra of $G(\mathbb{A}_\infty)$, and $\mathbf{K}_\infty$ is a maximal compact subgroup of $G(\mathbb{A}_\infty)$.

Theorem 5.1. *1. Let σ be infinite dimensional and*

(a) $L(\sigma, 1/2) \neq 0$, where $L(\sigma, s)$ is the Jacquet-Langlands L-function,

(b) $\epsilon(\sigma_v \otimes \omega_{K_v/k_v}, 1/2) = \delta_v \omega_{K_v/k_v}(-1)\epsilon(\sigma_v, 1/2)$ for all places v. Here ω_{K_v/k_v} is the quadratic character associated to K_v/k_v and $\epsilon(\sigma_v, 1/2)$ is the Jacquet-Langlands ϵ-factor which is independent of a choice of non-trivial character of k_v, and

$$\delta_v = \begin{cases} 1 & \text{if } v \notin S_B \\ -1 & \text{if } v \in S_B. \end{cases}$$

Then $\Theta(\sigma, B, \eta)$ is non-zero, irreducible, non-tempered and cuspidal if B is not isomorphic to D.

2. For the local decomposition $\Theta(\sigma, B, \eta) \simeq \bigotimes'_v \Theta(\sigma, B, \eta)_v$, $\Theta(\sigma, B, \eta)_v$ can be determined as a representation for any v. (This description of local factors will be seen as elements of local A-packets later.)

This theorem is proved by using the condition of non-vanishing of Shimura correspondence in [9]. From the description of all $\Theta(\sigma, B, \eta)_v$ the A-parameter of $\Theta(\sigma, B, \eta)$ should be

$$\psi_\sigma = \left((\phi_\sigma \otimes \mathrm{St}) \oplus (\mathbf{1}_{\mathcal{L}_k} \otimes \mathbf{1}_{SL(2,\mathbb{C})})\right) \times p_k$$

where ϕ_σ is the Langlands parameter of σ. This A-parameter is of Saito-Kurokawa type.

6 CAP REPRESENTATIONS OF SOUDRY TYPE

Let $V = V_\xi$ be the one-dimensional skew-hermitian space over (D, ι) defined by $\xi \in D$ with $\tau(\xi) = 0$. Let $\delta = \det V_\xi = \nu(\xi) = -\xi^2 \mod (k^\times)^2$ and $k' = k(\xi) \simeq k(\sqrt{-\delta})$. $G(V)$ and $G_0(V)$ denote the unitary group and special unitary group of V, respectively. Then $G_0(V)$ is isomorphic to the norm torus for the quadratic extension k'/k. Since $(G(V), G)$ is a dual reductive pair we can consider the Weil representation $\omega_{V,\mu}$ of $G(V_{\mathbb{A}}) \times G(\mathbb{A})$.

Let $\chi = \prod_v \chi_v$ be a non-trivial character of $G_0(V_k)\backslash G_0(V_{\mathbb{A}})$ and put $S_\chi = \{v \mid \chi_v^2 = 1\}$. Since

$$\mathrm{Ind}_{G_0(V_{\mathbb{A}})}^{G(V_{\mathbb{A}})} \chi \subset L^2_{\mathrm{disc}}(G(V)) = L^2(G(V))$$

we want to construct an irreducible automorphic representation of $G(\mathbb{A})$ by the theta lift from $\mathrm{Ind}_{G_0(V_\mathbb{A})}^{G(V_\mathbb{A})}\chi$. However $\mathrm{Ind}_{G_0(V_\mathbb{A})}^{G(V_\mathbb{A})}\chi$ is not irreducible. Therefore the description of its irreducible decomposition is needed. As for its local component we have

$$\mathrm{Ind}_{G_0(V_v)}^{G(V_v)}\chi_v \simeq \begin{cases} \tilde{\chi}_v^+ \oplus \tilde{\chi}_v^- & v \in S_\chi \cap S_D{}^c, \\ \tilde{\chi}_v & \text{otherwise.} \end{cases}$$

Here $\tilde{\chi}_v^+$, $\tilde{\chi}_v^-$ are characters not isomorphic to each other, and $\tilde{\chi}_v$ is χ_v if $v \in S_D$ and a two-dimensional irreducible representation otherwise. Fix a $\gamma_0 \in O(k', N_{k'/k})\backslash SO(k', N_{k'/k})$ and embed γ_0 in $G(V_v) \simeq O(k'_v, N_{k'_v/k_v})$ for all $v \notin S_D$. For $v \in S_\chi \cap S_D{}^c$ we may assume $\tilde{\chi}_v^+(\gamma_0) = 1$, which characterizes $\tilde{\chi}_v^+$ and $\tilde{\chi}_v^-$. Then an irreducible component of the above induced representation is of form,

$$\tau = (\textstyle\bigotimes_{v\in S} \tilde{\chi}_v^-) \otimes (\bigotimes'_{v\in S_\chi\backslash S} \tilde{\chi}_v^+) \otimes (\bigotimes'_{v\notin S_\chi} \tilde{\chi}_v)$$

for some finite set $S \subset S_\chi \cap S_D{}^c$. In this case write $\tau = \tau_S$. For any $v \in S_\chi \cap S_D{}^c$ define

$$\mathcal{S}^\pm(V_v) = \{f \in \mathcal{S}(V_v) \mid f(\gamma_0 \cdot) = \pm f\}$$

where $\mathcal{S}(V_v)$ is the space of Schwartz-Bruhat functions on V_v. For $f \in \mathcal{S}(V_\mathbb{A})$, define

$$\theta(f,h,g) = \sum_{x\in V_k} \omega_{V,\psi}(h,g)f(x) \qquad (g \in G(\mathbb{A}), h \in G(V_\mathbb{A}))$$

The theta lift from τ_S is defined as follows.

(I) $\underline{\chi^2 \neq 1}$

The theta integral is defined by

$$\theta(f,\chi)(g) = \int_{G_0(V_k)\backslash G_0(V_\mathbb{A})} \theta(f,h,g)\chi(h)dh.$$

The theta lift $\Theta(V,\chi,S)$ from τ_S is defined by $\Theta(V,\chi,S) = \{\theta(f,\chi) \mid f \in \mathcal{S}_S(V_\mathbb{A})\}$ where $\mathcal{S}_S(V_\mathbb{A}) = (\bigotimes_{v\in S} \mathcal{S}^-(V_v)) \otimes (\bigotimes'_{v\in S_\chi\backslash S} \mathcal{S}^+(V_v)) \otimes (\bigotimes'_{v\notin S_\chi} \mathcal{S}(V_v))$.

(II) $\underline{\chi^2 = 1}$

In this case τ_S is one-dimensional. The theta integral is defined by

$$\theta(f,\chi)(g) = \int_{G(V_k)\backslash G(V_\mathbb{A})} \theta(f,h,g)\tau_S(h)dh,$$

The theta lift $\Theta(V,\chi,S)$ from τ_S is defined by $\Theta(V,\chi,S) = \{\theta(f,\chi) \mid f \in \mathcal{S}(V_\mathbb{A})\}$.

In any case, $\Theta(V,\chi,S)$ becomes a $G(\mathbb{A}_f) \times (\mathfrak{g}_\infty, \mathbf{K}_\infty)$-module by right regular action.

Theorem 6.1. *1. $\Theta(V,\chi,S)$ is non-zero, irreducible, non-tempered and cuspidal.*

2. For the local decomposition $\Theta(V,\chi,S) \simeq \bigotimes'_v \Theta(V,\chi,S)_v$, $\Theta(V,\chi,S)_v$ can be determined as a representation for any v. (This description of local factors will be seen as elements of local A-packets later.)

$\Theta(V,\chi,S)$ is an inner form analogue of the following representation of $Sp(4,\mathbb{A})$.

$$\begin{array}{lccr} & & \text{quot. of } \mathrm{Ind}_{P_K(\mathbb{A})}^{Sp(4,\mathbb{A})}(\theta(k',\chi)\otimes\omega_{k'/k}|\cdot|_{\mathbb{A}}) & : Sp(4) \\ & & \nearrow^{\text{theta lift}} & \\ O(k'): & \mathrm{Ind}_{SO(k')(\mathbb{A})}^{O(k')(\mathbb{A})}\chi & \xrightarrow[\text{Shalika-Tanaka}]{} \theta(k',\chi) : \text{cuspidal} & : SL(2). \end{array}$$

Here $O(k')$ is the orthogonal group of the 2-dimensional quadratic space $(k', N_{k'/k})$ where k' is a quadratic extension of k, and P_K is the Klingen parabolic subgroup of $Sp(4)$. This fact is used to prove the above theorem. As for the multiplicity $m(\Theta(V,\chi,S))$ of $\Theta(V,\chi,S)$ in $L^2_{\text{disc}}(G)$ we have the following evaluation.

Proposition 6.2.

$$m(\Theta(V,\chi,S)) \geq \begin{cases} 2^{|S_\chi\cap S_D|-1} & \textit{if } \chi^2\neq 1,\ S_D\cap S_\chi\neq\emptyset, \\ 2^{|S_D|-2} & \textit{if } \chi^2=1,\ S_D\cap S_\chi\neq\emptyset, \\ 1 & \textit{if } S_D\cap S_\chi=\emptyset. \end{cases}$$

This result is caused by the failure of Hasse's principle for skew-hermitian spaces. This proposition is shown by using the difference of Fourier coefficients arising from the failure of Hasse's principle.

The A-parameter of $\Theta(V,\chi,S)$ must be same to that of the representation of $Sp(4,\mathbb{A})$ constructed above. By Adams conjecture [1], this A-parameter should be given by $\psi_{k',\chi}$ in the following diagram.

$$\begin{array}{lccr} & & \psi_{k',\chi} = (\mathrm{Ind}_{W_{k'}}^{W_k}\chi\otimes\mathbf{1})\oplus(\omega_{k'/k}\otimes\mathrm{Sym}^2) & : Sp(4) \\ & & \nearrow^{\text{theta lift}} & \\ O(k'): & \chi\otimes\mathbf{1} & \xrightarrow[\text{Shalika-Tanaka}]{} \mathrm{Ind}_{W_{k'}}^{W_k}\chi\otimes\mathbf{1} & : SL(2). \end{array}$$

Here the Langlands parameter associated to χ is also written by χ. $\psi_{k',\chi}$ is an A-parameter of Soudry type.

7 Conjecture of Hiraga and Saito

Let F be a local field of characteristic 0 and $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Rewrite $G^* = Sp(4)$. We have the following bijection [8].

$$\begin{array}{ccc} \{\text{inner forms of } G^*\}/_{\sim} & \approx & H^1(F, G^*{}_{\text{ad}}) \\ \Psi & & \Psi \\ G' & \longleftrightarrow & u_{G'} : \Gamma\ni\gamma\mapsto\eta_{G'}^{-1}\circ{}^{\gamma}\eta_{G'} \end{array}$$

Here $\sim$ means isomorphy and $\eta_{G'} : G^*(\overline{F})\to G'(\overline{F})$ is an inner twist. In addition, if F is non-archimedean then from [6]

$$\begin{array}{ccc} H^1(F, G^*{}_{\text{ad}}) & \approx & \pi_0(Z(\widehat{G^*}_{\text{sc}})^{\Gamma})^D \\ \Psi & & \Psi \\ u_{G'} & \longleftrightarrow & \chi_{G'}. \end{array}$$

Here $\widehat{G^*}_{\rm sc}$ is the simply connected cover of $\widehat{G^*} = SO(5,\mathbb{C})$ so that $\widehat{G^*}_{\rm sc} = Sp(4,\mathbb{C})$ and $(\ )^D$ means Pontrjagin dual. Write $j_{\rm sc} : \widehat{G^*}_{\rm sc} \to \widehat{G^*}$ for the covering map. The local Langlands group $\mathcal{L}_F$ is defined by

$$\mathcal{L}_F = \begin{cases} W_F \times SU(2,\mathbb{R}) & F : \text{non-archimedean}, \\ W_F & F : \text{archimedean}, \end{cases}$$

where W_F is the Weil group of F. A local A-parameter $\psi : \mathcal{L}_F \times SL(2,\mathbb{C}) \to {}^LG^*$ is defined similarly for the global case. For a local A-parameter ψ and an inner form G' of G^* suppose the existence of local A-packet $\Pi_\psi^{G'}$ [2], which becomes a finite set of irreducible admissible representations of $G'(F)$. For a global or local A-parameter ψ, $\mathcal{S}_\psi$ denotes $j_{\rm sc}^{-1}(C_\psi)$. $\mathcal{S}_\psi$ is defined by $\pi_0(S_\psi) = S_\psi/S_\psi^0$. For an inner form G' of G^* the following condition is called the relevance condition for (G',ψ):

$$\mathrm{Ker}\,\chi_{G'} \supset Z(\widehat{G^*}_{\rm sc})^\Gamma \cap S_\psi^0.$$

Since

$$Z_\psi^\Gamma := \mathrm{Im}(Z(\widehat{G^*}_{\rm sc})^\Gamma \to \mathcal{S}_\psi) \simeq Z(\widehat{G^*}_{\rm sc})^\Gamma/(Z(\widehat{G^*}_{\rm sc})^\Gamma \cap S_\psi^0),$$

if (G',ψ) satisfies the relevance condition then $\chi_{G'}$ can be regarded as a character of Z_ψ^Γ. The conjecture of Hiraga and Saito is described as follows.

Conjecture 7.1 ([3]). *Let F be non-archimedean. For a local A-parameter $\psi : \mathcal{L}_F \times SL(2,\mathbb{C}) \to {}^LG^*$ there exists a pairing*

$$\langle\, ,\, \rangle_F : \mathcal{S}_\psi \times \Big(\coprod_{G' \in H^1(F, G^*_{\rm ad})} \Pi_\psi^{G'} \Big) \to \mathbb{C}$$

which satisfies the following condition:
For any inner form G' of G^* there exists

$$\begin{array}{cccl} \rho : & \Pi_\psi^{G'} & \to & \Pi(\mathcal{S}_\psi, \chi_{G'}) = \{\textit{irred. repre.}\ \sigma \textit{ of } \mathcal{S}_\psi \mid \sigma|_{Z_\psi^\Gamma} = \chi_{G'}\}/\!\sim \\ & \cup\!\!\!\!\!\!\!\! & & \cup \\ & \pi & \mapsto & \rho_\pi \end{array}$$

such that $\langle s, \pi\rangle_F = \mathrm{Tr}\,\rho_\pi(s)$ for all $s \in \mathcal{S}_\psi$.

If F is non-archimedean then the set of inner forms of G^* consists of G^* and non-split group G_F. If F is real it consists of G^*, $G_F = Sp(1,1)$ and compact group $Sp(4)$, and if F is complex it consists of only G^*. In any case put $\Pi_\psi^{\rm s} = \Pi_\psi^{G^*}$, $\Pi_\psi^{\rm ns} = \Pi_\psi^{G_F}$, where $\Pi_\psi^{\rm ns} = \emptyset$ if F is complex. Since my results of residual spectrum and CAP representations (Theorem 4.1, 5.1, 6.1 and Proposition 6.2) do not contain the case of compact $Sp(4)$ at real place, we will forget the case of real F and compact $Sp(4)$.

We will go back to the global case. For an elliptic A-parameter ψ the associated local A-parameter ψ_v is given for any place v by the hypothetical homomorphism $\mathcal{L}_{k_v} \to \mathcal{L}_k$. Also homomorphism $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$ is given. Assume that the pairing $\langle\, ,\, \rangle_v : \mathcal{S}_{\psi_v} \times (\Pi_{\psi_v}^{\rm s} \sqcup \Pi_{\psi_v}^{\rm ns}) \to \mathbb{C}$ satisfying the above conjecture is given for any v. Then the global pairing $\langle\, ,\, \rangle = \prod_v \langle\, ,\, \rangle_v$:

$\mathcal{S}_\psi \times \Pi_\psi^G \to \mathbb{C}$ is defined. Let $\epsilon_\psi : \mathcal{S}_\psi \to \{\pm 1\}$ be the character defined in [2]. For $\pi \in \Pi_\psi^G$ set

$$m_\psi(\pi) = \frac{1}{|\mathcal{S}_\psi|} \sum_{s \in \mathcal{S}_\psi} \epsilon_\psi(s)\langle s, \pi\rangle.$$

Arthur's multiplicity conjecture is described as follows.

Conjecture 7.2 ([2]). *The multiplicity of* π *in* $L^2_{\mathrm{disc}}(G)$ *is equal to* $\sum_{\psi \in \Psi_0(G)} m_\psi(\pi)$.

8 Multiplicity conjecture

The results of section 4, 5 and 6 give a speculation of the description of the multiplicity of non-tempered automorphic representations of $G(\mathbb{A})$. For an irreducible automorphic representation π of $G(\mathbb{A})$ the multiplicity of π in $L^2_{\mathrm{disc}}(G)$ is denoted by $m(\pi)$.

Expectation 8.1. *1. (Saito-Kurokawa type) Suppose that an irreducible cuspidal representation* σ *of* $GL(2, \mathbb{C})$*, a quaternion algebra* B *and* $\eta \in D$ *satisfy the condition of Theorem 5.1, 1. Then* $m(\Theta(\sigma, B, \eta)) = 1$.

2. (Soudry type)

$$m(\Theta(V, \chi, S)) = \begin{cases} 2^{|S_\chi \cap S_D| - 1} & \text{if } \chi^2 \neq 1,\ S_D \cap S_\chi \neq \emptyset, \\ 2^{|S_D| - 2} & \text{if } \chi^2 = 1,\ S_D \cap S_\chi \neq \emptyset, \\ 1 & \text{if } S_D \cap S_\chi = \emptyset. \end{cases}$$

These expected multiplicities can be rewritten in terms of Arthur's conjectural multiplicity. In other words, there is a pairing $\langle\, ,\, \rangle$ such that $m(\pi) = m_\psi(\pi)$ for $\pi \in \Pi_\psi^G$ and all $\langle\, ,\, \rangle_v$ satisfy the conjecture of Hiraga and Saito. Finally, we will see the dscription.

8.1 Saito-Kurokawa type

An A-parameter ψ of Saito-Kurokawa type is written by the form

$$\psi = \psi_\sigma = \left((\phi_\sigma \otimes \mathrm{St}) \oplus (\mathbf{1}_{\mathcal{L}_k} \otimes \mathbf{1}_{SL(2,\mathbb{C})})\right) \times p_k$$

where $\sigma \simeq \bigotimes_v \sigma_v$ is an infinite dimensional irreducible cuspidal representation of $PGL(2, \mathbb{A})$.

(1) $\underline{v \notin S_D}$

Write V_v^{hyp} for the 4-dimensional hyperbolic quadratic space over k_v. $SO(V_v^{\mathrm{hyp}})$ is isomorphic to

$$\{(g_1, g_2) \in GL(2, k_v) \times GL(2, k_v) \mid \det(g_1) = \det(g_2)^{-1}\}/\{(z, z^{-1}) \mid z \in k_v^\times\}.$$

$\theta(\sigma_v, V_v^{\mathrm{hyp}})$ denotes the Howe correspondent of $\mathrm{Ind}_{SO(V_v^{\mathrm{hyp}})}^{O(V_v^{\mathrm{hyp}})}(\sigma_v \otimes \mathbf{1})$, which is an irreducible representation of $G(k_v)$. Write V_v^{ani} for the 4-dimensional anisotropic quadratic space over k_v if v is non-archimedean, and $V_v^{\pm}$ for the 4-dimensional positive and negative definite quadratic spaces over k_v if v is real. Since the special orthogonal groups of these quadratic spaces are isomorphic to

$$\{(g_1, g_2) \in D_{k_v}^\times \times D_{k_v}^\times \mid \nu_{D_{k_v}}(g_1) = \nu_{D_{k_v}}(g_2)^{-1}\}/\{(z, z^{-1}) \mid z \in k_v^\times\}$$

where D_{k_v} is the quaternion division algebra over k_v, $\theta(JL(\sigma_v), V_v^{\mathrm{ani}})$ and $\theta(JL(\sigma_v), V_v^{\pm})$ are defined similarly for $\theta(\sigma_v, V_v^{\mathrm{hyp}})$. Here $JL(\sigma_v)$ is the Jacquet-Langlands correspondent of σ_v.

$$\Pi^{\mathrm{s}}_{\psi_v} = \begin{cases} \{\tau_0 = \theta(\sigma_v, V_v^{\mathrm{hyp}}),\ \tau_1 = \theta(JL(\sigma_v), V_v^{\mathrm{ani}})\} & v : \text{non-arch. and } \phi_\sigma : \text{irreducible}, \\ \{\tau_0 = \theta(\sigma_v, V_v^{\mathrm{hyp}}),\ \tau_1^{\pm} = \theta(JL(\sigma_v), V_v^{\pm})\} & v : \text{real and } \phi_\sigma : \text{irreducible}, \\ \{\tau_0 = \theta(\sigma_v, V_v^{\mathrm{hyp}})\} & \text{otherwise}. \end{cases}$$

Any τ_0 is a quotient of $\mathrm{Ind}_{P(k_v)}^{G(k_v)}(|\det|_v^{1/2}\sigma_v)$.

(2) $\underline{v \in S_D}$

Write V_v for the 2-dimensional skew-hermitian space over D_v of determinant 1. Since $G(V_v) = G_0(V_v)$ and $G_0(V_v)$ is isomorphic to

$$\{(g_1, g_2) \in D_v^\times \times GL(2, k_v) \mid \nu(g_1) = \det(g_2)^{-1}\}/\{(z, z^{-1}) \mid z \in k_v^\times\}$$

the Howe correspondents $\theta(\sigma_v, V_v)$ and $\theta(JL(\sigma_v), V_v)$ are defined.

$$\Pi^{\mathrm{ns}}_{\psi_v} = \begin{cases} \{\tau_0' = \theta(\sigma_v, V_v),\ \tau_1' = \theta(JL(\sigma_v), V_v)\} & \phi_\sigma : \text{irreducible}, \\ \{\tau_0' = \theta(\sigma_v, V_v)\} & \phi_\sigma : \text{reducible}. \end{cases}$$

$S_\psi \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and

$$S_{\psi_v} \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \phi_{\sigma_v} : \text{irreducible}, \\ \{1\} \times \mathbb{Z}/2\mathbb{Z} & \phi_{\sigma_v} : \text{reducible}. \end{cases}$$

Define a pairing $\langle\ ,\ \rangle_v$ as

$$\begin{aligned} \langle\, \cdot\,, \tau_\epsilon \,(\text{or } \tau_\epsilon^{\pm})\rangle_v &= \mathrm{sgn}^\epsilon \otimes \mathbf{1} && \text{if } v \notin S_D, \\ \langle\, \cdot\,, \tau_\epsilon' \rangle_v &= \mathrm{sgn}^\epsilon \otimes \mathrm{sgn} && \text{if } v \in S_D. \end{aligned}$$

$$\epsilon_\psi = \begin{cases} \mathbf{1} & \epsilon(1/2, \phi_\sigma) = 1, \\ \mathrm{sgn} \otimes \mathbf{1} & \epsilon(1/2, \phi_\sigma) \neq 1, \end{cases}$$

where $\epsilon(1/2, \phi_\sigma)$ is the value of Jacquet-Langlands ϵ-function of σ at 1/2. Then the Arthur's conjectural multiplicity is described by

$$m_\psi(\pi) = \frac{1}{2}(1 + \epsilon(\frac{1}{2}, \phi_\sigma)\langle(-1, 1), \pi\rangle) \quad (\pi \in \Pi_\psi^G).$$

If π is represented by the form $\Theta(\sigma, B, \eta)$ satisfying the condition of Theorem 5.1,1 then $m_\psi(\pi) = 1$.

8.2 Soudry type

An A-parameter ψ of Soudry type is written by the form

$$\psi = \psi_{k', \chi} = \left((\mathrm{Ind}_{W_{k'}}^{W_k} \chi \otimes \mathbf{1}) \oplus (\omega_{k'/k} \otimes \mathrm{Sym}^2)\right) \times p_k,$$

for some k' and χ.
(1) $\underline{v \notin S_D}$

$$\Pi^{\mathrm{s}}_{\psi_v} = \begin{cases} \{\theta(V_v^{\pm}, \tilde{\chi}_v)\} & \chi_v^2 \neq 1 \text{ and } \delta_v \neq -1, \\ \{\theta(\mathbb{H}_v, \tilde{\chi}_v)\} & \chi_v^2 \neq 1 \text{ and } \delta_v = -1, \\ \{\theta(V_v^{\pm}, \tilde{\chi}_v^{\pm})\} & \chi_v^2 = 1 \text{ and } \delta_v \neq -1, \\ \{\theta(\mathbb{H}_v, \tilde{\chi}_v^{\pm})\} & \chi_v^2 = 1 \text{ and } \delta_v = -1. \end{cases}$$

Here $V_v^{\pm}$ is the two-dimensional quadratic space over k_v with determinant δ and Hasse invariant ± 1, $\mathbb{H}_v$ is the two-dimensional hyperbolic space over k_v, and $\theta(V_v, \lambda_v)$ denotes the Howe correspondent of the representation λ_v of $G(V_v)$. The correspondent from $\tilde{\chi}_v^-$ is supercuspidal and the others are of the form of a quotient of $\mathrm{Ind}_{P_K(k_v)}^{Sp(2,k_v)}(\omega_{k_v'/k_v}|\cdot|_v \otimes \tau_v)$ for some irreducible representation τ_v of $SL(2, \mathbb{A})$.
(2) $\underline{v \in S_D}$

$$\Pi^{\mathrm{ns}}_{\psi_v} = \begin{cases} \{\theta(V_v, \chi_v), \theta(V_v, \chi_v^{-1})\} & \chi_v^2 \neq 1, \\ \{\theta(V_v, \chi_v)\} & \chi_v^2 = 1. \end{cases}$$

Elements of $\Pi^{\mathrm{ns}}_{\psi_v}$ are supercuspidal except for $\chi_v = 1$.

$$\mathcal{S}_\psi \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \chi^2 \neq 1, \\ D_4 & \chi^2 = 1, \end{cases}$$

where D_4 is the dihedral group with 8 elements. If k_v' is a quadratic extension of k_v then

$$\mathcal{S}_{\psi_v} \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \chi_v^2 \neq 1, \\ D_4 & \chi_v^2 = 1, \end{cases}$$

and if $k_v' \simeq k_v \oplus k_v$ then

$$\mathcal{S}_{\psi_v} \simeq \begin{cases} \{1\} & \chi_v^2 \neq 1, \\ \mathbb{Z}/2\mathbb{Z} & \chi_v^2 = 1. \end{cases}$$

Define a pairing $\langle\, ,\, \rangle_v$ as follows. If $v \in S_D$ and $\chi_v^2 = 1$ then

$$\langle s, \theta(V_v, \chi_v)\rangle_v = \begin{cases} 2 & s = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

and otherwise

$$\langle\, \cdot\, , \theta(V_v^{\eta}, \tilde{\chi}_v^{\epsilon})\rangle_v = \mathrm{sgn}^{\epsilon} \otimes \mathrm{sgn}^{\eta},$$

where we regard $\mathbb{H}_v = V_v^+$ and $\tilde{\chi}_v = \tilde{\chi}_v^+$. In case of Soudry type, $\epsilon_\psi = \mathbf{1}$. Then the Arthur's conjectural multiplicity is described by for an irreducible automorphic representation $\pi \in \Pi^G_\psi$,

$$m_\psi(\pi) = \begin{cases} 2^{|S_\chi \cap S_D| - 1} & \text{if } \chi^2 \neq 1,\ S_D \cap S_\chi \neq \emptyset, \\ 2^{|S_D| - 2} & \text{if } \chi^2 = 1,\ S_D \cap S_\chi \neq \emptyset, \\ 1 & \text{if } S_D \cap S_\chi = \emptyset. \end{cases}$$

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