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SMOOTH SPACES OF DISCRETE SERIES REPRESENTATIONS AND THE HANKEL TRANSFORM

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ABSTRACT. This is a summary of our results on the Hankel inversion formula and the smooth space of the discrete series representations of $SL(2,\mathbb{R})$. We give a complete description of the smooth space of the Kirillov model for these representations.

1. INTRODUCTION

The Kirillov model of irreducible unitary representations of $GL(2,\mathbb{R})$ has played a significant role in the theory of automorphic forms. It has been noticed by [2] (See also [4],[6],[5], [1]) that the action of the Weyl element in the Kirillov model of an irreducible unitary representation of $GL(2,\mathbb{R})$ is given by a certain integral transform and that in the case of the discrete series this is a classical Hankel transform of integer order. In particular, this implies in theory that the Hankel transform is of order two when the Weyl element has an action of order two. We use this fact to give a new proof of the famous Hankel inversion formula which holds for a certain Schwartz space of functions. Using this we show that this Schwartz space is exactly the smooth space of the Kirillov model. We will work with the group $SL(2,\mathbb{R})$ but similar results hold for the group $GL(2,\mathbb{R})$.

2. THE HANKEL INVERSION FORMULA

Let $J_\nu(x)$ be the classical J-Bessel function defined by

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}.$$

The Hankel inversion formula is classically stated as follows ([7] p.453). Let ϕ be a complex valued function defined on the positive real line. Then under certain assumptions on ϕ and ν (See [7]) we have

$$\phi(z)=\int_0^\infty\int_0^\infty \phi(x)J_\nu(xy)J_\nu(yz)xydxdy$$

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In more modern notation we define the Hankel transform of order ν of ϕ to be

$$h_\nu(\phi)(y) = \int_0^\infty \phi(x) J_\nu(xy) x dx$$

Then under certain assumptions on ϕ and ν the Hankel transform is self reciprocal, that is, $h_\nu^2 = Id$. Under a change of variables $x \to x^2$ and $f(x) = \sqrt{x}\phi(\sqrt{x})/2$, the Hankel inversion formula is equivalent to $\mathcal{H}_\nu^2 = Id$ where

$$\mathcal{H}_\nu(f)(y) = \int_0^\infty f(x)\sqrt{xy} J_\nu(2\sqrt{xy}) \frac{dx}{x}.$$

2.1. **The Schwartz space.** Let $S([0,\infty))$ be the Schwartz space of functions on $[0,\infty)$. That is, $f : [0,\infty) \to \mathbb{C}$ is in $S([0,\infty))$ if f is smooth on $[0,\infty)$ and f and all its derivatives are rapidly decreasing at ∞. Let

$$S_\nu([0,\infty)) = \{f : [0,\infty) \to \mathbb{C} \mid f(x) = x^{1/2+\nu/2} f_1(x) \text{ and } f_1 \in S([0,\infty))\}$$

Our first theorem is the following:

Theorem 2.1. *Assume that $Re(\nu) > -1$. Then $\mathcal{H}_\nu$ is an isomorphism of order two of $S_\nu([0,\infty))$. Moreover, when ν is real $\nu > -1$, $\mathcal{H}_\nu$ is an $L^2((0,\infty), dx/x)$ isometry.*

Remark 2.2. This theorem was proved in [8] when ν is real, $\nu > -1/2$ and in [3] when ν is real, $\nu > -1$. Our proof is different and is based on the following integral identity:

Assume $\mathrm{Re}(\nu) > -1$. For $f \in S_\nu([0,\infty))$ we define $T_\nu(f) = (x^{-1/2+\nu/2} f)^\vee$. That is,

$$T_\nu(f)(z) = (2\pi)^{-1/2} \int_0^\infty x^{-1/2+\nu/2} f(x) e^{ixz} dx.$$

for a function $\phi : \mathbb{R}^* \to \mathbb{C}$ we define

$$\mathcal{W}_\nu(\phi)(x) = |x|^{-\nu-1} e^{\mathrm{sgn}(x)\pi i(\nu+1)/2} \phi(-1/x).$$

Theorem 2.3. *Let $f \in S_\nu([0,\infty))$ then $T_\nu \circ \mathcal{H}_\nu(f) = \mathcal{W}_\nu \circ T_\nu(f)$.*

The above theorem is proved by changing the order of integration after introducing a convergence factor. We now show that it implies Theorem 2.1

Theorem 2.4. *Assume $Re(\nu) > -1$ and $f \in S_\nu([0,\infty))$. Then*

$$\mathcal{H}_\nu \circ \mathcal{H}_\nu(f) = f$$

Proof. This follows immediately from Theorem 2.3 and the fact that $\mathcal{W}_\nu \circ \mathcal{W}_\nu = Id$ which is easy to check. The argument is as follows. Let $f \in S_\nu([0,\infty))$. Then

$$T_\nu \circ \mathcal{H}_\nu \circ \mathcal{H}_\nu(f) = \mathcal{W}_\nu \circ T_\nu \circ \mathcal{H}_\nu(f) = \mathcal{W}_\nu \circ \mathcal{W}_\nu \circ T_\nu(f) = T_\nu(f).$$

Since T_ν is one to one, it follows that $\mathcal{H}_\nu \circ \mathcal{H}_\nu(f) = f$. □

We let I_ν be a space of functions defined by

$$I_\nu = \{\phi : \mathbb{R} \to \mathbb{C} \mid \phi \text{ is smooth on } \mathbb{R} \text{ and } \mathcal{W}_\nu(\phi) \text{ is smooth on } \mathbb{R}\}.$$

Proposition 2.5. *T_ν maps $S_\nu([0,\infty))$ into I_ν.*

3. A DISCRETE SERIES REPRESENTATION INSIDE AN INDUCED SPACE

We now consider the discrete series representations of $SL(2,\mathbb{R})$ with trivial central character. Similar results hold for the case of nontrivial central character. Let $G = SL(2,\mathbb{R})$. Let N and A be subgroups of G defined by

$$N = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

$$A = \left\{ s(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{R}^* \right\}.$$

and

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We let $\nu = 2k - 1$ where k is a positive integer . The space I_{2k-1} is the space of smooth functions ϕ on $\mathbb{R}$ such that

$$\mathcal{W}_{2k-1}(\phi)(x) = (-1)^k |x|^{-2k} \phi(-1/x) = (-1)^k x^{-2k} \phi(-1/x)$$

is smooth. This is also the space of smooth functions ϕ such that $w(\phi)(x) = x^{-2k}\phi(-1/x)$ is smooth. We now define a representation π_{2k-1} of G on I_{2k-1} in the following way: Let $\phi \in I_{2k-1}$.

$$\begin{aligned} (w\phi)(x) &= x^{-2k}\phi(-1/x) \\ (n(y)\phi)(x) &= \phi(x+y) \\ (s(z)\phi)(x) &= z^{-2k}\phi(z^{-2}x) \end{aligned}$$

More generally we have

$$\left(\pi_{2k-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi \right)(x) = (cx+a)^{-2k} \phi\left(\frac{dx+b}{cx+a}\right) \tag{3.1}$$

Proposition 3.1. *The representation π_{2k-1} of G on I_{2k-1} is isomorphic to the smooth space of a representation of G which is induced from a Borel subgroup. As such, it inherits the standard Fréchet topology.*

Theorem 3.2. *The space $T_{2k-1}(S_{2k-1}([0,\infty)))$ is a closed invariant subspace of I_{2k-1} which is isomorphic to a discrete series representation.*

Remark 3.3. It follows from Theorem 2.3 that the image of $S_{2k-1}([0,\infty))$ is invariant under the action of G. Thus it is enough to prove that this space is closed and that the space of K-finite vectors is irreducible as a $(\mathfrak{g}, K)$ module.

4. THE KIRILLOV MODEL

We are now ready to describe the Kirillov model. The action of G on the image of $S_{2k-1}([0,\infty))$ under T_{2k-1} induces the following two actions of G

on S_{2k-1}: For each $f \in S_{2k-1}([0,\infty))$ let

$$(D^+_{2k-1}(n(y))f)(x) = e^{iyx}f(x)$$
$$(D^+_{2k-1}(s(z))f)(x) = f(z^2x)$$
$$(D^+_{2k-1}(w)f)(x) = (-1)^k\mathcal{H}_{2k-1}(f)(x)$$

$$(D^-_{2k-1}(n(y))f)(x) = e^{-iyx}f(x)$$
$$(D^-_{2k-1}(s(z))f)(x) = f(z^2x)$$
$$(D^-_{2k-1}(w)f)(x) = (-1)^k\mathcal{H}_{2k-1}(f)(x)$$

From Theorem 3.2 we get the following corollary:

Corollary 4.1. *D^+_{2k-1} (D^-_{2k-1}) is an irreducible smooth representation of $SL(2,\mathbb{R})$. It is isomorphic to (the smooth space of) a discrete series representation with the lowest weight (highest weight) vector of weight $2k-1$.*

It is easy to see that the above action of G can be extended to a Hilbert space representation of G on the space $\mathbf{H}^{\pm}_{2k-1} = L^2((0,\infty), dx/x)$. Our main result is the following:

Theorem 4.2. *$D^{\pm}_{2k-1}$ is an irreducible unitary representation on the space $\mathbf{H}^{\pm}_{2k-1} = L^2((0,\infty), dx/x)$. The smooth space of this representation is $S_{2k-1}([0,\infty))$.*

References

1. Ehud Moshe Baruch and Zhengyu Mao, *Bessel identities in the Waldspurger correspondence over the real numbers*, Israel J. Math. **145** (2005), 1–81.
2. James W. Cogdell and Ilya Piatetski-Shapiro, *The arithmetic and spectral analysis of Poincaré series*, Perspectives in Mathematics, vol. 13, Academic Press Inc., Boston, MA, 1990.
3. Antonio J. Duran, *On Hankel transform*, Proc. Amer. Math. Soc. **110** (1990), no. 2, 417–424.
4. I. M. Gel′fand, M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, Generalized Functions, vol. 6, Academic Press Inc., Boston, MA, 1990, Translated from the Russian by K. A. Hirsch, Reprint of the 1969 edition.
5. Yoichi Motohashi, *A note on the mean value of the zeta and L-functions. XII*, Proc. Japan Acad. Ser. A Math. Sci. **78** (2002), no. 3, 36–41.
6. N. Ja. Vilenkin, *Special functions and the theory of group representations*, Translated from the Russian by V. N. Singh. Translations of Mathematical Monographs, Vol. 22, American Mathematical Society, Providence, R. I., 1968.
7. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1995, Reprint of the second (1944) edition.
8. A.H. Zemanian, *A distributional Hankel transformation*, SIAM J. Appl. Math. **14** (1966), 561–576.

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