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APPLICATIONS OF AN ARITHMETIC TRACE FORMULA

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ABSTRACT. Recently we have found a trace formula comparing periods and special values of automorphic forms and L-functions. In the first part of the talk we recall the formula, which depends on comparing two different spectral decompositions of an Eisenstein series of Siegel type. In the second part we talk about applications.

1. ARITHMETIC TRACE FORMULA

This is an extended version of a talk given at the conference:

Automorphic representations, automorphic L-functions and arithmetic,

organized by Yoshi-Hiro Ishikawa and Masao Tsuzuki at the RIMS Institute Kyoto, January 19 - 23, 2009 [1].

Let $S_k = S_k(\Gamma)$ be the vector space of cuspidal modular forms of integer weight k with respect to $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Let $\| \; \|$ be the Petersson norm. Let $g \in S_k(\Gamma)$ be a primitive Hecke eigenform with Satake parameters $\alpha_p = \alpha_p(g), \beta_p = \beta_p(g)$, normalized by $\alpha_p \beta_p = p^{k-1}$, then the Hecke- and the symmetric square L-function of g are defined by the Euler products

$$L(g,s) := \prod_p \left\{ (1-\alpha_p p^{-s})(1-\beta_p p^{-s}) \right\}^{-1}$$

$$L(\mathrm{Sym}^2(g),s) := \prod_p \left\{ (1-\alpha_p^2 p^{-2s})(1-\alpha_p\beta_p p^{-2s})(1-\beta_p^2 p^{-2s}) \right\}^{-1}.$$

The completions of these L-functions at infinity are given by

$$\widehat{L}(g,s) := \Gamma_{\mathbb{C}}(s)\, L(g,s)$$

$$\widehat{L}(\mathrm{Sym}^2(g),s) := \Gamma_{\mathbb{R}}(s-k+2)\,\Gamma_{\mathbb{C}}(s)\, L(\mathrm{Sym}^2(g),s),$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$. It is well-known that the Hecke L-function continues to an entire function on $\mathbb{C}$ with functional equation $s \mapsto 1-s$

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with sign $(-1)^{\frac{k}{2}}$. By a theorem of Shimura [Sh], the symmetric square-L-function has an analytical continuation to $\mathbb{C}$, and satisfies the functional equation

$$\widehat{L}(\mathrm{Sym}^2(g), 2k-1-s) = \widehat{L}(\mathrm{Sym}^2(g), s).$$

Then we define

$$(1.1) \qquad \widehat{L}(\mathrm{Sym}^2(g), 1)_{\mathrm{alg}} = \widehat{L}(\mathrm{Sym}^2(g), 2k-2)_{\mathrm{alg}} := \frac{\widehat{L}(\mathrm{Sym}^2(g), 2k-2)}{\pi^{\frac{k}{2}-1} \parallel g \parallel^2},$$

which is known to be a non-zero, totally real algebraic number [Sh],[Z1]. Also the related traces are interesting arithmetic objects:

$$(1.2) \qquad \mathrm{trace}_k\left(\widehat{L}(\mathrm{Sym}^2, 2k-2)_{\mathrm{alg}}\right) := \sum_g \widehat{L}(\mathrm{Sym}^2(g), 2k-2)_{\mathrm{alg}} \in \mathbb{Q},$$

where g runs through a primitive Hecke eigenbasis of $S_k(\Gamma)$. The functional equation implies that this is equal to

$$(1.3) \qquad \mathrm{trace}_k\left(\widehat{L}(\mathrm{Sym}^2, 1)_{\mathrm{alg}}\right).$$

They can be explicitly calculated. The doubling method related to Siegel type Eisenstein series [Ga] provides a good way to do this.

For example let $k = 12, 16, 18, 20, 22, 24$. Then we have for the traces (1.3) the values:

$$\left(\frac{2^{24}\cdot 3\cdot 5\cdot 7}{23\cdot 691}\right)_{k=12} \qquad \left(\frac{2^{30}\cdot 3^2\cdot 5\cdot 7^3\cdot 11}{31\cdot 3617}\right)_{k=16}$$

$$\left(\frac{2^{37}\cdot 5^3\cdot 7\cdot 11\cdot 13}{43867}\right)_{k=18} \qquad \left(\frac{2^{36}\cdot 3^3\cdot 7^3\cdot 11\cdot 71^2}{283\cdot 617}\right)_{k=20}$$

$$\left(\frac{2^{42}\cdot 3\cdot 5^3\cdot 7^2\cdot 13\cdot 17\cdot 61\cdot 103}{11\cdot \underline{43}\cdot 131\cdot 593}\right)_{k=22} \qquad \left(\frac{2^{42}\cdot 11^2\cdot 59\cdot 691\cdot 2294824233197}{3\cdot 13\cdot 47\cdot 103\cdot 2294797}\right)_{k=24}.$$

Here the $\underline{43}$ indicates that the prime 43 does not occur and hence the numerator and denominator of the trace in the case $k = 22$ are coprime to 43. In all the other cases above, whenever $2k-1$ is a prime, this prime occurs in the denominator. The other *big* primes in the denominators are irregular primes related to the k-th Bernoulli number B_k. The pattern $p = 2k-1$ can be directly understood by studying the class numbers $h(\sqrt{-p})$ of $\mathbb{Q}(\sqrt{-p})$ for primes $p \equiv 3 \pmod 4$.

Let further

$$A_p(g) := \begin{pmatrix} \alpha_p(g) & 0 \\ 0 & \beta_p(g) \end{pmatrix},$$

then the Rankin triple L-function $L(f \otimes g \otimes h, s)$ for three primitive Hecke eigenforms f, g, h is given by the infinite product

$$(1.4) \qquad \prod_{p \text{ prime}} \left\{\det\left(1_8 - A_p(f) \otimes A_p(g) \otimes A_p(h)\, p^{-s}\right)\right\}^{-1}.$$

It is well-known that the Rankin triple L-function at the critical value $2k-2$ satisfies Deligne's conjecture. Let

$$\widehat{L}(f \otimes g \otimes h, s) := \Gamma_{\mathbb{C}}(s)\,\Gamma_{\mathbb{C}}(s-k+1)^3 L(f \otimes g \otimes h, s)$$

be the completed Rankin triple L-function. Then

$$(1.5) \qquad \widehat{L}(f \otimes g \otimes h, 2k-2)_{\mathsf{alg}} := \frac{\widehat{L}(f \otimes g \otimes h, 2k-2)}{\| f \|^2 \| g \|^2 \| h \|^2}$$

is a totally real algebraic number. This result is due to Garrett by employing pullback formulas of Siegel type Eisenstein series of degree 3.

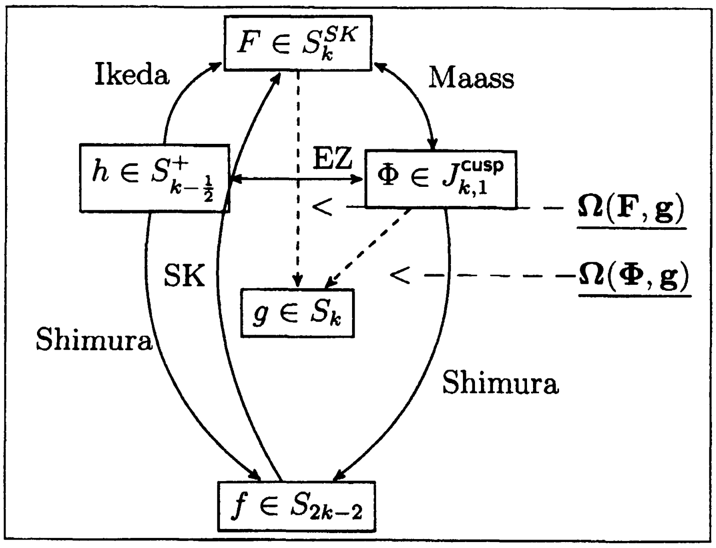
Consider periods $\Omega(F, g)$ attached to Saito-Kurokawa lifts F and cuspidal elliptic newforms $g \in S_k$ of weight k:

$$(1.6) \qquad \int_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} F\Big(j(\tau, \tilde{\tau})\Big)\, \overline{g(\tau)\, g(\tilde{\tau})}\, \operatorname{Im}(\tau)^{k-2} \operatorname{Im}(\tilde{\tau})^{k-2}\, d\tau\, d\tilde{\tau}.$$

Here j denotes the diagonal embedding of $\mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}$ into the Siegel upper half-space $\mathbb{H}_2$ of degree 2. Ichino [Ich] proved that the square of $\Omega(F, g)$ is essentially equal to the central value of the L-function

$$(1.7) \qquad L(f \otimes \operatorname{Sym}^2(g), s).$$

Here $f \in S_{2k-2}(\Gamma)$ is related to F via the Saito-Kurokawa (SK) correspondance. The underlying picture of the spaces and automorphic forms involved in the Saito-Kurokawa lifting is the following:



The described isomorphisms are compatible with the action of the underlying Hecke algebras. We have an isomorphism between the plus space $S^+_{k-\frac{1}{2}}$ of modular forms of half-integral weight $k-1/2$ and the space of Jacobi forms $J_{k,1}^{\text{cusp}}$ of weight k and index 1 due to Eichler and Zagier. We denote by S_k^{SK} the subspace of Siegel modular forms of degree 2 generated by Saito-Kurokawa lifts. We note that Ikeda [Ik1] obtained an interesting construction of the Saito-Kurokawa lifts of modular forms of half-integral weight leading to his construction of the now so-called Ikeda lifts. Mainly Maass and Andrianov considered the explicit isomorphism between the space of Jacobi forms and the space S_k^{SK} explicitely. This lift can be used to define in a canonical way periods of Jacobi forms $\Omega(\Phi, g)$. They are directly related with the periods $\Omega(F, g)$. The forms h, Φ, f, and F correspond to each other. The correspondence of the space of Saito-Kurokawa lifts and S_{2k-2} is abstract and given via eigenvalues.

Let $(g_a)_a$ run through a primitive Hecke eigenbasis of S_{2k-2}. Let $\mathbb{K}_{2k-2}$ denote the totally real number field generated by the Hecke eigenvalues and $\mathcal{O}_{2k-2}$ the ring of integers, in which the Fourier coefficients are contained. It is known that a Hecke eigenbasis of

$$S_{2k-2},\ J_{k,1}^{\text{cusp}},\ S^+_{k-\frac{1}{2}},\ S_k^{\text{SK}}$$

can be choosen such that the Fourier coefficients are contained in $\mathbb{K}_{2k-2}$. We fix such a basis

$$(f_i)_i\,,\ (\Phi_i)_i\,,\ (h_i)_i\,,\ (F_i)_i\,, \tag{1.8}$$

and indicate the correspondance by the index. Let $\| H \|$ be the Petersson norm of the form H. Then

$$\widehat{L}(f_i, 2k-3)_{\mathsf{alg}} := \frac{\widehat{L}(f_i, 2k-3)}{\| \Phi_i \|^2}, \quad \Omega(F_i, g_a)_{\mathsf{alg}} := \frac{\Omega(F_i, g_a)}{\| g_a \|^4} \in \mathbb{K}_{2k-2}.$$

Let $\widehat{\zeta}(s)$ be the completed Riemann zeta function. Then we have for the values at the positive even integers:

$$\widehat{\zeta}(k)_{\mathsf{alg}} := \frac{\widehat{\zeta}(k)}{\pi^{\frac{k}{2}}} \in \mathbb{Q}^{\times} \quad (k \in \mathbb{N}, \text{ even}). \tag{1.9}$$

Now we state the announced formula [H1], which relates special values of different kind of L-functions at points inside and outside the convergent domain of the Euler product.

Theorem: [Arithmetic Trace Formula]
Let k be a positive even integer. Then we have

$$\begin{aligned}
&\frac{2^{3k}}{2k-2} \frac{1}{\widehat{\zeta}(k)_{\mathsf{alg}}} \sum_d \widehat{L}(\mathsf{Sym}^2(g_d), 1)_{\mathsf{alg}}\, g_d \otimes g_d \\
&+(-1)^{\frac{k}{2}} 2^{k-2} \sum_{a,b,c} \widehat{L}(g_a \otimes g_b \otimes g_c, 2k-2)_{\mathsf{alg}}\, g_a \otimes g_b \\
&= \frac{(-1)^{\frac{k}{2}}\, 2^{2k+1}}{2k-2} \frac{1}{\widehat{\zeta}(2k-2)_{\mathsf{alg}}} \frac{\Gamma(k-1)\,\Gamma(k)^2}{\Gamma(2k-1)\,\Gamma(k/2)^2} \\
&\quad \sum_{a,b} \widehat{L}(\mathsf{Sym}^2(g_a), 1)_{\mathsf{alg}}\, \widehat{L}(\mathsf{Sym}^2(g_b), 1)_{\mathsf{alg}}\, g_a \otimes g_b \\
&+ \sum_{a,b} \sum_{\Phi} \left\{ 2^{3k-3} \cdot \widehat{L}(f, 2k-3)_{\mathsf{alg}} \cdot \Omega(F, g_a)_{\mathsf{alg}} \cdot \Omega(F, g_b)_{\mathsf{alg}} \right\} g_a \otimes g_b.
\end{aligned}$$

2. Applications

2.1. Congruences for primes $2k-1$.
In the following let $p := 2k-1$ be a prime. The normalized trace of the special value of the Rankin triple L-function

$$\sum_{a,b,c=1}^{\dim S_k} \left(\frac{\widehat{L}(g_a \otimes g_b \otimes g_c, k)_{\mathsf{alg}}}{\widehat{\zeta}(2k-2)_{\mathsf{alg}}} \right)$$

is congruent mod p to a quadratic polynomial evaluated at the trace of the special value of the symmetric-square L-function

$$\sum_{d=1}^{\dim S_k} \left(\frac{\widehat{L}(\mathrm{Sym}^2(g_d),1)_{\mathrm{alg}}}{\widehat{\zeta}(2k-2)_{\mathrm{alg}}} \right). \tag{2.1}$$

We have:

Theorem: [Congruence Relations of Special Values]
Let p be a prime with $p \equiv 3 \pmod 4$, $p \geq 23$ and $k = \frac{p+1}{2}$. Let $(g_a)_a$ be a primitive Hecke eigenbasis of $S_k(\Gamma)$. Then

$$\frac{(-1)^{(1+h(\sqrt{-p}))/2}}{2^k} \sum_{a,b,c=1}^{\dim S_k} \left(\frac{\widehat{L}(g_a \otimes g_b \otimes g_c, k)_{\mathrm{alg}}}{\widehat{\zeta}(2k-2)_{\mathrm{alg}}} \right) \equiv$$

$$\left\{ \sum_{a=1}^{\dim S_k} \frac{2}{\frac{k}{2}!} \left(\frac{\widehat{L}(\mathrm{Sym}^2(g_a),1)_{\mathrm{alg}}}{\widehat{\zeta}(2k-2)_{\mathrm{alg}}} \right) \right\}^2 - \frac{1}{h(\sqrt{-p})} \sum_{a=1}^{\dim S_k} \frac{2}{\frac{k}{2}!} \left(\frac{\widehat{L}(\mathrm{Sym}^2(g_a),1)_{\mathrm{alg}}}{\widehat{\zeta}(2k-2)_{\mathrm{alg}}} \right) \pmod p.$$

The following *non-vanishing results* make the congruence relation powerful. In a joint project with Neil Dummigan [DH], motivated by the results from above, we obtained the following three results.

Theorem I: Appearence of the prime $2k-1$.
Let p be a prime ($p \geq 23$) and $p \equiv 3 \pmod 4$. Then

$$\mathrm{trace}_{\frac{p+1}{2}} \left(\widehat{L}(\mathrm{Sym}^2, 1) \right)_{\mathrm{alg}} \in p^{-1}\mathbb{Z}_{(p)}^{\times} \tag{2.2}$$

if and only if $h(\sqrt{-p}) > 1$.

Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p. In a way, the p in the denominator has nothing to do with the class number. But when $h(\sqrt{-p}) = 1$, a subtle cancellation occurs:

Theorem II: Class number one.
Let p a prime ($p \geq 23$), $p \equiv 3 \pmod 4$, and the class number $h(\sqrt{-p}) = 1$. Then

$$\mathrm{trace}_{\frac{p+1}{2}} \left(\widehat{L}(\mathrm{Sym}^2, 1) \right)_{\mathrm{alg}} \in \mathbb{Z}_{(p)}^{\times}. \tag{2.3}$$

Next we turn to congruences of modular forms. Let $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)\, q^n$ be the unique normalised cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. Wilton

[Wi] proved the following congruences. Let $\ell \neq 23$ be a prime.

$$\tau(\ell) \equiv \begin{cases} 0 \pmod{23} & \text{if } \left(\frac{\ell}{23}\right) = -1; \\ 2 \pmod{23} & \text{if } \left(\frac{\ell}{23}\right) = 1 \text{ and } \ell = u^2 + 23v^2; \\ -1 \pmod{23} & \text{otherwise.} \end{cases}$$

Swinnerton-Dyer [SD] considered more generally a normalised, cuspidal Hecke eigenform $g = \sum_{n=1}^{\infty} a_n q^n$ for $\mathrm{SL}_2(\mathbb{Z})$, of weight k. For simplicity suppose that the a_n are rational. This probably means that $k = 12, 16, 18, 20, 22$ or 26. He showed that if p is a prime, and if for all primes ℓ such that $\left(\frac{\ell}{p}\right) = -1$ we have $a_\ell \equiv 0 \pmod{p}$, then necessarily $p < 2k$. In the case $p = 2k - 1$ (if it is prime), he observed that such congruences hold for $k = 12$ ($p = 23$, i.e. Wilton's case), and also for $k = 16$ ($p = 31$), but not for $k = 22$ ($p = 43$). In fact, we have the following.

Theorem III: Dihedral congruences.
Let k be an even integer such that $p := 2k - 1$ is prime. There exists a normalised, cuspidal Hecke eigenform $g = \sum_{n=1}^{\infty} a_n q^n$ for $\mathrm{SL}_2(\mathbb{Z})$, of weight k, and a prime $\mathfrak{p} \mid p$ of $\mathbb{Q}(\{a_n\})$ such that $a_\ell \equiv 0 \pmod{\mathfrak{p}}$ for all primes ℓ with $\left(\frac{\ell}{p}\right) = -1$, if and only if $h(\sqrt{-p}) > 1$.

Theorems I and III may appear to describe two unrelated consequences of the condition $h(\sqrt{-p}) > 1$. We can explain that they are linked by the Bloch-Kato conjecture on special values of L-functions. The Galois representation behind Theorem III is used to produce a non-zero $\mathfrak{p}$-torsion element in some "global torsion" group whose order appears in the denominator of the conjectural formula for the ratio of $L(\mathsf{Sym}^2(g), 1)$ to a canonical Deligne period. This may be viewed as explaining the non-integrality in Theorem I.

What happens in the case $h(\sqrt{-p}) = 1$? The solution involves periods $\Omega(F, g)$ attached to Saito-Kurokawa lifts F (of primitive Hecke eigenforms $f \in S_{2k-2}(\Gamma)$ via the Saito-Kurokawa (SK) correspondance [Z2]) and primitive Hecke eigenforms $g \in S_k(\Gamma)$.

Theorem [Class number #1]
Let p be a prime, and let $S_k(\Gamma)$ for $k = \frac{p+1}{2}$ be non-trivial. If the class number of $\mathbb{Q}(\sqrt{-p})$ is one, then

$$-\frac{2^{k+1}}{\frac{k}{2}!} \sum_{d=1}^{\dim S_k(\Gamma)} \widehat{L}(\mathsf{Sym}^2(g_d), 1)_{\mathsf{alg}} + \sum_{a,b,c=1}^{\dim S_k(\Gamma)} \widehat{L}(g_a \otimes g_b \otimes g_c, k)_{\mathsf{alg}} \equiv (-1)^{\frac{k+2}{2}} 2^{2-k}$$

$$\sum_{a,b=1}^{\dim S_k(\Gamma)} \sum_{i=1}^{\dim S_{2k-2}(\Gamma)} \left\{ 2^{3k-3} \cdot \widehat{L}(f_i, 2k-3)_{\mathsf{alg}} \cdot \Omega(F_i, g_a)_{\mathsf{alg}} \cdot \Omega(F_i, g_b)_{\mathsf{alg}} \right\} \pmod{p}.$$

2.2. **Periods.** *In the following let*

$$\omega(g_a, g_b) := \sum_i \frac{\widehat{L}(f_i, 2k-3)}{\| \Phi_i \|^2} \frac{\Omega(F_i, g_a)\ \Omega(F_i, g_b)}{\| g_a \|^4 \ \| g_b \|^4}. \tag{2.4}$$

Theorem: Vanishing of periods.
Let p be an irregular primes. Let k be an even positive integer with $p | \frac{B_k}{k}$ and $p \nmid \frac{B_{2k-2}}{2k-2}$. Then we have

$$\sum_{a,b=1}^{\dim S_k(\Gamma)} p\ \omega(g_a, g_b)\ g_a \otimes g_b \equiv 0 \pmod{p}. \tag{2.5}$$

Remark:
There exist infinitely many primes p with this property ([H2]).

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