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Stark's units, CM-periods and multiple gamma functions

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Abstract

Yoshida's class invariant $X(c)$ is defined by special values of Barnes's multiple gamma functions for each ideal class c of a totally real field. In this paper, we will show some "monomial relations" on $\exp(X(c))$. On the other hand, Shintani's formula expresses derivative values of partial zeta functions in terms of the invariant $X(c)$. As a result, our "monomial relations" relates to Stark's conjecture over totally real fields. More, Yoshida's conjecture expresses any CM-period, which is the transcendental part of a critical value of L-function associated with an algebraic Hecke character, by the values of $X(c)$. Therefore, we may consider the "relation" between some monomial relations on CM-periods and algebraicity of Stark's units.

Introduction

Let F be a totally real field and M an finite abelian extension of F. For an arbitrary element $\tau \in \mathrm{Gal}(M/F)$, we denote by $\zeta(s,\tau)$ the partial zeta function, which is defined by

$$\zeta(s,\tau) := \sum_{\left(\frac{M/F}{\mathfrak{a}}\right)=\tau} N\mathfrak{a}^{-s}.$$

Here $\left(\frac{M/F}{\mathfrak{a}}\right)$ is the Artin symbol and $\mathfrak{a}$ runs over integral ideals of F which are relatively prime to the conductor of M/F. Note that the series converges if $\mathrm{Re}(s)$ is large enough, can be meromorphically continued to the whole complex plane, and is analytic at $s=0$. Assume that there exists an embedding

$$\iota : M \hookrightarrow \mathbf{R}.$$

Then Stark's conjecture states that there exists $u \in M^\times$ such that

$$\frac{d}{ds}\zeta(s,\tau)\big|_{s=0} = -\frac{1}{2}\log u^{\iota\circ\tau}. \tag{0.1}$$

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for all $\tau \in \mathrm{Gal}(M/F)$. We call the element u Stark's unit. (Indeed in many cases $u \in \mathcal{O}_M^\times$.) We note that Stark's conjecture when $F = \mathbf{Q}$ is proved by using Euler's reflection formula;

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{0.2}$$

In the proof, we also use Lerch's formula for the gamma function and algebraicity of the division values of sine function. For example, let $F = \mathbf{Q}$, $M = \mathbf{Q}(\sqrt{2})$. We have a canonical isomorphism $(\mathbf{Z}/8\mathbf{Z})^\times/\{\pm 1\} \cong \mathrm{Gal}(M/F)$, $a \mapsto \tau_a$ $(a = 1, 3)$, $\tau_1 = \mathrm{id}$, $\tau_3 \neq \mathrm{id}$. Then we get

$$\begin{aligned}
\zeta'(0, \tau_a) &= \log\left(\Gamma(\frac{a}{8})\Gamma(\frac{8-a}{8})\right) - \log(2\pi) \\
&= -\log\left(2\sin(\frac{a\pi}{8})\right) \\
&= \begin{cases} -\frac{1}{2}\log(2-\sqrt{2}) & \text{if } a = 1, \\ -\frac{1}{2}\log(2+\sqrt{2}) & \text{if } a = 3. \end{cases}
\end{aligned}$$

Shintani expressed the derivative values $\zeta'(0, \tau)$ in terms of Barnes's multiple gamma functions, when F is totally real. We may consider his formula as a generalization of Lerch's formula. Therefore it is natural to ask whether one can prove Stark's conjecture by generalizing Euler's reflection formula to multiple gamma functions.

In this paper, we treat Yoshida's class invariants instead of each special value of a multiple gamma function. Associated with an ideal class c of any totally real field, Yoshida defined the invariant $X(c)$ as a finite sum of the log of special values of multiple gamma functions (+ some correction terms). Our main result is a "monomial relation"

$$\exp(X(c))\exp(X(c')) \in \overline{\mathbf{Q}}^\times$$

for some pair (c, c') of ideal classes (Theorem 2.1.1). This Theorem is valid if $F \neq \mathbf{Q}$, and in the case of $[F : \mathbf{Q}] = 2$, it is due to Yoshida. Unfortunately, in order to apply such a result to algebraicity of Stark's units, we need one more monomial relation among $\exp(X(c))$ (conjectural equation (2.2) in §2.2). In §1, we introduce Yoshida's class invariant $X(c)$, and in §2, we state our main results. In §3, we also see a "relation" between Stark's units and Shimura's CM-periods. Roughly speaking, assuming Yoshida's conjecture on CM-periods, the algebraicity of Stark's units follows from some monomial relations of CM-periods. We also see a p-adic analogue of monomial relations among $\exp(X(c))$, which is Theorem 4.1.1 in §4. It is interesting that we use both archimedean and p-adic functions to express the Stark unit (Conjecture 4.2.1). We omit the complete proof because of lack of space and give the sketch of proof in §5.

1 A generalization of gamma values

Our object is to generalize the following classical result

$$\pi^{-1}\Gamma\Big(\frac{a}{m}\Big)\Gamma\Big(\frac{m-a}{m}\Big) \in \overline{\mathbf{Q}}^\times, \qquad (a, m, m-a \in \mathbf{Z}_{>0}). \tag{1.1}$$

For that purpose, we introduce Barnes's multiple gamma function and Yoshida's class invariant. For detail, see [Yo3].

1.1 Barnes's multiple gamma function

First, we introduce the Barnes multiple *zeta* function, which is a generalization of the Hurwitz zeta function.

Definition 1.1.1. *Let $z, v_1, v_2, \ldots, v_r \in \mathbf{R}_{>0}$ with a positive integer r. We define*

$$(1.2)\qquad \zeta_r(s,(v_1,v_2,\ldots,v_r),z) := \sum_{m_1,m_2,\ldots,m_r\in\mathbf{Z}_{\geq 0}} (z+m_1v_1+m_2v_2+\cdots+m_rv_r)^{-s}$$

for $\mathrm{Re}(s) > r$. *One can continue it meromorphically to the whole s-plane, and it is analytic at $s=0$.*

Definition 1.1.2. *We define the multiple gamma function $\Gamma_r(z,(v_1,v_2,\ldots,v_r))$ and the correction term $\rho_r((v_1,v_2,\ldots,v_r))$ for $z,v_1,v_2,\ldots,v_r\in\mathbf{R}_{>0}$, $r\in\mathbf{Z}_{>0}$ by*

$$(1.3)\qquad \begin{aligned} \log\frac{\Gamma_r(z,(v_1,v_2,\ldots,v_r))}{\rho_r((v_1,v_2,\ldots,v_r))} &= \frac{d}{ds}\zeta_r(s,(v_1,v_2,\ldots,v_r),z)\Big|_{s=0}, \\ \log\rho_r((v_1,v_2,\ldots,v_r)) &= -\lim_{x\to+0}\left[\frac{d}{ds}\zeta_r(s,(v_1,v_2,\ldots,v_r),x)\Big|_{s=0}+\log x\right]. \end{aligned}$$

In this paper to simplify the notations, we use the symbols $\zeta(s;Z)$, $L\Gamma(Z)$ for a "good" subset Z of $\mathbf{R}_{>0}$ in the following sense.

Definition 1.1.3. *We call a subset $Z\subset\mathbf{R}_{>0}$ "good" when we can write*

$$Z=\sqcup_{j=1}^{k}\{z_j+m_1v_{j,1}+m_2v_{j,2}+\cdots+m_{r(j)}v_{j,r(j)}\mid m_i\in\mathbf{Z}_{\geq 0}\}$$

with $z_j, v_{i,j}\in\mathbf{R}_{>0}$, $r_j,k\in\mathbf{Z}_{>0}$. Then we put

$$\zeta(s;Z) := \sum_{j=1}^{k}\zeta_{r_j}(s,(v_{j,1},v_{j,2},\ldots,v_{j,r(j)}),z_j),$$

$$L\Gamma(Z) := \sum_{j=1}^{k}\log\frac{\Gamma_r(z,(v_{j,1},v_{j,2},\ldots,v_{j,r(j)}))}{\rho_{r(j)}((v_{j,1},v_{j,2},\ldots,v_{j,r(j)}))}.$$

Note that the function $\zeta(s;Z)$ is the meromorphic continuation of the series $\sum_{z\in Z}z^{-s}$ and that we have $L\Gamma(Z)=\zeta'(0;Z)$. With a slight abuse of notations, we also call $\zeta(s;Z)$ (resp. $L\Gamma(Z)$) the multiple zeta function (resp. the log *multiple gamma function).*

Remark 1.1.1. *One can consider $L\Gamma(Z)$ as a generalization of the* log *of the gamma function since we have, for $z,v\in\mathbf{R}_{>0}$,*

$$L\Gamma(\{z+mv\mid m\in\mathbf{Z}_{\geq 0}\}) = \log\frac{\Gamma_1(z,(v))}{\rho_1((v))} = \log\Gamma(\frac{z}{v})-\frac{1}{2}\log 2\pi-(\frac{1}{2}-\frac{z}{v})\log v,$$

where π is the circular constant.

Remark 1.1.2. *Shintani expressed the derivative values of partial zeta functions in term of special values of Barnes's multiple gamma functions. He also gave an explicit formula which expresses special values of Barnes's multiple zeta functions by some elementary terms. In particular we have*

$$\zeta_r(0,(v_1,\ldots,v_r),z)\in\mathbf{Q}(v_1,\ldots,v_r,z).$$

1.2 Yoshida's class invariant

Yoshida defined the class invariant $X(c)$ for any ideal class c (of an arbitrary conductor) of any totally real field by special values of log multiple gamma functions. Below, we briefly sketch the definition. Let F be a totally real number field of degree n, F^+ the set of all totally positive elements in F, and $\mathcal{O}_F$ the ring of integers of F. We denote by $\mathfrak{C}_\mathfrak{f}$ the narrow ideal class group modulo $\mathfrak{f}$ with an integral ideal $\mathfrak{f}$ of F. Namely, we put

$$\begin{aligned}
\mathrm{I}_\mathfrak{f} &:= \text{ the set of all fractional ideals of } F \text{ relatively prime to } \mathfrak{f}, \\
\mathrm{P}_\mathfrak{f} &:= \{(\alpha) \in \mathrm{I}_\mathfrak{f} \mid \alpha \in F^+,\ \alpha \equiv 1 \bmod^* \mathfrak{f}\}, \\
\mathfrak{C}_\mathfrak{f} &:= \mathrm{I}_\mathfrak{f}/\mathrm{P}_\mathfrak{f}.
\end{aligned}$$

Let $v_1, v_2, \ldots, v_r \in F^\times$. We assume that these are linearly independent in $F \otimes \mathbf{R}$ over $\mathbf{R}$. Then we define the (r-dimensional open simplicial) cone $C(v_1, v_2, \ldots, v_r)$ with basis $v_1, v_2, \ldots, v_r$ by

$$C(v_1, v_2, \ldots, v_r) := \{x = v_1 \otimes x_1 + v_2 \otimes x_2 + \ldots v_r \otimes x_r \mid x_i \in \mathbf{R}_{>0}\ (i = 1, 2, \ldots, r)\}.$$

We denote by E_F (resp. E_F^+) the unit group (resp. totally positive unit group) of F. We put $(F \otimes \mathbf{R})^+ := \{x \in (F \otimes \mathbf{R})^+ \mid \sigma(x) > 0,\ {}^\forall\sigma \in \mathrm{Hom}_{\mathbf{R}\text{-alg}}(F \otimes \mathbf{R}, \mathbf{R})\}$. Then E_F^+ acts on $(F \otimes \mathbf{R})^+$ through the embeddings $E_F^+ \hookrightarrow (F \otimes \mathbf{R})^+$, $z \mapsto z \otimes 1$. We consider fundamental domains of $(F \otimes \mathbf{R})^+/E_F^+$ satisfying the following conditions. The existence of such a fundamental domain is due to Shintani.

Definition 1.2.1. *We say $\mathcal{D}$ is a Shintani domain if there exist $v_{j,i} \in F^+$ ($j \in J$, J is a finite set of indices, $i = 1, 2, \ldots, r(j)$, $r(j) \in \mathbf{Z}_{>0}$) which satisfy*

$$\begin{aligned}
(F \otimes \mathbf{R})^+ &= \sqcup_{\epsilon \in E_F^+} \epsilon \mathcal{D}, \\
\mathcal{D} &= \sqcup_{j \in J} C(v_{j,1}, v_{j,2}, \ldots, v_{j,r(j)}).
\end{aligned}$$

Definition 1.2.2. *For a Shintani domain $\mathcal{D}$, an ideal class $c \in \mathfrak{C}_\mathfrak{f}$, and a fractional ideal $\mathfrak{A}_c$ of F with $\mathfrak{A}_c = c \bmod \mathrm{P}_{(1)}$, we define a subset $Z(c; \mathcal{D}, \mathfrak{A}_c)$ of F^+ by*

$$Z(c; \mathcal{D}, \mathfrak{A}_c) := \{z \in \mathcal{D} \cap \mathfrak{A}_c^{-1} \mid z\mathfrak{A}_c \in c\}.$$

We note that for any $\iota \in \mathrm{Hom}(F, \mathbf{R})$, the image $\iota(Z(c; \mathcal{D}, \mathfrak{A}_c)) \subset \mathbf{R}_{>0}$ is "good" in the sense of Definition 1.1.3. Indeed, let $v_{j,i}$ be as in Definition 1.2.1 and take $L \in \mathbf{Z}_{>0}$ so that $Lv_{j,i} \in \mathfrak{f}$ for all j, i. We put

$$\begin{aligned}
C_j &:= C(v_{j,1}, \ldots, v_{j,r(j)}), \\
R(c; C_j, \mathfrak{A}_c, L) &:= \{z = x_1 v_{j,1} + x_2 v_{j,2} + \cdots + x_{r(j)} v_{j,r(j)} \mid z \in \mathfrak{A}_c^{-1},\ z\mathfrak{A}_c \in c,\ 0 < x_i \leq L\}.
\end{aligned}$$

Then $R(c; C_j, \mathfrak{A}_c, L)$ is finite and we have

$$Z(c; \mathcal{D}, \mathfrak{A}_c) = \sqcup_{j \in J} \sqcup_{z \in R(c; C_j, \mathfrak{A}_c, L)} \{z + m_1 v_{j,1} + m_2 v_{j,2} + \cdots + m_{r(j)} v_{j,r(j)} \mid m_i \in \mathbf{Z}_{\geq 0}\}.$$

Definition 1.2.3. *Let $\mathcal{D}, c, \mathfrak{A}_c, Z(c;\mathcal{D},\mathfrak{A}_c)$ be as above. For $\tau,\tau' \in \mathrm{Hom}(F,\mathbf{R})$ we put*

$$v_{\tau,\tau'} := \frac{d}{ds}\Big[\sum_{z\in Z(c;\mathcal{D},\mathfrak{A}_c)} ((\tau(z)\tau'(z))^{-s} - \tau(z)^{-s} - \tau'(z)^{-s})\Big]\Big|_{s=0}.$$

Note that the above series in [ ] converges when $\mathrm{Re}(s)$ is large enough, can be meromorphically continued to the whole complex plane, and is analytic at $s=0$. For $\iota \in \mathrm{Hom}(F,\mathbf{R})$, we put

$$\begin{aligned}
G(c,\iota;\mathcal{D},\mathfrak{A}_c) &:= L\Gamma(\iota(Z(c;\mathcal{D},\mathfrak{A}_c))),\\
W(c,\iota;\mathcal{D},\mathfrak{A}_c) &:= -\frac{1}{n}\zeta(0;\iota(Z(c;\mathcal{D},\mathfrak{A}_c)))\log \mathrm{N}\mathfrak{A}_c,\\
V(c,\iota;\mathcal{D},\mathfrak{A}_c) &:= \frac{2}{n}\sum_{\iota\neq\tau\in\mathrm{Hom}(F,\mathbf{R})} v_{\iota,\tau} - \frac{1}{n^2}\sum_{\tau,\tau'\in\mathrm{Hom}(F,\mathbf{R}),\ \tau\neq\tau'} v_{\tau,\tau'},\\
X(c,\iota;\mathcal{D},\mathfrak{A}_c) &:= G(c,\iota;\mathcal{D},\mathfrak{A}_c) + W(c,\iota;\mathcal{D},\mathfrak{A}_c) + V(c,\iota;\mathcal{D},\mathfrak{A}_c).
\end{aligned}$$

The definition of the invariant V seems to be complicated, but these invariants have good properties as follows.

Theorem 1.2.1. *(Yoshida.) There exist finite elements $a_i \in F$, $\epsilon_i \in E_F^+$, depending on the choice of $c, \iota, \mathcal{D}, \mathfrak{A}_c$, such that*

$$V(c,\iota;\mathcal{D},\mathfrak{A}_c) = \sum_i \iota(a_i)\log\iota(\epsilon_i). \tag{1.4}$$

Moreover, the value of $\exp(X(c,\iota;\mathcal{D},\mathfrak{A}_c))$ does not depend on the choice of $\mathcal{D},\mathfrak{A}_c$ up to algebraic numbers. More precisely for any Shintani domains $\mathcal{D},\mathcal{D}'$ and for any fractional ideal $\mathfrak{A}_c, \mathfrak{A}'_c$ with $\mathfrak{A}_c \equiv \mathfrak{A}'_c \equiv c \bmod \mathrm{P}_{(1)}$, there exist a rational number r and an element $f \in F^+$ satisfying

$$X(c,\iota;\mathcal{D},\mathfrak{A}_c) - X(c,\iota;\mathcal{D}',\mathfrak{A}'_c) = r\log\iota(f).$$

For the invariant G, we can write

$$G(c,\iota;\mathcal{D},\mathfrak{A}_c) - G(c,\iota;\mathcal{D}',\mathfrak{A}'_c) = \sum_i \iota(\alpha_i)\log\iota(\beta_i)$$

with some elements $\alpha_i, \beta_i \in F$.

For simplicity, we fix an embedding $\mathrm{id} : F \hookrightarrow \mathbf{R}$ and put $G(c;\mathcal{D},\mathfrak{A}_c) := G(c,\mathrm{id};\mathcal{D},\mathfrak{A}_c)$, etc. More we write $X(c) := X(c;\mathcal{D},\mathfrak{A}_c)$, or $X(c,\iota) := X(c,\iota;\mathcal{D},\mathfrak{A}_c)$ when we consider them up to algebraic numbers.

1.3 Shintani's formula

We recall Shintani's methods to investigate partial zeta functions. Associated with an arbitrary ideal class $c \in \mathfrak{C}_\mathfrak{f}$, we define the partial zeta function $\zeta(s,c)$ by

$$\zeta(s,c) := \sum_{\mathfrak{a}\subset\mathcal{O}_F,\ \mathfrak{a}\in c} \mathrm{N}\mathfrak{a}^{-s}.$$

For an abelian extension M of F and for $\tau \in \mathrm{Gal}(M/F)$, we also define the partial zeta function $\zeta(s,\tau)$ by

$$\zeta(s,\tau) := \sum_{\mathfrak{a}\subset\mathcal{O}_F,\ (\mathfrak{a},\mathfrak{f}_M)=1,\ (\frac{M/F}{\mathfrak{a}})=\tau} \mathrm{N}\mathfrak{a}^{-s}, \tag{1.5}$$

where we denote the finite part of the conductor of M/F by $\mathfrak{f}_M$. It is clear that if we denote the Artin map by $\mathrm{Art}:\mathfrak{C}_{\mathfrak{f}_M}\to\mathrm{Gal}(M/F)$ then we have

$$\zeta(s,\tau) = \sum_{c\in\mathfrak{C}_{\mathfrak{f}_M},\ \mathrm{Art}(c)=\tau} \zeta(s,c).$$

Therefore, the following results concerning partial zeta functions associated with ideal classes can be translated to properties of partial zeta functions associated with $\tau \in \mathrm{Gal}(M/F)$.

Let notations be as in the previous subsection. Since the map

$$Z(c;\mathcal{D},\mathfrak{A}_c)\to\{\mathfrak{a}\in c \mid \mathfrak{a}\subset\mathcal{O}_F\},\ z\mapsto z\mathfrak{A}_c$$

is bijective, we have

$$\zeta(s,c) = \mathrm{N}\mathfrak{A}_c^{-s}\sum_{z\in Z(c;\mathcal{D},\mathfrak{A}_c)} \mathrm{N}z^{-s}.$$

Using this expression, Shintani related the derivative value $\zeta'(0,c)$ to special values of Barnes' multiple gamma functions. By using Yoshida's class invariants X, we can rewrite Shintani's formula as follows.

Theorem 1.3.1. *(Shintani' formula.)*

$$\zeta'(0,c) = \sum_{\iota\in\mathrm{Hom}(F,\mathbf{R})} X(c,\iota;\mathcal{D},\mathfrak{A}_c).$$

Remark 1.3.1. *For example, let $F=\mathbf{Q}$, $\mathcal{D}=\mathbf{R}_{>0}$. Take the ideal class $c := a \bmod m$ with $(a,m)=1$, $0<a<m$, which is an element $\in \mathfrak{C}_{(m)} = (\mathbf{Z}/m\mathbf{Z})^\times$. Take $\mathfrak{A}_c=(1)=\mathbf{Z}$. Then we have*

$$\zeta(s,c) = \sum_{n\geq 0}(a+mn)^{-s},$$
$$\zeta'(0,c) = X(c;\mathcal{D},\mathfrak{A}_c) = \log(\Gamma(a/m)) - \frac{1}{2}\log 2\pi - (\frac{1}{2}-\frac{a}{m})\log m.$$

Therefore we have

$$\exp(X(c)) \equiv \Gamma(a/m)\pi^{-1/2} \bmod \overline{\mathbf{Q}}^\times.$$

Hence we may consider the value $\exp(X(c))$ as a generalization of classical gamma values $\Gamma(a/m)$ up to some correction terms.

2 Main results

In order to state our main results, we need some notations. Let F be a totally real field of degree n and $\mathfrak{f}$ an integral ideal of F.

Definition 2.0.1. *For each embedding $\sigma \in \mathrm{Hom}(F, \mathbf{R})$, take an element $v_\sigma \in \mathcal{O}_F$ so that $v_\sigma \equiv 1 \bmod \mathfrak{f}$, $v_\sigma^\sigma < 0$, $v_\sigma^{\sigma'} > 0$ $(\sigma \neq {}^\forall\sigma' \in \mathrm{Hom}(F, \mathbf{R}))$. Then we put $s_\sigma := (v_\sigma) \bmod P_\mathfrak{f} \in \mathfrak{C}_\mathfrak{f}$.*

We note that the class $s_\sigma \in \mathfrak{C}_\mathfrak{f}$ does not depend on the choice of such an element v_σ. More if $M, \mathfrak{f}_M, \mathrm{Art}$ are as in (1.5) then $\mathrm{Art}(s_\sigma) \in \mathrm{Gal}(M/F)$ is the complex conjugation associated with a lift $\widetilde{\sigma} : M \hookrightarrow \mathbf{C}$ of $\sigma : F \hookrightarrow \mathbf{R}$.

2.1 Monomial relations on $\exp(X_F(c))$

The following Theorem is our main result in this paper. In the case of $[F : \mathbf{Q}] = 2$, this is due to Yoshida.

Theorem 2.1.1. *For $\iota, \sigma \in \mathrm{Hom}(F, \mathbf{R})$ with $\iota \neq \sigma$ and for $c \in \mathfrak{C}_\mathfrak{f}$ we have*

$$\exp(X(c, \iota)) \exp(X(cs_\sigma, \iota)) \in \overline{\mathbf{Q}}^\times.$$

More precisely there exist $r \in \mathbf{Q}$, $\alpha \in F^+$, depending on c, σ, $\mathcal{D}$ and $\mathfrak{A}_c$, which satisfy

$$X(c, \iota; \mathcal{D}, \mathfrak{A}_c) + X(cs_\sigma, \iota; \mathcal{D}, \mathfrak{A}_c) = r \log \iota(\alpha)$$

for any $\iota \neq \sigma$.

Remark 2.1.1. *In fact, we can write $r \in \mathbf{Q}$, $\alpha \in F$ in the above Theorem explicitly by the Bernoulli numbers and the bases of cones in a Shintani domain, but it is rather complicated.*

Remark 2.1.2. *For $\sigma, \sigma' \in \mathrm{Hom}(F, \mathbf{R})$, that $s_\sigma = s_{\sigma'}$ may happen even if $\sigma \neq \sigma'$. For example, assume that the maximal ray class field $H_\mathfrak{f}$ modulo $\mathfrak{f}\infty_1 \ldots \infty_n$ is a CM-field. Then $s_\sigma = s_{\sigma'}$ for any σ, σ'. Therefore if $F \neq \mathbf{Q}$ and if $H_\mathfrak{f}$ is a CM-field, then we have*

$$\exp(X(c, \iota)) \exp(X(cs_\sigma, \iota)) \in \overline{\mathbf{Q}}^\times.$$

without assuming that $\iota \neq \sigma$. Similarly, assume that an abelian extension K of F is a CM-field. Let $\mathfrak{f}$ be the conductor of K/F and $\mathrm{Art} : \mathfrak{C}_\mathfrak{f} \to \mathrm{Gal}(K/F)$ the Artin map. Note that K has the unique complex conjugation ρ. For $\tau \in \mathrm{Gal}(K/F)$, we put

$$X(\tau, \iota) := \sum_{c \in \mathfrak{C}_\mathfrak{f},\ \mathrm{Art}(c) = \tau} X(c, \iota).$$

Then we can show that

$$\exp(X(\tau, \iota)) \exp(X(\tau\rho, \iota)) \equiv 1 \bmod \overline{\mathbf{Q}}^\times$$

for any $\tau \in \mathrm{Gal}(K/F)$, $\iota \in \mathrm{Hom}(F, \mathbf{R})$. If $F = \mathbf{Q}$, then this formula follows from the classical result (1.1) and Remark 1.3.1.

2.2 The relation to Stark's conjecture

Let F be a totally real field of degree n, $\mathfrak{f}$ an integral ideal of F, and $H_\mathfrak{f}$ the narrow ideal class field modulo $\mathfrak{f}$. Then we have the Artin map Art : $\mathfrak{C}_\mathfrak{f} \cong \mathrm{Gal}(H_\mathfrak{f}/F)$ with the narrow ideal class group $\mathfrak{C}_\mathfrak{f}$ modulo $\mathfrak{f}$. We fix an embedding id : $F \hookrightarrow \mathbf{R}$ and take the ideal class $s_{\mathrm{id}} \in \mathfrak{C}_\mathfrak{f}$ as above. Since the fixed subfield $M := H_\mathfrak{f}^{\langle s_{\mathrm{id}} \rangle}$ under s_{id} has a real place lying above id, Stark's conjecture states that for all $c \in \mathfrak{C}_\mathfrak{f}$, we have

$$\exp(\zeta'(0,c) + \zeta'(0, cs_{\mathrm{id}})) \in \overline{\mathbf{Q}}^\times. \tag{2.1}$$

Conversely, we can show that the algebraicity of any Stark's unit over F follows from the algebraicity (2.1) for all c. Now, by using Shintani's formula (Theorem 1.3.1), we may relate the derivative values $\zeta'(0,c)$ to Yoshida's class invariants $X(c)$. Namely we have

$$\exp(\zeta'(0,c) + \zeta'(0, cs_{\mathrm{id}})) \equiv \prod_{\iota \in \mathrm{Hom}(F,\mathbf{R})} \exp(X(c,\iota)) \exp(X(cs_{\mathrm{id}},\iota)) \bmod \overline{\mathbf{Q}}^\times.$$

Therefore by Theorem 2.1.1, "the algebraicity of Stark's units" is equivalent to

$$\exp(X(c)) \exp(X(cs_{\mathrm{id}})) \equiv 1 \bmod \overline{\mathbf{Q}}^\times \tag{2.2}$$

for all c.

3 The relation to CM-periods

Yoshida formulated a conjecture which expresses any CM-period as a product of (rational powers of) $\exp(X(c))$. Assuming his conjecture, we can investigate the relation between "monomial relations on multiple gamma functions", "monomial relations on CM-periods", and "algebraicity of Stark's units".

3.1 CM-periods and the relation to multiple gamma functions

First we recall Shimura's CM-period symbol p_K. For detail, see [S]. Let K be a CM-field, I_K the $\mathbf{Q}$ vector space formally generated by the embeddings $K \hookrightarrow \mathbf{C}$. Then Shimura showed that

Theorem 3.1.1. *There exists a bilinear map*

$$p_K : I_K \times I_K \to \mathbf{C}^\times/\mathbf{Q}^\times$$

which is characterized by the following property; Let χ be an arbitrary algebraic Hecke character of K with the infinite type $\sum_{\sigma \in \mathrm{Hom}(K,\mathbf{C})} l_\sigma \sigma$, $l_\sigma \in \mathbf{Z}$. For simplicity we assume that

1. *$l_\sigma + l_{\rho\circ\sigma} = 0$ for all σ.*
2. *$l_\sigma \in 2\mathbf{Z} - \{0\}$ for all σ.*

Here we denote the complex conjugation on $\mathbf{C}$ *by* ρ. *Then* $s = 0$ *is a critical value of the associated Hecke L-function* $L(s, \chi)$ *to* χ *and we have*

$$L(0,\chi) \equiv \pi^{\sum_{l_\sigma \in \Phi_\chi} \frac{l_\sigma}{2}} p_K(T_\chi, \Phi_\chi) \bmod \overline{\mathbf{Q}}^\times,$$

where we put $\Phi_\chi := \sum_{l_\sigma > 0} \sigma$, $T_\chi := \sum_{l_\sigma \in \Phi_\chi} l_\sigma \sigma$.

Remark 3.1.1. *Shimura expressed any critical value of any Hecke L-function of* K *in terms of a rational power of* π *and his symbol* p_K *up to algebraic numbers. He also showed some "monomial relations" among* $p_K(\sigma, \sigma')$, *e.g.,*

$$p_K(\sigma, \rho \circ \sigma') \equiv p_K(\rho \circ \sigma, \sigma') \equiv p_K(\sigma, \sigma')^{-1} \bmod \overline{\mathbf{Q}}^\times \tag{3.1}$$

for any $\sigma, \sigma' \in \mathrm{Hom}(K, \mathbf{C})$.

By the following Yoshida's conjecture, we can express any CM-period in terms of Yoshida's invariant $X(c)$. For detail, see [Yo3].

Conjecture 3.1.1. *(Yoshida's conjecture.) Let* K *be a CM-field and* F *a totally real field. We assume that* K/F *is abelian, and put* $G := \mathrm{Gal}(K/F)$, $\widehat{G}^- :=$ *the set of all odd characters of* G. *For any* $\chi \in \widehat{G}^-$, *we denote by* K_χ *the fixed subfield of* K *under* $\mathrm{Ker}\,\chi$, *by* $\mathfrak{f}_\chi$ *the conductor of* K_χ/F. *Then for any* $\tau \in G$ *we have*

$$p_K(\mathrm{id}, \tau) \equiv \pi^{-\frac{\mu(\tau)}{2}} \exp\Big(\frac{1}{|G|} \sum_{\chi \in \widehat{G}^-} \frac{\chi(\tau) \sum_{c \in \mathfrak{C}_{\mathfrak{f}_\chi}} \chi(c) X(c)}{L(0,\chi)}\Big) \bmod \overline{\mathbf{Q}}^\times.$$

Here we put $\mu(\tau) = 1, -1, 0$ *if* $\tau = \mathrm{id}, \rho$, *otherwise, respectively.*

3.2 A slight generalization of Yoshida's conjecture.

First we generalize Shimura's period symbol p_K to any number field K as follows. For a number field K, we denote by K_{CM} the maximal CM-subfield. (If K has no CM-subfield, we put $K_{CM} :=$ the maximal totally real subfield.) We denote by I_K^0 the image of the linear map $\mathrm{Inf} : I_{K_{CM}} \to I_K$ defined by $\sigma \mapsto \sum_{\sigma' \in \mathrm{Hom}(K,\mathbf{C}),\ \sigma'|_{K_{CM}} = \sigma} \sigma'$. As a corollary of Theorem 3.1.1 and results of Harder and Schappacher, we get

Corollary 3.2.1. *For any number field* K, *there exists a bilinear map*

$$p_K : I_K \times I_K^0 \to \mathbf{C}^\times / \mathbf{Q}^\times$$

characterized by the property;

$$L(0,\chi) \equiv \pi^{\sum_{l_\sigma \in \Phi_\chi} \frac{l_\sigma}{2}} p_K(T_\chi, \Phi_\chi) \bmod \overline{\mathbf{Q}}^\times$$

for any algebraic Hecke character χ *of* K *satisfying that* $l_\sigma + l_{\rho\circ\sigma} = 0$ *for all* σ *and for all complex conjugations* ρ *on* K^σ *and that* $l_\sigma \in 2\mathbf{Z} - \{0\}$ *for all* σ. *Here* $l_\sigma, \Phi_\chi, T_\chi$ *are the same as in Theorem 3.1.1.*

Remark 3.2.1. *By using results of Harder and Schappacher, we can write critical values of Hecke L-functions of K in terms of those of K_{CM}, up to algebraic numbers. Therefore no new period appears even if we generalize the p_K-symbol as above.*

By using Theorem 2.1.1 (in particular, the following Remark 2.1.2), Yoshida's conjecture can be rewrite as follows; Let F be a totally real field, K be an abelian extension of F, $\mathfrak{f}$ be the conductor of K/F and $G := \mathrm{Gal}(K/F)$. For $\tau \in G$, we put

$$\Gamma(\tau) := \prod_{c \in \mathfrak{C}_{\mathfrak{f}},\ \mathrm{Art}(c)=\tau} \exp(X(c)),$$

which is well-defined up to algebraic numbers. Then Yoshida's conjecture states that

$$\Gamma(\tau) \equiv \pi^{\zeta(0,\tau)} p_K(\tau, \sum_{\sigma \in G} \zeta(0,\sigma)\sigma) \bmod \overline{\mathbf{Q}}^\times \tag{3.2}$$

if K is a CM-field. Now we can generalize his conjecture slightly as follows.

Conjecture 3.2.1. *The above formula (3.2) holds even if K is not a CM-field.*

Remark 3.2.2. *Assuming the above conjecture, "the algebraicity of Stark's units" (2.2) follows from monomial relations among CM-periods. Let $K = H_{\mathfrak{f}}$. Then $\Gamma(\mathrm{Art}(c)) = \exp(X(c))$. Conjecture 3.2.1 states that*

$$\exp(X(c))\exp(X(cs_{\mathrm{id}})) \equiv \pi^{\zeta(0,\tau)+\zeta(0,\rho\circ\tau)} p_K(\tau + \rho\circ\tau, \sum_{\sigma\in G} \zeta(0,\sigma)\sigma) \bmod \overline{\mathbf{Q}}^\times.$$

In general, we have $\zeta(0,\tau) + \zeta(0,\rho\circ\tau) = 0$. That $p_K(\tau + \rho\circ\tau, \sigma) \equiv 1 \bmod \overline{\mathbf{Q}}^\times$ follows from monomial relations (3.1) of CM-periods.

4 p-adic analogues

In this section, we formulate a p-adic analogue of Theorem 2.1.1. More, by using both archimedean and p-adic multiple gamma functions, we can construct a refined invariant, which is well-defined (without modulo algebraic numbers). For simplicity, we fix embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$. We normalize the p-adic absolute value $|z|_p$ on $z \in \mathbf{C}_p$ by $|z|_p = 1/p$.

4.1 The p-adic class invariant $X_p(c)$

We recall the definition of the symbol $X_p(c)$ which is the p-adic counterpart of Yoshida's invariant $X(c)$. For detail, see [KY2].

Definition 4.1.1. *We call a subset $Z \subset \overline{\mathbf{Q}}$ "p-good" (with respect to ι_∞, ι_p) when we can write*

$$Z = \sqcup_{j=1}^k \{z_j + m_1 v_{j,1} + m_2 v_{j,2} + \cdots + m_{r(j)} v_{j,r(j)} \mid m_i \in \mathbf{Z}_{\geq 0}\}$$

with $r_j, k \in \mathbf{Z}_{>0}$, $z_j, v_{i,j} \in \overline{\mathbf{Q}}$ *satisfying* $\iota_\infty(z_j), \iota_\infty(v_{i,j}) \in \mathbf{R}_{>0}$, $|\iota_p(z_j)|_p > |\iota_p(v_{i,j})|_p$. *In this case we define*

$$\zeta_{(p)}(s;Z) := \sum_{z\in Z}(|\iota_p(z)|_p\iota_\infty(z))^{-s}$$

for $s \in \mathbf{C}$ *with* $\mathrm{Re}(s)$ *large enough. It can be continued meromorphically to the whole complex plane, and is analytic at* $s = 0$. *Moreover there exists the p-adic interpolation function* $\zeta_p(s;Z)$ *which is characterized by*

$$\zeta_p(-k;Z) = \zeta_{(p)}(-k;Z)$$

for $k \in \mathbf{Z}_{\geq 0}$, $k \equiv 0 \bmod N$ *with a suitable integer* N. *We also put*

$$L\Gamma_p(Z) := \zeta_p'(0;Z).$$

Remark 4.1.1. *Pierrette Cassou-Noguès constructed the p-adic interpolation function* $\zeta_{p,r}(s,(v_1,\ldots,v_r),z)$ *of Barnes's multiple zeta function* $\zeta_r(s,(v_1,\ldots,v_r),z)$ *under some condition in [CN1]. As the derivative value* $\zeta_{p,r}'(0,(v_1,\ldots,v_r),z)$, *the author defined the p-adic* log *multiple gamma function* $L\Gamma_{p,r}(z,(v_1,\ldots,v_r))$ *and investigated its properties in [Ka]. We can write the above functions* $\zeta_p(s;Z)$, $L\Gamma_p(Z)$ *as a finite sum of these functions* $\zeta_{p,r}(s,(v_1,\ldots,v_r),z)$, $L\Gamma_{p,r}(z,(v_1,\ldots,v_r))$, *respectively. We call* $L\Gamma_p(Z)$ *the p-adic* log *multiple gamma function, also.*

Let F be a totally real field. For an embedding $\iota \in \mathrm{Hom}(F,\overline{\mathbf{Q}})$, we denote by $\mathfrak{p}_\iota$ the prime ideal of F associated with $\iota_p \circ \iota : F \hookrightarrow \mathbf{C}_p$. Other notations are as above. Then we have

Lemma 4.1.1. *If* $c \in \mathfrak{C}_\mathfrak{f}$ *with* $\mathfrak{p}_\iota \mid \mathfrak{f}$, *then the subset* $\iota(Z(c;\mathcal{D},\mathfrak{A}_c))$ *is p-good.*

Definition 4.1.2. *Assume that* $\mathfrak{p}_\iota$ *divides* $\mathfrak{f}$. *For an ideal class* $c \in \mathfrak{C}_\mathfrak{f}$, *a Shintani domain* $\mathcal{D}$ *and a fractional ideal* $\mathfrak{A}_c$ *with* $\mathfrak{A}_c = c \bmod \mathrm{P}_{(1)}$, *we put*

$$\begin{aligned}
G_p(c,\iota;\mathcal{D},\mathfrak{A}_c) &:= L\Gamma_p(\iota(Z(c;\mathcal{D},\mathfrak{A}_c))),\\
W_p(c,\iota;\mathcal{D},\mathfrak{A}_c) &:= -\frac{1}{n}\zeta_p(0;\iota(Z(c;\mathcal{D},\mathfrak{A}_c)))\log_p \mathrm{N}\mathfrak{A}_c,\\
V_p(c,\iota;\mathcal{D},\mathfrak{A}_c) &:= \sum_i \iota_p\circ\iota(a_i)\log_p \iota_p\circ\iota(\epsilon_i),\\
X_p(c,\iota;\mathcal{D},\mathfrak{A}_c) &:= G_p(c,\iota;\mathcal{D},\mathfrak{A}_c) + W_p(c,\iota;\mathcal{D},\mathfrak{A}_c) + V_p(c,\iota;\mathcal{D},\mathfrak{A}_c),
\end{aligned}$$

where we take elements $a_i \in F$, $\epsilon_i \in E_F^+$ *satisfying (1.4). For simplicity, we use the notation* $G_p(c) := G_p(c;\mathcal{D},\mathfrak{A}_c) := G_p(c,\iota;\mathcal{D},\mathfrak{A}_c)$, *etc.*

The following Theorem is a p-adic analogue of Theorem 2.1.1.

Theorem 4.1.1. *Assume that* $\mathfrak{p}_\iota$ *divides* $\mathfrak{f}$. *For* $\sigma \in \mathrm{Hom}(F,\mathbf{R})$ *(without assuming* $\sigma \neq \iota_\infty \circ \iota$*), and for* $c \in \mathfrak{C}_\mathfrak{f}$ *we have*

$$X_p(c) + X_p(cs_\sigma) \in \log_p \overline{\mathbf{Q}}^\times.$$

More precisely there exist $r \in \mathbf{Q}$, $\alpha \in F^\times$, *depending on* c, σ, $\mathcal{D}$ *and* $\mathfrak{A}_c$, *which satisfy*

$$X_p(c,\iota;\mathcal{D},\mathfrak{A}_c) + X_p(cs_\sigma,\iota;\mathcal{D},\mathfrak{A}_c) = r\log_p \iota_p\circ\iota(\alpha)$$

for any ι *with* $\mathfrak{p}_\iota \mid \mathfrak{f}$.

Remark 4.1.2. *In the archimedean case, Theorem 2.1.1 holds when $\sigma \neq \iota$. Moreover in the case of $\sigma = \iota$, it involves Stark's conjecture. In the p-adic version, Theorem 4.1.1 holds even when $\sigma = \iota_\infty \circ \iota$.*

4.2 A refined version of Conjecture (2.2)

By using archimedean and p-adic multiple gamma functions simultaneously, we conjecturally write the Stark unit strictly.

Definition 4.2.1. *Let M be an abelian extension of F with the conductor $\mathfrak{f}$. Assume that M has a real place. Take $\iota \in \mathrm{Hom}(M, \overline{\mathbf{Q}})$ so that $\iota_\infty \circ \iota$ corresponds to the real place, and take p, ι_p so that $\mathfrak{p}_\iota \mid \mathfrak{f}$. Then by Theorem 4.1.1, for $\tau \in \mathrm{Gal}(M/F)$ there exist $r \in \mathbf{Q}$, $\alpha \in F^\times$ such that*

$$\sum_{c \in \mathfrak{C}_\mathfrak{f},\ \mathrm{Art}(c)=\tau} X_p(c, \iota; \mathcal{D}, \mathfrak{A}_c) = r \log_p \iota_p \circ \iota(\alpha).$$

We may assume that $|\iota_p \circ \iota(\alpha)|_p = 1$, $\iota_\infty \circ \iota(\alpha) > 0$. For $\tau \in \mathrm{Gal}(M/F)$, we put

$$\Gamma_{\infty/p}(\tau) := \exp\Big(\sum_{c \in \mathfrak{C}_\mathfrak{f},\ \mathrm{Art}(c)=\tau} X(c, \iota; \mathcal{D}, \mathfrak{A}_c)\Big)/\alpha^r.$$

Theorem 4.2.1. *The definition of $\Gamma_{\infty/p}(\tau)$ does not depend on the choice of $\mathcal{D}, \mathfrak{A}_c$.*

Conjecture 4.2.1. *The definition of $\Gamma_{\infty/p}(\tau)$ depends only on τ and $\iota_\infty \circ \iota$. Moreover $\Gamma_{\infty/p}(\tau) = \iota_\infty \circ \iota \circ \tau(u)$, where u is the Stark unit in (0.1).*

5 A Sketch of the proof

We need the following key Lemma to prove our main Theorems. We omit the proof of this Lemma in this paper. Let F be a totally real field of degree n. Put $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} := \mathrm{Hom}(F, \mathbf{R})$. We embed $F \hookrightarrow \mathbf{R}^n$ by $z \mapsto (z^{\sigma_i})_i$.

Lemma 5.0.1. *Assume that $n = [F : \mathbf{Q}] \geq 2$. Then we can take a Shintani domain $\mathcal{D}$, an element $\nu \in F$, finite cones C_i, $C_{i'}$ whose basis are in $F \cap (\mathbf{R}_+ \times \mathbf{R}^\times \times \mathbf{R}_+^{n-2})$, and totally positive units $u_i \in E_F^+$ satisfying the following properties;*

1. *$\nu^{\sigma_i} < 0$ if $i = 2$, $\nu^{\sigma_i} > 0$ otherwise,*
2. *we can decompose $\mathcal{D} = \sqcup_{m \in M} D_m$ so that $X_m := \sqcup_{i \in I_m} C_i \supset D_m, \nu D_m$, $Y_m := \sqcup_{i' \in I'_m} C_{i'} = X_m - D_m - \nu D_m$,*
3. *there exists a one to one correspondence $\phi : \sqcup_{m \in M} I_m \to \sqcup_{m \in M} I'_m$ such that $C_{\phi(i)} = u_i C_i$,*

where M, I_m, I'_m are finite sets of indices.

We shall give the sketch of the proof of Theorem 2.1.1. For simplicity we see only the main idea to calculate special values of multiple gamma functions (that is, $G(c)$ terms) and assume that there exists an $\epsilon_0 \in E_F$ such that $\epsilon_0 \nu_{\sigma_2} \in F^+$. We can prove Theorem

4.1.1 in the same manner. We may assume that $\iota = \sigma_1 = \mathrm{id}$, $\sigma = \sigma_2$. Take $\nu, \mathcal{D}$ as in the above Lemma. Recall

$$G(c; \mathcal{D}, \mathfrak{A}_c) = \frac{d}{ds}\left[\sum_{z \in Z(c;\mathcal{D},\mathfrak{A}_c)} z^{-s}\right]_{s=0},$$
$$Z(c; \mathcal{D}, \mathfrak{A}_c) = \{z \in \mathcal{D} \cap \mathfrak{A}_c^{-1} \mid z\mathfrak{A}_c \in c\}.$$

Note that $\epsilon_0^{-1}\nu\mathcal{D}$ is another Shintani domain. We can write

$$\sum_{z \in Z(c;\mathcal{D},\mathfrak{A}_c)} z^{-s} + \epsilon_0^{-s} \sum_{z \in Z(c;\epsilon_0^{-1}\nu\mathcal{D},\mathfrak{A}_c)} z^{-s} = \sum_{z \in (\mathcal{D} \sqcup \nu D) \cap \mathfrak{A}_c^{-1},\ z\mathfrak{A}_c \in c} z^{-s}.$$

By assumption, we get $(\nu_{\sigma_2}) = (\nu_{\sigma_2}\epsilon_0) \in P_{(1)}$. Therefore for the ideal class cs_{σ_2}, we can take $\mathfrak{A}_{cs_{\sigma_2}}$ so that $\mathfrak{A}_{cs_{\sigma_2}} = \mathfrak{A}_c$ and we get

$$\sum_{z \in Z(cs_{\sigma_2};\mathcal{D},\mathfrak{A}_c)} z^{-s} + \epsilon_0^{-s} \sum_{z \in Z(cs_{\sigma_2};\epsilon_0^{-1}\nu\mathcal{D},\mathfrak{A}_c)} z^{-s} = \sum_{z \in (\mathcal{D} \sqcup \nu D) \cap \mathfrak{A}_c^{-1},\ z\mathfrak{A}_c \in cs_{\sigma_2}} z^{-s}.$$

Put $R := \{z \in \mathfrak{A}_c^{-1} \mid z\mathfrak{A}_c \in c \text{ or } cs_{\sigma_2}\}$. Then we have

$$\sum_{z \in Z(c;\mathcal{D},\mathfrak{A}_c)} z^{-s} + \epsilon_0^{-s} \sum_{z \in Z(c;\epsilon_0^{-1}\nu\mathcal{D},\mathfrak{A}_c)} z^{-s} + \sum_{z \in Z(cs_{\sigma_2};\mathcal{D},\mathfrak{A}_c)} z^{-s} + \epsilon_0^{-s} \sum_{z \in Z(cs_{\sigma_2};\epsilon_0^{-1}\nu\mathcal{D},\mathfrak{A}_c)} z^{-s}$$
$$= \frac{2}{\operatorname{ord} s_{\sigma_2}} \sum_{z \in R \cap (\mathcal{D} \sqcup \nu \mathcal{D})} z^{-s}.$$

The derivative value at $s = 0$ of the left hand side is equal to

$$\begin{aligned}
&2(G(c; \mathcal{D}, \mathfrak{A}_c) + G(cs_{\sigma_2}; \mathcal{D}, \mathfrak{A}_c)) \\
&- (G(c; \mathcal{D}, \mathfrak{A}_c) - G(c; \epsilon_0^{-1}\nu\mathcal{D}, \mathfrak{A}_c)) \\
&- (G(cs_{\sigma_2}; \mathcal{D}, \mathfrak{A}_c) - G(cs_{\sigma_2}; \epsilon_0^{-1}\nu\mathcal{D}, \mathfrak{A}_c)) \\
&- (\zeta(0; Z(c; \epsilon_0^{-1}\nu\mathcal{D}, \mathfrak{a}_c)) + \zeta(0; Z(cs_2; \epsilon_0^{-1}\nu\mathcal{D}, \mathfrak{a}_c))) \log \epsilon_0.
\end{aligned}$$

By the above Lemma, we can write

$$\sum_{z \in R \cap (\mathcal{D} \sqcup \nu \mathcal{D})} z^{-s} = \sum_{m \in M} \sum_{i \in I_m} (1 - u_i^{-s}) \sum_{z \in R \cap C_i} z^{-s}.$$

We can show that each set $R \cap C_i$ is good in the sense of Definition 1.1.3. Therefore we get

$$\begin{aligned}
&G(c; \mathcal{D}, \mathfrak{A}_c) + G(cs_{\sigma_2}; \mathcal{D}, \mathfrak{A}_c) \\
&= \frac{1}{2}(G(c; \mathcal{D}, \mathfrak{A}_c) - G(c; \epsilon_0^{-1}\nu\mathcal{D}, \mathfrak{A}_c)) + \frac{1}{2}(G(cs_{\sigma_2}; \mathcal{D}, \mathfrak{A}_c) - G(cs_{\sigma_2}; \epsilon_0^{-1}\nu\mathcal{D}, \mathfrak{A}_c)) \\
&+ \frac{1}{2}(\zeta(0, Z(c; \epsilon_0^{-1}\nu D, \mathfrak{a}_c)) + \zeta(0, Z(cs_2; \epsilon_0^{-1}\nu D, \mathfrak{a}_c))) \log \epsilon_0 \\
&+ \frac{1}{\operatorname{ord} s_2} \sum_{m \in M} \sum_{i \in I_m} \zeta(0; R \cap C_i) \log u_i.
\end{aligned}$$

By Theorem 1.2.1 and Remark 1.1.2, we can write the right hand side in the form of $\sum_i \alpha_i \log \beta_i$, $\alpha, \beta \in F$ explicitly. We can also write other terms $V(c), W(c)$ explicitly, and summing up them, we get the assertion of Theorem 2.1.1.

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