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Explicit formulas for the twisted Koecher-Maaß series for the Saito-Kurokawa lift and their applications

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1 Introduction

The theory of explicit formulas for the Koecher-Maaß series is initiated by Böcherer [1] and Ibukiyama and the first named author [5], [6], [7]. So far, there are some applications of these explicit formulas to the theory of modular forms. For example, we can refer to [2], [7], [4], [9]. In our talk, we announced a new result in this direction, that is an explicit formula for the twisted Koecher-Maaß series associated with the Saito-Kurokawa lift was given and their applications were presented.

As for "twist" by Dirichlet characters χ, in view of Saito [10] for example, one of the most natural one seems to be

$$L^*(s,F,\chi)=\sum_{T}\frac{\chi(\det(2T))c_F(T)}{\epsilon(T)(\det T)^s},$$

where T runs over a complete set of representatives of $SL_n(\mathbf{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree n, $c_F(T)$ is the T-th Fourier coefficient of a Siegel modular form F on $\Gamma_n=Sp_n(\mathbf{Z})$

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and $\epsilon(T) = \#\{U \in SL_n(\mathbf{Z}); T[U] = T\}$. We will sometimes call $L^*(s, F, \chi)$ the twisted Koecher-Maaß series of the second kind.

On the other hand, Choie-Kohnen [3] introduced a different type of "twist". For a positive integer N, let $SL_{n,N}(\mathbf{Z}) = \{U \in SL_n(\mathbf{Z}); U \equiv 1_n \mod N\}$ and $\epsilon_N(T) = \#\{U \in SL_{n,N}(\mathbf{Z}); T[U] = T\}$. For a primitive Dirichlet character χ mod N, the Koecher-Maaß series $L(s, F, \chi)$ of F twisted by χ is defined to be

$$L(s, F, \chi) = \sum_{T} \frac{\chi(\mathrm{tr}(T)) c_F(T)}{\epsilon_N(T)(\det T)^s},$$

where T runs over a complete set of representatives of $SL_{n,N}(\mathbf{Z})$-equivalence classes of positive definite half-integral symmetric matrices of degree n. In [3], Choie and Kohnen proved a meromorphic continuation of $L(s, F, \chi)$ to the whole s-plane and a functional equation. Moreover they got a result on the algebraicity of its special values.

Theorem 1. (Choie-Kohnen) *Let F be an element in the space $S_k(\Gamma_n)$ of all Siegel cusp forms of weight k on Γ_n. Put*

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^{n} \pi^{i-1}/2\Gamma(s - (i-1)/2),$$

and put

$$\Lambda(s, F, \chi) = N^{2s} \tau(\chi)^{-1} L(s, F, \chi) \quad (\mathrm{Re}(s) >> 0),$$

where $\tau(\chi)$ is the Gauss sum of χ. Then $\Lambda(s, F, \chi)$ has a analytic continuation to the whole s-plane and has the following functional equation:

$$\Lambda(k - s, F, \chi) = (-1)^{nk/2} \chi(-1) \Lambda(s, F, \bar{\chi}).$$

Theorem 2. (Choie-Kohnen) *Let $F \in S_k(\Gamma_n)$ with k even. Then there exists a $\mathbf{Z}$-module $M_f \subset \mathbf{C}$ of finite rank such that*

$$\frac{N L(m, F, \chi)}{\tau(\chi)(2\pi\sqrt{-1})^{nm}} \in M_f \otimes_{\mathbf{Z}} \mathbf{Z}[\chi]$$

for any primitive character χ and any integer m such that $(n+1)/2 \leq m \leq k - (n+1)/2$, where $\mathbf{Z}[\chi]$ is the $\mathbf{Z}$-module obtained from by adjoining the values of χ.

We shall call $L(s,F,\chi)$ the twist of the first kind. Our main results in this report are explicit formulas for the twisted Koecher-Maaß series associated with the Saito-Kurokawa lift, namely for the twist in the sence of Choie-Kohnen (i.e. the twist of the first kind). In order to state our results, take a cusp form $h(z), z \in H_1 = \{z = x+\sqrt{-1}y; y>0\}$ in the Kohnen plus space $S^+_{k-1/2}(\Gamma_0(4))$ with k even. By definition this has a Fourier expansion of the form

$$h(z) = \sum_{m\geq 1,\ m\equiv 0,3 \pmod 4} c(m)\mathbf{e}(mz) \in S^+_{k-1/2}(\Gamma_0(4)).$$

Define a function on $H_2 = \{Z = {}^tZ \in M_2(\mathbf{C}); \Im Z > O\}$ by

$$M(h)(Z) = \sum_{T\in\mathcal{L}_{2>0}} \left(\sum_{d|e(T)} d^{k-1}c\left(\frac{\det 2T}{d^2}\right)\right)\mathbf{e}(tr(TZ)),$$

where T runs over $\mathcal{L}_{2>0}$ the set of all positive definite half-integral symmetric matrices of degree 2 and $e(T) = G.C.D(n,r,m)$ for $T = \begin{pmatrix} n & r/2 \\ r/2 & m\end{pmatrix}$. As is well known, $M(h)(Z)$ is a Siegel cusp form of degree two and even weight k, called the Saito-Kurokawa lift of $h(z)$.

It is quite easy to get an explicit form of the twist of the second kind. In fact, by using the same argument as in Böcherer [1], one easily obtains

$$L^*(s,M(h),\chi) = 2^{2s}L(2s-k+1,\chi^2)\sum_{d=1}^{\infty}\frac{c(d)\chi(d)H_1(d)}{d^s}, \tag{1}$$

where for any $D \in \mathbf{Z}_{>0}$, we put

$$H_1(D) = \sum_{A\in\mathcal{L}_{2>0}(D)/SL_2(\mathbf{Z})} \frac{1}{\epsilon(A)}$$

with $\mathcal{L}_{2>0}(D) = \{A \in \mathcal{L}_{2>0}; 4\det A = D\}$.

On the other hand, it seems non-trivial to get that of the first kind. Our Theorem 3 and 4 are concerned with it. Moreover, combining with Choie-Kohnen and Shimura's results, the resulting explicit formula gives a new kind of applications to the Rankin-Selberg convolutions of modular forms of half-integral weight. To be more precise, we need some notation. For a Dirichlet character η mod N, let $E^{(\eta)}_{3/2}(z)(z = x+\sqrt{-1}y \in H_1)$ be the twist by η of Zagier's Eisenstein series of weight 3/2. It has the Fourier expansion

$$E^{(\eta)}_{3/2}(z) = 2\sum_{m=0}^{\infty}\eta(m)H_1(m)\mathbf{e}(mz) + y^{-1/2}\sum_{n=-\infty}^{\infty}\beta(4\pi n^2y)\mathbf{e}(-n^2z)\eta(-n^2),$$

where we put $H_1(0) = -\frac{1}{24}$ and

$$\beta(x) = \frac{1}{16\pi} \int_1^{\infty} u^{-3/2} e^{-xu} du.$$

If η is primitive, the Eisenstein series $E_{3/2}^{(\eta)}(z)$ belongs to $M_{3/2}^{\infty}(\Gamma_0(4N^2), \eta^2)$, the space of C^{∞} modular forms of weight 3/2, character η^2 and level $4N^2$. Note that there exist constants $A, a, b > 0$ such that $|E_{3/2}^{(\eta)}(z)| \leq A(y^a + y^{-b})$ for any $z = x + \sqrt{-1}y \in H_1$. Let $N = p_1^{e_1} \cdots p_r^{e_r}$ be the prime decomposition of N and put $\tilde{N} = p_1 \cdots p_r$. We then define $\mathcal{E}_{3/2}^{(\eta)}(z)$ by

$$\mathcal{E}_{3/2}^{(\eta)}(z) = \sum_{M|\tilde{N}} E_{3/2}^{((\frac{*}{M})\eta)}(z),$$

where $(\frac{*}{M})$ denotes the Jacobi symbol. We note here that if $(\frac{*}{M})\eta$ are primitive of conductor N for all $M|\tilde{N}$ and if $p \equiv -1$ mod 4 for some prime factor p of N, then $\mathcal{E}_{3/2}^{(\eta)}(z)$ is holomorphic on H_1 and therefore by the above growth condition, we see that it belongs to $M_{3/2}(\Gamma_0(4N^2), \eta^2)$.

Now for $h_1(z) = \sum_{m=0}^{\infty} c_1(m)\mathbf{e}(mz) \in M_{k-1/2}(\Gamma_0(4))$ and an element $h_2(z)$ of $M_{l-1/2}^{\infty}(\Gamma_0(4N^2), \eta^2)$ with the Fourier expansion

$$h_2(z) = 2\sum_{m=0}^{\infty} c_2(m)(mz) + y^{-1/2} \sum_{n=-\infty}^{\infty} b(n,y)\mathbf{e}(-n^2 z),$$

we define the convolution product $L(s, h_1, h_2)$ by

$$L(s, h_1, h_2) = L(2s - k + l + 3, \eta^2) \sum_{m=1}^{\infty} \frac{c_1(m)c_2(m)}{m^s}.$$

This type of Dirichlet series for two half-integral weight homomorphic modular forms was introduced by Shimura [12]. Let N be a positive integer, and $N = p^{e_1} \cdots p^{e_r}$ be the prime decomposition of N. Let χ be a Dirichlet character mod N. Fix a prime factor p of N. Let $\chi^{(p)}$ be the p-factor of χ so that

$$\chi = \prod_{p|N} \chi^{(p)}.$$

Our first main result is

Theorem 3. *Let h be a Hecke eigenform in $S^+_{k-1/2}(\Gamma_0(4))$. Let N be an odd positive integer, and $N = p_1^{e_1} \cdots p_r^{e_r}$ be the prime decomposition of N. Let χ be a primitive Dirichlet character mod N.*

(1) *If $\chi^{(p_i)}(-1) = -1$ for some i. Then we have $L(s, M(h), \chi) = 0$.*

(2) *Assume that $\chi^{(p_i)}(-1) = 1$ for any i. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^2 = \chi$. Then we have*

$$L(s, M(h), \chi) = 2^{2s} \prod_{i=1}^{r} \left(1 - (\frac{-1}{p_i}) p_i^{-1}\right) N^2 (\frac{-1}{N}) L(s, h, \mathcal{E}_{3/2}^{(\tilde{\chi})}).$$

We note here that the expression in (2) of the above theorem does not depend on the choice of $\tilde{\chi}$. An application to the Rankin-Selberg convolution of modular forms of half-integral weight will be given in Section 3.

Our calculations are also applicable to the Siegel-Eisenstein series of degree 2 and even weight $k \geq 4$ defined by

$$E_{2,k}(Z) = \sum_{\{C,D\}} \det(CZ + D)^{-k}, \qquad Z \in H_2,$$

where the sum is taken over all non-associated coprime symmetric pairs $\{C, D\}$ of degree 2.

For a non-negative integer m, the Cohen function $H(k-1, m)$ is given by $H(k-1, m) = L_{-m}(2-k)$. Here

$$L_D(s) = \begin{cases} \zeta(2s-1), & D = 0 \\ L(s, \chi_{D_K}) \sum_{a|f} \mu(a) \chi_{D_K}(a) a^{-s} \sigma_{1-2s}(f/a), & D \neq 0,\ D \equiv 0, 1 \bmod 4 \\ 0, & D \equiv 2, 3 \bmod 4, \end{cases}$$

where the natural number f is defined by $D = D_K f^2$ with the discriminant D_K of $K = \mathbf{Q}(\sqrt{D})$, χ_{D_K} is the Kronecker symbol, μ is the Möbius function and $\sigma_s(n) = \sum_{d|n} d^s$. For $k \geq 4$, the Cohen Eisenstein series $\mathcal{H}_{k-1}(z)$ is

$$\mathcal{H}_{k-1}(z) = \sum_{m=0}^{\infty} H(k-1, m) \mathbf{e}(mz).$$

It is known that $\mathcal{H}_{k-1}(z)$ is a modular form of weight $k-1/2$ belonging to the Kohnen plus space and that the Saito-Kurokawa lift of $\mathcal{H}_{k-1}(z)$ coincides with $E_{2,k}(Z)$ up to a scalar multiple, namely the T-th Fourier coefficient of $E_{2,k}(Z)$ for a positive definite T is (B_k: the k-th Bernoulli number)

$$\frac{4k(k-1)}{B_k B_{2k-2}} \sum_{d|e(T)} d^{k-1} H\left(k-1, \frac{\det 2T}{d^2}\right).$$

By the same argument proving Theorem 3, we have

Theorem 4. *Let N and χ be as in Theorem 3. We have*

$$L^*(s, E_{2,k}, \chi) = \frac{4k(k-1)}{B_k B_{2k-2}} 2^{2s} L(s, \mathcal{H}_{k-1}, E_{3/2}^{(\chi)}),$$

and moreover

(1) *If $\chi^{(p_i)}(-1) = -1$ for some i. Then we have $L(s, E_{2,k}, \chi) = 0$.*

(2) *Assume that $\chi^{(p_i)}(-1) = 1$ for any i. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^2 = \chi$. Then we have*

$$\begin{aligned} & L(s, E_{2,k}, \chi) \\ = \; & \frac{4k(k-1)}{B_k B_{2k-2}} 2^{2s} \prod_{i=1}^{r} \left(1 - \left(\frac{-1}{p_i}\right) p_i^{-1}\right) N^2 \left(\frac{-1}{N}\right) L(s, \mathcal{H}_{k-1}, \mathcal{E}_{3/2}^{(\tilde{\chi})}). \end{aligned}$$

In joint works with Ibukiyama ([5], [6]), the first named author got an explicit formula of $L(s, F, \chi_0)$ when χ_0 is the principal character, and F is the Klingen Eisenstein lift and the Ikeda lift of an elliptic cuspidal Hecke eigenform, respectively. It is interesting to generalize these results to the twisted cases of degree n.

2 Sketch of the proof

Our Theorems 3 and 4 follows from (1) and the following proposition.

Proposition 1. *Let F be an element of $M_k(\Gamma_2)$. Let N be an odd positive integer, and $N = p_1^{e_1} \cdots p_r^{e_r}$ be the prime decomposition of N. Let χ be a primitive Dirichlet character mod N.*

(1) *If* $\chi^{(p_i)}(-1) = -1$ *for some* i. *Then we have* $L(s,F,\chi) = 0$.

(2) *Assume that* $\chi^{(p_i)}(-1) = 1$ *for any* i. *Fix a character* $\tilde{\chi}$ *such that* $\tilde{\chi}^2 = \chi$. *Then we have*

$$L(s,F,\chi) = \prod_{i=1}^{r}\left(1 - (\frac{-1}{p_i})p_i^{-1}\right) N^2(\frac{-1}{N}) \sum_{M|\tilde{N}} L^*(s,F,(\frac{*}{M})\tilde{\chi}),$$

where $\tilde{N} = p_1 \cdots p_r$.

In order to prove this, we first note that $L(s,F,\chi)$ can be written as

$$L(s,F,\chi) = \sum_{A \in \mathcal{L}_{n>0}/SL_n(\mathbf{Z})} \frac{c_F(A)h(A,\chi)}{\epsilon(A)(\det A)^s}, \tag{2}$$

where

$$h(A,\chi) = \sum_{\bar{U} \in SL_n(\mathbf{Z}/N\mathbf{Z})} \chi(\mathrm{tr}(A[U])).$$

From now on, we restrict ourselves to the case of $n = 2$ and A is an element of $\mathcal{L}_{2>0}$. For each $c \in \mathbf{Z}$, put

$$R_N(A,c) = \{\mathbf{x} = (x_1,x_2,x_3,x_4) \in (\mathbf{Z}/N\mathbf{Z})^4; (A \perp A)[\mathbf{x}] - c \equiv 0 \bmod N$$

$$\text{and } x_1x_4 - x_2x_3 - 1 \equiv 0 \bmod N\}.$$

Then we have

$$h(A,\chi) = \sum_{c=1}^{N} \chi(c)\#R_N(A,c).$$

To determine $h(A,\chi^{(p_i)})$, we shall compute $\#R_N(A,c)$ for N being a power of a prime.

Lemma 1. *Let* p *be an odd prime number. Let* A *be a symmetric matrix of degree* 2 *with entries in* $\mathbf{Z}_p$. *Assume that* $A \not\equiv O \bmod p$, *and that* $(\frac{4\det A}{p}) = -1$ *or* $(\frac{4\det A}{p}) = 0$ *Then for any* $c \in F_p^\times$, *we have*

$$\#R_{p^e}(A,c) = p^{2(e-1)}\left(p - (\frac{-1}{p})\right)\left(p - (\frac{-4\det A}{p})\right).$$

Lemma 2. *Let A be as in the previous lemma. Assume that $(\frac{4\det A}{p}) = 1$. For any $c \in \mathbf{Z}$ let $r = \mathrm{ord}_p(4\det A - c^2)$. Then we have*

$$\#R_p(A,c) = p^{2e-2}\left(p - (\frac{-1}{p})\right)\left(p\sum_{i=e-r}^{e}(\frac{-1}{p})^{e-i} - \sum_{i=e-r-1}^{e-1}(\frac{-1}{p})^{e-i}\right)$$

if $r \leq e-1$, and

$$\#R_p(A,c) = p^{2e-2}\left(p - (\frac{-1}{p})\right)\left(p\sum_{i=0}^{e}(\frac{-1}{p})^{e-i} - \sum_{i=0}^{e-1}(\frac{-1}{p})^{e-i}\right)$$

if $r = e$.

Suppose that $N = p^{e_1}\cdots p^{e_r}$ is the prime decomposition of N. By the Chinese remainder theorem, $h(A,\chi)$ has the form

$$h(A,\chi) = \prod_{i=1}^{r} h(A,\chi^{(p_i)}).$$

This combined with above two lemmas implies that

Proposition 2. *Let N be an odd positive integer and $N = p^{e_1}\cdots p^{e_r}$ be the prime decomposition of N. Let χ be a primitive Dirichlet character mod N. Let $\chi^{(p_i)}$ be the primitive Dirichlet character mod p^{e_i} such that $\chi = \chi^{(p_1)}\cdots\chi^{(p_r)}$.*

(1) *Assume that $\chi^{(p_i)}(-1) = -1$ for some i. Then we have*

$$h(A,\chi) = 0.$$

(2) *Assume that $\chi^{(p_i)}(-1) = 1$ for any i. Then we have*

$$h(A,\chi) = \prod_{i=1}^{r}\left\{\left(1 + (\frac{4\det A}{p_i})\right)\left(1 - (\frac{-1}{p_i})p_i^{-1}\right)\right\} N^2(\frac{-1}{N})\tilde{\chi}(4\det A)),$$

where $\tilde{\chi}$ is a character such that $\tilde{\chi}^2 = \chi$.

Proposition 2 combined with (2) implies Proposition 1.

3 Special values of twisted Koecher-Maaß series and convolution products of half-integral modular forms

For a holomorphic modular form g of integral or half-integral weight, we denote by $\mathbf{Q}(g)$ the field generated over $\mathbf{Q}$ by all the Fourier coefficients of g. First we recall the following results due to Shimura [11], [12].

Proposition 3. **(Shimura)** (1) *Let f be a Hecke eigenform in $S_{2k-2}(\Gamma_1)$. Then there exist complex numbers $u_{\pm}(f)$ uniquely determined up to $\mathbf{Q}(f)^{\times}$ multiple such that $\dfrac{\Gamma(m)L(m,f,\chi)}{\tau(\chi)(2\pi\sqrt{-1})^m u_j(f)} \in \mathbf{Q}(f)(\chi)$ for any integer $0 < m \leq 2k-3$ and a Dirichlet character χ such that $j = (-1)^m\chi(-1)$.*
(2). Let h be a Hecke eigenform in $S^{+}_{k-1/2}(\Gamma_0(4))$ and $S(h)$ the normalized Hecke eigenform in $S^{+}_{2k-2}(\Gamma_1)$ corresponding to h under the Shimura correspondence. Furthermore, for an integer l such that $k > l \geq 2$, and a primitive character ξ of conductor M, let g be an element of $M_{l/2-1}(\Gamma_0(4M),\xi)$. Then $\dfrac{L(m/2,h,g)}{u_-(S(h))\pi^{m-1}\sqrt{-1}} \in \mathbf{Q}(h)\mathbf{Q}(g)$ for any odd integer m such that $1 \leq m \leq 2k-3$ and a Dirichlet charcater η.

Proposition 4. **(Shimura)** *Let h be a Hecke eigenform in $S^{+}_{k-1/2}(\Gamma_0(4))$ and $S(h)$ be the normalized Hecke eigenform in $S^{+}_{2k-2}(\Gamma_1)$ corresponding to h under the Shimura correspondence. Assume that all the Fourier coefficients of h belong to $\mathbf{Q}(S(h))$. Let χ be a Dirichlet character of conductor N. Assume that χ^2 is primitive, and that $p \equiv -1 \bmod 4$ for some prime factor p of N. Then for any odd integer m such that $1 \leq m \leq 2k-3$, the value $L(m/2, M(h), \chi^2)$ belongs to the vector space $\mathbf{Q}(S(h))(\chi)u_-(S(h))\pi^{m-1}\sqrt{-1}$.*

Applying these two propositions to our explicit formula, we obtain

Theorem 5. *Let h be a Hecke eigenform in $S^{+}_{k-1/2}(\Gamma_0(4))$ and $S(h)$ be the normalized Hecke eigenform in $S^{+}_{2k-2}(\Gamma_1)$ corresponding to h under the Shimura correspondence. Assume that all the Fourier coefficients of h belong to $\mathbf{Q}(S(h))$. Let χ be a Dirichlet character of conductor N. Assume that χ^2*

is primitive, and that $p \equiv -1 \bmod 4$ *for some prime factor* p *of* N*. Then for any odd integer* m *such that* $1 \leq m \leq 2k-3$*, the value* $L(m/2, M(h), \chi^2)$ *belongs to the vector space* $\mathbf{Q}(S(h))(\chi)u_{-}(S(h))\pi^{m-1}\sqrt{-1}$.

We note that the above theorem is not a special case of Choie and Kohnen's result. Actually they treated the values of Koecher-Maaß series at integers. On the other hand, we treat the values at half-integers. It seems interesting to know whether this type of algebraicity holds for the twisted Koecher-Maaß series of any cusp form of even degree.

Theorem 6. *There exists a positive integer* $r = r_h$ *such that the values* $L(l, h, \mathcal{E}_{3/2}^{(\chi_1)}), ..., L(l, h, \mathcal{E}_{3/2}^{(\chi_{r+1})})$ $(i = 1, \cdots, r+1)\}$ *are linearly dependent over* $\bar{\mathbf{Q}}$ *for any integer* $1 \leq l \leq k-2$ *and Dirichlet characters* $\chi_1, ..., \chi_{r+1}$ *of odd conductors such that* $\chi_1^2, ..., \chi_r^2, \chi_{r+1}^2$ *are primitive. In particular, the values* $L(l, h, \mathcal{E}_{3/2}^{(\chi_1)}), ..., L(l, h, \mathcal{E}_{3/2}^{(\chi_{r+1})})$ *are linearly dependent over* $\bar{\mathbf{Q}}$ *for any non-quadratic characters* $\chi_1, ..., \chi_{r+1}$ *of odd prime conductors.*

We note that the algebracity of special values of convolution products of half-integral weight modular froms at half-integers were deeply investigated by Shimura as stated above. However, as far as we know, there is no result about the special values of them at integers, and the above result seems a little bit surprising. We hope the above result shed a new right on this subject.

Applying Choie-Kohnen's functional equation to our explicit formula, we obtain

Theorem 7. *Put*

$$\Lambda(s, h, \mathcal{E}_{3/2}^{(\chi)}) = N^{2s}\pi^{-2s}\tau(\chi^2)^{-1}L(s, h, \mathcal{E}_{3/2}^{(\chi)}).$$

Assume that χ^2 *is primitive. Then* $\Lambda(s, h, \mathcal{E}_{3/2}^{(\chi)})$ *has a analytic continuation to the whole s-plane and has the following functional equation:*

$$\Lambda(k-s, h, \mathcal{E}_{3/2}^{(\chi)}) = \Lambda(s, h, \mathcal{E}_{3/2}^{(\bar{\chi})}).$$

The meromorphy of this type of series was derived in [12] by using so called the Rankin-Selberg integral expression in more general setting, and the functional equation may also be derived by using it.

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