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SELMER GROUPS FOR TENSOR PRODUCT L-FUNCTIONS

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ABSTRACT. We explain how, given a large prime divisor of a normalised critical value of a modular, symmetric square or tensor product L-function, an element of that order in an associated Selmer group may be constructed, in accord with the Bloch-Kato conjecture. The construction uses the 4-dimensional Galois representations attached to automorphic representations of $GSp_4(\mathbb{A}_{\mathbb{Q}})$. The only new case here is the tensor product L-function, where we use anticipated congruences between Yoshida lifts and non-lifts. In this case we can only deal with near-central critical values. We explain the apparent fundamental difficulty with the other critical values.

1. VARIOUS L-FUNCTIONS

Let $f=\sum_{n=0}^{\infty}a_n(f)q^n\in S_{k'}(\Gamma_0(N))$ be a normalised ($a_1=1$) cuspidal Hecke eigenform, with the weight $k'\geq 2$. Let $\mathbb{Q}(f)$ be the finite extension of $\mathbb{Q}$ generated by the Hecke eigenvalues $a_n(f)$.

Theorem 1.1 (Deligne). *For any number field $K\supset\mathbb{Q}(f)$ and prime divisor $\lambda\mid\ell$, there is a 2-dimensional K_λ-vector space $V_{f,\lambda}$ and a continuous linear representation*

$$\rho_{f,\lambda}:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\mathrm{Aut}(V_{f,\lambda}),$$

unramified outside ℓN, such that if $p\nmid\ell N$ is a prime, and Frob_p is an arithmetic Frobenius element, then

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_p^{-1}))=a_f(p),\ \det(\rho_f(\mathrm{Frob}_p^{-1}))=p^{k'-1}.$$

The condition on the determinant is a consequence of the existence of an alternating Galois equivariant pairing $V_{f,\lambda}\times V_{f,\lambda}\to K_\lambda(1-k')$, where the right hand side is a Tate twist of the trivial representation. This shows that the dual of $V_{f,\lambda}$ is $V_{f,\lambda}(k'-1)$.

The L-function $L(f,s)=\sum_{n=1}^{\infty}a_n(f)n^{-s}$ (for $\Re s>(k+1)/2$) is the L-function attached to the compatible system $(\rho_{f,\lambda})_\lambda$ of Galois representations, in the sense that whenever $p\neq\ell$, the p-factor in the Euler product is the reciprocal of

$$\det(I-\rho_{f,\lambda}(\mathrm{Frob}_p^{-1})|V_\lambda^{I_p}),$$

where I_p is an inertia subgroup at p.

By a well-known theorem of Hecke, $L(f,s)$ has an analytic continuation to the whole of $\mathbb{C}$, and satisfies a precise functional equation relating its values at s and $k'-s$. Its critical values (in the sense of Deligne [De]) are at integers t such that $1\leq t\leq k'-1$.

One may choose a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant O_λ-lattice $T_{f,\lambda}$ in $V_{f,\lambda}$, then define $W_{f,\lambda}:=V_{f,\lambda}/T_{f,\lambda}$. Let $W_f[\lambda]\simeq T_{f,\lambda}/\lambda T_{f,\lambda}$ be the λ-torsion in $W_{f,\lambda}$. This is a 2-dimensional

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$\mathbb{F}_\lambda$-vector space affording a representation $\overline{\rho}_{f,\lambda}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In general it depends on the choice of $T_{f,\lambda}$, though its semisimplification does not. Let $W_f := \oplus_\lambda W_{f,\lambda}$.

Following [BK] (Section 3), for $p \neq \ell$ (including $p = \infty$, where $\lambda \mid \ell$) let

$$H^1_f(\mathbb{Q}_p, V_{f,\lambda}(t)) := \ker\big(H^1(D_p, V_{f,\lambda}(t)) \to H^1(I_p, V_{f,\lambda}(t))\big).$$

Here D_p is a decomposition subgroup at a prime above p, I_p is the inertia subgroup, and $V_{f,\lambda}(t)$ is a Tate twist of $V_{f,\lambda}$, etc. The cohomology is for continuous cocycles and coboundaries. For $p = \ell$ let

$$H^1_f(\mathbb{Q}_\ell, V_{f,\lambda}(t)) := \ker\big(H^1(D_\ell, V_{f,\lambda}(t)) \to H^1(D_\ell, V_{f,\lambda}(t) \otimes_{\mathbb{Q}_\ell} B_{crys})\big).$$

(See §1 of [BK] or §2 of [Fo1] for the definition of Fontaine's ring B_{crys}.) Let $H^1_f(\mathbb{Q}, V_{f,\lambda}(t))$ be the subspace of those elements of $H^1(\mathbb{Q}, V_{f,\lambda}(t))$ that, for all primes p, have local restriction lying in $H^1_f(\mathbb{Q}_p, V_{f,\lambda}(t))$. There is a natural exact sequence

$$0 \longrightarrow T_{f,\lambda}(t) \longrightarrow V_{f,\lambda}(t) \stackrel{\pi}{\longrightarrow} W_{f,\lambda}(t) \longrightarrow 0.$$

Let $H^1_f(\mathbb{Q}_p, W_{f,\lambda}(t)) = \pi_* H^1_f(\mathbb{Q}_p, V_{f,\lambda}(t))$. Define the λ-Selmer group $H^1_f(\mathbb{Q}, W_{f,\lambda}(t))$ to be the subgroup of elements of $H^1(\mathbb{Q}, W_{f,\lambda}(t))$ whose local restrictions lie in $H^1_f(\mathbb{Q}_p, W_{f,\lambda}(t))$ for all primes p. Note that the condition at $p = \infty$ is superfluous unless $\ell = 2$. Define the Shafarevich-Tate group

$$Ш(t) = \bigoplus_\lambda \frac{H^1_f(\mathbb{Q}, W_{f,\lambda}(t))}{\pi_* H^1_f(\mathbb{Q}, V_{f,\lambda}(t))}.$$

Conjecture 1.2 (Case of Bloch-Kato). *Suppose that* $1 \le t \le k'-1$. *Then we have the following equality of fractional ideals of* K*:*

$$\frac{L(f,t)}{(2\pi i)^t \Omega^{(-1)^t}} = \frac{\prod_{p\le\infty} c_p(t)\ \#Ш(t)}{\#H^0(\mathbb{Q}, W_f(t))\#H^0(\mathbb{Q}, \check{W}_f(1-t))}.$$

We shall not define the Deligne periods $\Omega^\pm$ or the Tamagawa factors $c_p(t)$. (In later sections we shall always refer to other papers for full details anyway.) They depend on various choices, but the ratio of the two sides of the conjecture is independent of all choices. Deligne's conjecture (that the left hand side belongs to K) is known to be true in this case. The Bloch-Kato conjecture [BK, Fo2] may be viewed as an integral refinement.

Let $V_{\mathrm{Sym}^2 f,\lambda} = \mathrm{Sym}^2 V_{f,\lambda}$, and if $g \in S_k(\Gamma_0(N))$ (with $k' > k \geq 2$) is another normalised newform then let $V_{f\otimes g,\lambda} = V_{f,\lambda} \otimes V_{g,\lambda}$. Let $L(\mathrm{Sym}^2 f, s)$ and $L(f\otimes g, s)$ be the L-functions associated to these compatible systems of Galois representations.

$L(\mathrm{Sym}^2 f, s)$ has an analytic continuation to $\mathbb{C}$, and a functional equation relating its values at s and $2k'-1-s$. Its critical points are the pairs $r+k'-1$ and $k'-r$ for odd r with $1 \le r \le k'-1$, and Deligne's conjecture holds [Ran, Sh1, Z].

$L(f\otimes g, s)$ has an analytic continuation to $\mathbb{C}$, and a functional equation relating its values at s and $k+k'-1-s$. Its critical points are integers t with $k \le t \le k'-1$, and Deligne's conjecture holds [Ran, Sh2]. There is a pair of critical points $t = (k'+k-2)/2$ and $t = (k'+k)/2$ either side of the centre of the functional equation.

Sometimes the Bloch-Kato conjecture, applied to critical values of $L(f,s)$, $L(\mathrm{Sym}^2 f, s)$ or $L(f\otimes g, s)$, demands non-zero elements of various associated Selmer groups. The goal of this expository article is to explain how such elements can sometimes be constructed, in part reviewing work of others. In each case (except in §4) there is some special Siegel modular form of genus 2 (a Saito-Kurokawa lift, a Klingen-Eisenstein series or a Yoshida lift) associated to the genus-1 form(s). Then, starting from the appearance of some large prime λ in a normalised L-value, one shows that there is a congruence (mod λ) of Hecke eigenvalues between the special form and some other genus-2 (cusp) form F. Now consider a reduction of the 4-dimensional λ-adic Galois representation attached to F by Laumon and Weissauer [L, W, T]. Using the relation between eigenvalues of Hecke operators and traces of Frobenius, the Galois interpretation of the congruence is the reducibility of this mod λ representation, with composition factors coming from the genus-1 form(s) or twists of the trivial representation. Non-trivial extensions among these factors, inside the 4-dimensional representation, give rise to the required classes in Galois cohomology. The irreducibility of the λ-adic representation is important here, though I will not mention it again. The basic idea originates in Ribet's construction of elements in class groups of cyclotomic fields, where the congruences are between Eisenstein series and cusp forms (genus-1) [Ri]. It has been much used since, [MW, BC, SU, Br, Klo, Be].

2. Galois representations attached to Siegel cusp forms of genus two.

Let $\mathfrak{H}_2$ be the Siegel upper half plane of 2 by 2 complex symmetric matrices with positive-definite imaginary part. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_2 := \mathrm{Sp}(4,\mathbb{Z})$ and $Z \in \mathfrak{H}_2$, let $M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$ and $J(M,Z) := CZ+D$. Let V be the space of the representation $\rho_{j,k} := \det^k \otimes \mathrm{Sym}^j(\mathbb{C}^2)$ of $\mathrm{GL}(2,\mathbb{C})$. A holomorphic function $f : \mathfrak{H}_2 \to V$ is said to belong to the space $M_{j,k} = M_{j,k}(\Gamma_2)$ of Siegel modular forms of genus two and weight $\rho_{j,k}$ if

$$f(M\langle Z\rangle) = \rho_{j,k}(J(M,Z))f(Z)\ \forall M \in \Gamma_2, Z \in \mathfrak{H}_2.$$

(We consider only even j, to avoid necessitating $f = 0$.) See §2.1 of [A] or (2.12) of [Sa] for the definition of Hecke operators $T(m)$ in genus 2.

Let $\Gamma_1 = \mathrm{SL}(2,\mathbb{Z})$ and $M_{k'}(\Gamma_1)$ be the space of modular forms of weight k' for Γ_1. The space $S_{j,k}$ of Siegel cusp forms of genus two and weight $\rho_{j,k}$ is defined to be the kernel of the Siegel operator $\Phi : M_{j,k}(\Gamma_2) \to M_{j+k}(\Gamma_1)$. See [A] for details.

Take $F \in S_{k,j}(\Gamma_2)$ a Hecke eigenform. To F may be associated a cuspidal automorphic form $\Phi_F \in L^2_0(Z(\mathbb{A}_\mathbb{Q})\mathrm{GSp}(4,\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{A}_\mathbb{Q}))$. This "lifting procedure" is described in detail in §3 of [AS] (§3.1 for the scalar-valued case, §3.5 for the vector-valued case). Let Π_F be any irreducible constituent of the unitary representation of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ generated by right translates of Φ_F, as in 3.4 of [AS]. They are all isomorphic, in fact this unitary representation is expected to be irreducible already. To this Π_F we shall shortly apply the following theorem, which is part of Theorem I of [W].

Theorem 2.1 (Weissauer). *Suppose that Π is a unitary, irreducible, automorphic representation of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ for which Π_∞ belongs to the discrete series of weight (k_1,k_2). Let S denote the set of ramified places of the representation Π. Put $w = k_1 + k_2 - 3$. Then there exists a number field E such that*

(1) *for any prime $p \notin S$, if $L_p(p^{-s}) = L_p(\Pi_p, s - w/2)$ is the local factor in the spinor L-function, then $L_p(X)^{-1} \in E[X]$;*

(2) *for any prime* λ *of* O_E, *there exists a finite extension* K *of* E *(and* K_λ *of* E_λ*), and a 4-dimensional semisimple Galois representation*

$$\rho_{\Pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(4, K_\lambda),$$

unramified outside $S \cup \{\ell\}$ *(where* $\lambda \mid \ell$*), such that for each prime* $p \notin S \cup \{\ell\}$,

$$L_p(\Pi_p, s - w/2) = \det(I - \rho_{\Pi,\lambda}(\mathrm{Frob}_p^{-1})p^{-s})^{-1}.$$

The main theorems in [W] depend on hypotheses (A and B), whose proofs are currently only in preprint form. Section 3 of [U] gives an alternative proof of the above, in the case that $\ell \notin S$, $k_1 > k_2 > 3$ and the Newton polygon of $L_\ell(X)^{-1}$ has four distinct ℓ-slopes.

Proposition 2.2. *Let* F *be as above, with* $k \geq 3$. *If* K_λ *is a sufficiently large extension of* $\mathbb{Q}_\ell$ *then there exists a 4-dimensional semisimple Galois representation*

$$\rho_{F,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(4, K_\lambda),$$

unramified outside $\{\ell\}$, *such that for each prime* $p \neq \ell$, $\det(I - \rho_{F,\lambda}(\mathrm{Frob}_p^{-1})p^{-s})^{-1}$ *is the local factor in the spinor* L *function of* F.

This is a direct consequence of Theorem 2.1, applied to Π_F. Note that here $k_1 = j + k$, $k_2 = k$, $w = j + 2k - 3$, and the condition $k \geq 3$ is necessary to ensure that Π_∞ is discrete series. Also Π_F is unramified at all primes p, since F is for the full modular group Γ_2.

Let $V_{F,\lambda}$ be the space of the representation $\rho_{F,\lambda}$, and let $T_{F,\lambda}$ be a choice of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant O_λ-lattice in $V_{F,\lambda}$. Let $W_F[\lambda] = T_{F,\lambda}/\lambda T_{F,\lambda}$. Let $\overline{\rho}_{F,\lambda}$ be the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $W_F[\lambda]$. It may depend on the choice of $T_{F,\lambda}$, though its semisimplification does not.

3. Brown's construction for $L(f,k)$: congruences between Saito-Kurokawa lifts and non-lifts.

For more details of the next two sections, see §§3 and 4 of [DIK]. Let k be even and suppose that $f \in S_{2k-2}(\Gamma_1)$ is a normalised Hecke eigenform, say $f = \sum_{n=1}^{\infty} a_n(f)q^n$. Associated to f is a Saito-Kurokawa lift $\hat{f} \in S_k(\Gamma_2)$. This $\hat{f}$ is only defined up to scaling. If $\mu_{\hat{f}}(p)$ denotes the eigenvalue of the genus-2 Hecke operator $T(p)$ acting on $\hat{f}$, then, for all primes p,

$$\mu_{\hat{f}}(p) = a_p(f) + p^{k-2} + p^{k-1}.$$

Now suppose that $\lambda' \mid \ell$ is a prime divisor in the field $\mathbb{Q}(f)$ generated by the Hecke eigenvalues of f, such that

$$\mathrm{ord}_{\lambda'} L_{\mathrm{alg}}(f,k) > 0,$$

where, for critical points $1 \leq t \leq 2k-3$, $L_{\mathrm{alg}}(f,t) := \frac{L(f,t)}{(2\pi i)^t \Omega^\pm}$, $\pm = (-1)^t$ and $\Omega^\pm$ are certain canonical periods. If $\ell > 2k-2$ then one may show that the only factor on the right-hand-side of the Bloch-Kato conjecture that could possibly account for the factor of λ' is Ш(k). Here is how, using the construction of [Br], to obtain the required non-zero element of the λ-Selmer group $H^1_f(\mathbb{Q}, W_{f,\lambda}(k))$ (under certain conditions), for $\lambda \mid \lambda'$ in an appropriate field $K \supset \mathbb{Q}(f)$.

(1) Show that there exists another Hecke eigenform $F \in S_k(\Gamma_2)$, not a Saito-Kurokawa lift, such that for all primes p,

$$\mu_F(p) \equiv \mu_{\hat{f}}(p) \equiv a_f(p) + p^{k-2} + p^{k-1} \pmod{\lambda},$$

with $\lambda \mid \lambda'$ in any finite extension of $\mathbb{Q}(f)\mathbb{Q}(F)$. This depends on a formula for the pullback to $\mathfrak{H}_2 \times \mathfrak{H}_2$ of a differentiated Siegel-Eisenstein series of genus 4, in which the coefficient of $G \otimes G$ (for a Hecke eigenform $G \in S_k(\Gamma_2)$) is essentially a normalised critical value of its standard L-function, $L(G, m, \mathrm{St})/\langle G, G\rangle$. The standard L function of $\hat{f}$ is $\zeta(s)L(f, s+k-1)L(f, s+k-2)$, whose value at an even integer $0 < m \leq k-2$ is an algebraic multiple of the product of $\langle f, f\rangle$ and a power of π. It turns out (combining formulas of [KS] and [KZ]) that $L(f, k)$ appears in a formula for the ratio of $\langle \hat{f}, \hat{f}\rangle$ to $\langle f, f\rangle$. Thus λ gets to appear in the denominator of the coefficient $L(\hat{f}, m, \mathrm{St})/\langle \hat{f}, \hat{f}\rangle$. Applying elements of the Hecke algebra to an appropriate partial Fourier coefficient in the pullback formula, it is possible then to prove that there must be an F for which the congruence holds. See [Ka] for details.

(2) Take $K \supset \mathbb{Q}(f)\mathbb{Q}(F)$ big enough that we may apply Proposition 2.2. The Galois interpretation of the congruence of Hecke eigenvalues is that the composition factors of the representation $\overline{\rho}_{F,\lambda}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are $\overline{\rho}_{f,\lambda}$ (if we assume this to be irreducible, which would follow from $\ell > 2k-2$ and $\ell \nmid B_{2k-2}$), and the Tate twists $\mathbb{F}_\lambda(2-k)$ and $\mathbb{F}_\lambda(1-k)$ of the trivial representation. It is possible to choose the invariant lattice $T_{F,\lambda}$ in such a way that $\overline{\rho}_{F,\lambda}$ contains a non-trivial extension of $\mathbb{F}_\lambda(2-k)$ by $\overline{\rho}_{f,\lambda}$, from which we get a non-trivial extension of $\mathbb{F}_\lambda$ by $\overline{\rho}_{f,\lambda}(k-2)$. This gives rise to a non-zero class in $H^1(\mathbb{Q}, W_f[\lambda](k-2))$, which maps to a non-zero class in $H^1(\mathbb{Q}, W_{f,\lambda}(k-2))$. It is possible to show that this class satisfies the local conditions to belong to the Bloch-Kato Selmer group $H^1_f(\mathbb{Q}, W_{f,\lambda}(k-2))$. By a theorem of Kato [Kato], this Selmer group is finite, so is equal to $Ш(k-2)$. By [Fl] we then get a non-zero element of λ-torsion in $Ш(k)$, as desired. (Alternatively, just consider $L_{\mathrm{alg}}(f, k-2)$ instead, bearing in mind the functional equation.)

4. Harder's conjecture and other critical values.

Let $f \in S_{k'}(\Gamma_1)$ be a normalised Hecke eigenform. In the previous section we considered the near-central, normalised critical value $L_{\mathrm{alg}}(f, (k'/2)+1)$. More generally, for $k'/2 < t \leq k'-1$, we consider $L_{\mathrm{alg}}(f, t)$. Choose $j, k \geq 0$ such that $t = j+k$ and $k' = j+2k-2$. In other words, $k = k'+2-t$, $j = 2t-2-k'$. Note that t is paired with $k-2$ by the functional equation of $L_f(s)$, and that $j = 0$ when $t = (k'/2)+1$ (in which case $k' = 2k-2$ and $t = k$).

In the case $j > 0$ there is no Saito-Kurokawa lift. The following, due to Harder, is Conjecture 3 in §26 of [vdG]. Special cases are discussed in [Ha]. It bypasses the Saito-Kurokawa lift, and necessitates the use of vector-valued forms.

Conjecture 4.1. *Let* $f = \sum a_n(f)q^n \in S_{k'}(\Gamma_1)$ *be a normalised eigenform, and suppose that a "large" prime* λ' *of* $\mathbb{Q}(f)$ *divides* $L_{\mathrm{alg}}(f, t)$, *with* $(k'/2) < t \leq k'-1$. *As above, let* $k = k'+2-t, j = 2t-2-k'$. *In the case* $j > 0$, *there exists an eigenform* $F \in S_{k,j}(\Gamma_2)$, *and a prime* $\lambda \mid \lambda'$ *in (any field containing)* $\mathbb{Q}(f)\mathbb{Q}(F)$ *such that, for all primes* p,

$$\mu_F(p) \equiv p^{k-2} + p^{j+k-1} + a_p(f) \pmod{\lambda}.$$

Numerical evidence obtained by Faber and van der Geer [vdG] supports the conjecture in several cases, including $k' = 22, t = 14, j = 4, k = 10, \ell = 41$.

The construction in the previous section did not really require $\hat{f}$, only the congruence, so if the congruence promised by Harder's conjecture holds then we may proceed in exactly the same way (under certain conditions) to construct the element of Ш(t) required by the Bloch-Kato conjecture.

5. Urban's construction for $L(\mathrm{Sym}^2 f, s)$: congruences between Klingen-Eisenstein series and cusp forms.

For more details on this section, see [Du]. With respect to a natural inner product, there is an orthogonal decomposition $M_{j,k} = S_{j,k} \oplus N_{j,k}$, and, in the case $j > 0$, a linear isomorphism $S_{j+k}(\Gamma_1) \simeq N_{j,k}$, where to $g \in S_{j+k}(\Gamma_1)$ we associate its vector-valued Klingen-Eisenstein series, denoted $[g]_j$. See §1 of [A] (which generalises [Kli]) for more details. In the case $j = 0$ there are still Klingen-Eisenstein series, but one needs also the Siegel Eisenstein series to complete a basis for $N_{j,k}$.

The Klingen-Eisenstein series $[f]_j$ satisfies $\Phi([f]_j) = f$. If f is a Hecke eigenform, with Hecke eigenvalues $a_f(m)$, then $[f]_j$ is also a Hecke eigenform. If $\mu_{[f]_j}(m)$ is the eigenvalue for $T(m)$ on $[f]_j$, then for prime p we have

$$\mu_{[f]_j}(p) = (1 + p^{k-2})a_f(p).$$

Consequently the Andrianov (spinor) L-function satisfies

$$L([f]_j, s, \mathrm{spin}) = L(f, s)L(f, s - (k-2)).$$

See §§2 and 3 of [A] for more details.

Proposition 5.1. *Let* f *a normalised generator of* $S_{k'}(\Gamma_1)$, *with* $k' = 16, 18, 20, 22$ *or* 26. *Suppose that* $\ell > 2k'$ *is a prime such that* $\ell \mid L(\mathrm{Sym}^2 f, r+k'-1)/\langle f, f\rangle \pi^{2r+k'-1}$, *with* r *odd and* $1 \le r \le k'-1$. *Suppose further that* $r \ge 7$, *if* $k'/2$ *is even, and* $r \ge 9$ *(but* $r \ne 11$*) if* $k'/2$ *is odd. Let* $j = k'-1-r$ *and* $k = k'-j = r+1$. *Then there exists a Hecke eigenform* $F \in S_{j,k}$ *such that, for all* $m \ge 1$,

$$\nu(m, F) \equiv \nu(m, [f]_j) \equiv a_f(p)(1 + p^{k-2}) \pmod{\lambda},$$

where λ *is a prime ideal dividing* ℓ *in a sufficiently large field.*

See [Du] for the proof. It depends on a formula of [BSY] for the Fourier coefficients of $[f]_j$, in which $L(\mathrm{Sym}^2 f, r+k'-1)$ appears in the denominator. Certain cases had been proved by Kurokawa [Ku], Mizumoto [M] ($j = 0$) and by Satoh ($j = 2$). Generalising the method of proof to the case $\dim S_{k'}(\Gamma_1) > 1$ seems problematical.

In this situation (i.e. $\ell \mid L(\mathrm{Sym}^2 f, r+k'-1)/\langle f, f\rangle \pi^{2r+k'-1}$), the Bloch-Kato conjecture predicts that there should be an element of order ℓ in a Shafarevich-Tate group $Ш(r+k'-1) = Ш(2k'-2-j)$. Slightly adapting a construction of Urban [U], given the congruence this may be done as follows.

It is necessary to suppose that the image of $\rho_{F,\lambda}$ lies in $GSp_4(K_\lambda)$. This is known to be true in some cases (e.g. when $\dim S_{j,k} = 1$), and is expected to be true in general. One must also suppose that $\ell \nmid B_{k'}$ (which, given $\ell > 2k'$, ensures that $\overline{\rho}_{f,\lambda}$ is irreduible). The Galois interpretation of the congruence of Hecke eigenvalues is that the composition factors of $\overline{\rho}_{F,\lambda}$ are $\overline{\rho}_{f,\lambda}$ and $\overline{\rho}_{f,\lambda}(2-k)$. It is possible to choose the invariant lattice $T_{F,\lambda}$ in such a way that $W_F[\lambda]$ is an extension of $W_f[\lambda]$ by $W_f[\lambda](2-k)$, and one can show that this extension is non-trivial. This gives rise to a non-zero cohomology class in $H^1(\mathbb{Q}, \mathrm{Hom}(W_f[\lambda], W_f[\lambda](2-k)))$.

Now the dual of $W_f[\lambda]$ is isomorphic to $W_f[\lambda](k'-1) = W_f[\lambda](j+k-1)$. Hence $\mathrm{Hom}(W_f[\lambda], W_f[\lambda](2-k)) \simeq W_f[\lambda]\otimes W_f[\lambda]((j+k-1)+(2-k)) = W_f[\lambda]\otimes W_f[\lambda](j+1)$. In fact, one can show that our cohomology class is in $H^1(\mathbb{Q}, (\mathrm{Sym}^2 W_f[\lambda])(j+1))$, and, under the condition $\ell > 2j+3k-2$, that it gives rise to a non-zero class in $H^1(\mathbb{Q}, W_{\mathrm{Sym}^2 f,\lambda}(j+1))$ satisfying the Bloch-Kato local conditions. This Selmer group is conjecturally finite (since the corresponding L-value is non-zero), so we should have now a non-zero element of λ-torsion in $Ш(j+1)$, which we could transfer to $Ш(2k'-2-j)$ using [Fl].

Note that the scalar-valued case $j = 0$ goes with the rightmost critical point $2k'-2$. By varying the decomposition $k' = j+k$ we can move leftwards through the other critical values $2k'-2-j$, using vector-valued Siegel modular forms.

6. $L(f\otimes g, (k_1+k_2-2)/2)$: CONGRUENCES BETWEEN YOSHIDA LIFTS AND NON-LIFTS.

Let $f \in S_{k'}(\Gamma_0(N)), g \in S_k(\Gamma_0(N))$ be normalised newforms (with $k' > k \geq 2$ and $N > 1$), such that, for each Atkin-Lehner involution, the eigenvalues of f and g are equal. Let $k' = 2\nu_1+2, k = 2\nu_2+2$. Choose a factorisation $N = N_1N_2$, with N_1 the product of an odd number of prime factors, and let D be the definite quaternion algebra over $\mathbb{Q}$, ramified at ∞ and primes dividing N_1. Let R be an Eichler order of level N in D. With notation as in §1 of [BS2], let $\phi_1 : D_{\mathbb{A}} \to U_{\nu_1}^{(0)}$ and $\phi_2 : D_{\mathbb{A}} \to U_{\nu_2}^{(0)}$ correspond to f and g respectively, under the Jacquet-Langlands correspondence. Let $F_{f,g} = F_{\phi_1,\phi_2}$ (which of course depends on the choice of N_1) be the Yoshida lift [Y, BS3]. Then $F \in S_\rho(\Gamma_0^{(2)}(N))$, with $\rho = \det^\kappa \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, $j = k-2$, $\kappa = 2+\frac{k'-k}{2}$. It is an eigenform for the local Hecke algebras at $p \nmid N$, and, for all primes $p \nmid N$,

$$\mu_{F_{f,g}}(p) = a_f(p) + p^{(k'-k)/2}a_g(p).$$

Equivalently,

$$L_S(F_{f,g}, s, \mathrm{spin}) = L_S(f,s)L_S(g, s-(k'-k)/2),$$

where we omit Euler factors at $p \mid N$.

Suppose that (for some $k \leq t \leq k'-1$), $\mathrm{ord}_\lambda\left(\frac{L(f\otimes g,t)}{\pi^{2t-(k-1)}\langle f,f\rangle}\right) > 0$, with $\ell \nmid N$ and $\ell > k'+k-1$, where $\lambda \mid \ell$ in a finite extension K of $\mathbb{Q}(f)\mathbb{Q}(g)$. Suppose also that $\overline{\rho}_{f,\lambda}$ and $\overline{\rho}_{g,\lambda}$ are irreducible, with minimal level N. Then it is possible to show that the Bloch-Kato conjecture predicts that $\mathrm{ord}_\lambda(\#Ш(t)) > 0$, so that the Selmer group $H^1_f(\mathbb{Q}, W_{f\otimes g,\lambda}(t))$ should be non-trivial. In the case that $t = (k'+k-2)/2$, here is how to construct a non-zero element of this group.

(1) Under certain conditions, it should be possible to prove that there is a cusp form F for $\Gamma_0^{(2)}(N)$, an eigenvector for all the local Hecke algebras at $p \nmid N$, not itself a Yoshida lift, such that there is a congruence (mod λ) of all Hecke eigenvalues (for $p \nmid N$) between F and $F_{f,g}$. In particular,

$$\mu_F(p) \equiv a_p(f) + p^{(k'-k)/2}a_p(g) \pmod{\lambda}, \quad \text{for all } p \nmid N.$$

A bit like in §3, this should result from the appearance of λ in a formula for the ratio $\langle F_{f,g}, F_{f,g}\rangle / \langle \phi_1\otimes\phi_2, \phi_1\otimes\phi_2\rangle$. Proposition 10.2(ii) of [BS1] is such a formula in the case $k' = k = 2$ (which is not under consideration), where F is scalar valued of weight 2. The proof involves Siegel's main theorem for a genus-4 Eisenstein series, differential operators and a pullback formula. Their method should generalise to our situation, and they are looking into this. Great care will be required with various constants and scalings. A factor of $L_S(F_{f,g}, 0, St)$ appears in the formula, but this is essentially $L(f \otimes g, (k' + k - 2)/2)$, which is how λ becomes involved. The formula may be viewed as a "Rallis inner-product formula", since it relates the norms of a theta lift and what was lifted [Ral].

(2) The Galois interpretation of the congruence of Hecke eigenvalues is that the composition factors of $\overline{\rho}_{F,\lambda}$ are $\overline{\rho}_{f,\lambda}$ and $\overline{\rho}_{g,\lambda}((k-k')/2)$. It is possible to choose $T_{F,\lambda}$ in such a way that the latter is a submodule and the former is a quotient, and one can show that the extension is non-trivial. Hence one obtains a non-zero element of $H^1(\mathbb{Q}, \mathrm{Hom}(W_f[\lambda], W_g[\lambda]((k-k')/2)))$ $= H^1(\mathbb{Q}, W_f[\lambda] \otimes W_g[\lambda](k'-1+(k-k')/2)) = H^1(\mathbb{Q}, W_{f\otimes g}[\lambda]((k'+k-2)/2))$. One may show that this gives a non-zero element of $H^1(\mathbb{Q}, W_{f\otimes g,\lambda}((k'+k-2)/2))$ satisfying the Bloch-Kato local conditions. The local conditions at $p \mid N$ are especially subtle, and one must suppose that the image of $\rho_{G,\lambda}$ lies in $GSp_4(K_\lambda)$. For the local condition at ℓ, we actually need $\ell > \frac{3k'+k-2}{2}$.

7. The problem with the other critical values.

The Hodge-Tate weights of $\rho_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ are

$$0 \text{ and } k'-1.$$

Those of $\rho_{F,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ are

$$0,\ k-2,\ j+k-1,\ j+2k-3,$$

where $F \in S_{j,k}(\Gamma_2)$. These numbers can be recovered from the $\pmod{\lambda}$ representations, using the Fontaine-Lafaille theory [FL], since the λ-adic representations are crystalline (for sufficiently large $\ell \nmid N$). They are the places where the filtration of the associated module jumps. The numbers $0, k-2, j+k-1, j+2k-3$ are arranged symmetrically about their centroid on the number line. This is necessarily the case, since $\rho_{F,\lambda}$ is self-dual up to twist.

In the case $L(f,s)$ which we looked at, the composition factors of $\overline{\rho}_{F,\lambda}$ are $\overline{\rho}_{f,\lambda}$ (with "Hodge-Tate weights" 0 and $j+2k-3$, since $k' = j+2k-2$) and $\mathbb{F}_\lambda(2-k), \mathbb{F}_\lambda(1-j-k)$ (with weights $k-2, j+k-1$ respectively). Fixing k', the weight of f, we were able to vary j and k subject to $j+2k-2 = k'$, thus to deal with different critical points (subject to Harder's conjecture). The difference between $j+2k-3$ and 0 is fixed at $k'-1$, but the difference between $j+k-1$ and $k-2$ is free to vary.

In the case $L(\mathrm{Sym}^2 f, s)$, 0 and $j+k-1$ are the weights of $\overline{\rho}_{f,\lambda}$, while the overlapping $k-2$ and $j+2k-3$ are the weights of $\overline{\rho}_{f,\lambda}(2-k)$. By varying j and k subject to $j+k = k'$, we were able to deal with different critical points. The differences $(j+2k-3)-(k-2)$ and $(j+k-1)-0$ are forced to be the same, by the symmetry. Both equal $k'-1$.

In the case $L(f \otimes g, s)$, the weights k' of f and k of g (beware the change in meaning of k) are different, so the symmetry forces $k'-1 = (j+2\kappa-3)-0$ and, inside that, $k-1 = (j+\kappa-1)-(\kappa-2)$. Since there are now two weights k' and k, we have lost a degree of freedom, and cannot vary j and κ. Therefore only the near-central critical points can be dealt with by this method, if the only Galois representations at our disposal are self-dual (up to twist). To deal with other critical points, perhaps one would need Galois representations attached to non-self-dual cuspidal automorphic representations of $GL_4(\mathbb{A}_\mathbb{Q})$, but it seems that nobody knows how to construct such Galois representations. Note that, in the case of near-central critical values of $L(f \otimes g, s)$, we are, in general, already dealing with vector-valued forms.

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