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# GEOMETRIC LIMITS VIEWED THROUGH MODEL MANIFOLDS

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## 1. INTRODUCTION

In this note, we shall present results which will appear in the part II of the author's joint project with Teruhiko Soma [10]. The aim of this project is to give a topological and geometric classification of geometric limits of finitely generated Kleinian groups. In the second part, we shall define geometric limits of hierarchies of tight geodesics in the sense of Masur-Minsky. Such limits may not be hierarchies in general, but as we shall see, they have properties similar to hierarchies. We call such objects bug-infested hierarchies. We shall then show that bug-infested hierarchies give rise to model manifolds for geometric limits, via slices and resolutions as in the case of surface Kleinian groups.

In this note, we shall restrict ourselves to bug-infested hierarchies and their geometric limits without mentioning model manifolds.

## 2. BUG-INFESTED HIERARCHIES

Before defining bug-infested hierarchies, we shall review some terminologies on curve complexes defined by Masur-Minsky [6]. We shall use the same notations in the definition below. Let $g = \{v_i\}_{i \in I}$, where $I$ is either finite or $\mathbb{Z}^-$ or $\mathbb{Z}^+$ or $\mathbb{Z}$, be a tight geodesic (segment or ray or line) in the curve complex $\mathcal{C}(X)$ for a domain $X$ (i.e., an open essential subsurface) of $S$ connecting a vertex in the initial marking $I(g)$ to a vertex in the terminal marking $T(g)$ or converging to $I(g)$ as $i \to -\infty$ or to $T(g)$ as $i \to \infty$ in the compactification $\mathcal{C}(X) \cup \mathcal{EL}(X)$. We say that $X$ is the support of $g$ then and denote it by $D(g)$. Recall that each $v_i$ is not necessarily a vertex but a simplex in $\mathcal{C}(X)$ except for the first and the last ones. A component domain at a simplex $v_j$ of $g$ is either a component of $X \setminus v_j$ or an open annulus which is a tubular neighbourhood of a component of $v_j$. For a simplex $v_j$ of $g$, we define its predecessor $\mathrm{pred}(v_j)$ to be $v_{j-1}$ if $j \neq 1$, and $I(g)$ if $j = 1$. Similarly we define the successor $\mathrm{succ}(v_j)$. We denote $\mathrm{pred}(v_j)|Y$ by $I(Y, g)$ and $\mathrm{succ}(v_j)|Y$ by $T(Y, g)$. For a component domain $Y$ of a simplex $v$ of $g$, if $T(Y, g) \neq \emptyset$, then we write $Y \overset{d}{\searrow} g$. Similarly we write $g \overset{d}{\swarrow} Y$

if $I(Y,g) \neq \emptyset$. We also write $Y \overset{d}{\searrow} (g,v)$ or $(g,v) \overset{d}{\swarrow} Y$ if we need to specify whose component domain $Y$ is. There is a case when $Y|I(g)$ or $Y|T(g)$ is an element of $\mathcal{EL}(D(g))$. This corresponds to the case when $Y$ supports a "solitary geodesic ray"in the definition of bug-infested hierarchy below. In the expression $Y \overset{d}{\searrow} g$ or $g \overset{d}{\swarrow} Y$, we are assuming implicitly that we think of $Y$ as a component domain at a simplex of $g$.

Since the same domain may appear as supports of distinct geodesics in bug-infested hierarchies which will be defined below, when we say that two domains are the same, we need to explicate whether we are just regarding them as subsurfaces on $S$ or as component domains of simplices. Therefore, we say that two domains are *isotopic* when we regard them only as subsurfaces of $S$. We regard two isotopic domains as being the same *only when they are component domains of simplices belonging to the same geodesic.* In spite of this distinction, as a convention, for a domain $X$ and a simplex $v$ of a geodesic whose support contains a surface $X'$ isotopic to $X$, we use the symbol $v|X$ to denote $v|X'$.

**Definition 2.1** (Bug-infested hierarchy). A *bug-infested hierarchy* $H_\infty$ on $S$ is a system of possibly infinitely many finite tight geodesics (segments or rays or lines) on domains of $S$ related by $\overset{d}{\swarrow}$ and $\overset{d}{\searrow}$, which has the following properties. We should remark that the same geodesic can appear more than once in the system, and that a bug-infested hierarchy is not just a collection of geodesics but geodesics together with the relation of subordination. We distinguish two appearances of the same geodesic, and regard them as different geodesics. Also, if we have relations $a_1 \overset{d}{\searrow} a_2 \overset{d}{\searrow} \dots \overset{d}{\searrow} a_n$ in the following, we shall write $a_1 \searrow a_n$. We use similarly the symbol $\swarrow$.

(1) There are two generalised markings $I(H_\infty)$ and $T(H_\infty)$ which are unmeasured laminations on $S$ with (possibly empty) transversals on closed leaves, and $H_\infty$ contains only one tight geodesic supported on $S$ itself, which is called the *main geodesic*, which may be a geodesic segment or a ray or a line. The main geodesic connects a vertex in base($I(H_\infty)$) to one in base($T(H_\infty)$) if it is a segment. When it is a ray or line, its open end tends to base($I(H_\infty)$ or base($T(H_\infty)$) in the Gromov compactification $\mathcal{C}(S) \cup \mathcal{EL}(S)$.
(2) Each geodesic $g$ in $H_\infty$ is either a tight geodesic segment in $\mathcal{C}(D(g))$ connecting a vertex in base($I(g)$) to a vertex in base($T(g)$)

or a tight geodesic ray in $\mathcal{C}(D(g))$ starting from a vertex of base$(I(g))$ or base$(T(g))$, where $D(g)$ is a domain in $S$, called the *support* of $g$, where $I(g)$ and $T(g)$ are markings on $D(g)$. We assume that $g_{H_\infty}$ contains at least two simplices.

(3) For any geodesic segment $b \in H_\infty$ that is not the main geodesic, there are (finite or infinite) geodesics $a, a' \in H_\infty$ and simplices $v \in a, v' \in a'$ such that $(a, v) \overset{d}{\swarrow} b$ and $b \overset{d}{\searrow} (a', v')$. For these, $D(b)$ is isotopic to component domains $X$ at $v$ of $a$ and $X'$ at $v'$ of $a'$ with $(a, v) \overset{d}{\swarrow} X$ and $X' \overset{d}{\searrow} (a', v')$, and $I(b) = I(X, a)$ and $T(b) = T(X', a')$. (It is possible that $(a, v)$ and $(a', v')$, hence $X$ and $X'$ coincide.) We say that $b$ is supported on $X$ and $X'$.

(4) For any geodesic ray $b \in H_\infty$ that is not the main geodesic, there are $a \in H_\infty$ and a simplex $v$ on $a$ with either $(a, v) \overset{d}{\swarrow} b$ or $b \overset{d}{\searrow} (a, v)$. (Only one of the two can hold.) In either case, $D(b)$ is a component domain $X$ at $v$ of $a$ with $a \overset{d}{\swarrow} X$ or $X \overset{d}{\searrow} a$ holds depending on whether $a \overset{d}{\swarrow} b$ or $b \overset{d}{\searrow} a$, and $I(b) = I(X, a)$ in the former case and $T(b) = T(X, a)$ in the latter case. The ray $b$ is called an *upward ray* in the former case, and a *downward ray* in the latter case.

(5) Suppose that $X$ is a component domain at a simplex $v$ of some geodesic $a \in H_\infty$, and that $I(X, a) \neq \emptyset$ and $T(X, a) \neq \emptyset$. Then one and only one of the following holds.
   (a) There is a unique finite tight geodesic $b \in H_\infty$ supported on $X$ with $(a, v) \overset{d}{\swarrow} b \overset{d}{\searrow} (a, v)$.
   (b) There is a unique pair, called a *matching pair*, of geodesic rays $b^-$ and $b^+$, called the *lower ray* and the *upper ray* of the matching pair, both of which are supported on $X$ and satisfy $(a, v) \overset{d}{\swarrow} b^-$ and $b^+ \overset{d}{\searrow} (a, v)$. (Note that a lower ray is upward and an upper ray is downward.) For any simplices $s^- \in b^-$ and $s^+ \in b^+$, we have $d_X(s^-, s^+) \geq 3$, and also $d_X(I(b^-), T(b^+)) \geq 3$.
   (c) The domain $X$ is an annulus and there is a matching pair of geodesic rays $c^-, c^+$ supported on some component domain $Y$ of $v$ such that $(a, v) \swarrow c^-, c^+ \searrow (a, v)$ and $X \cap \bar{Y} \neq \emptyset$ as domains on $S$. In this case, $X$ supports no geodesics.

Let $X$ and $X'$ be distinct component domains at $v$ of $g$ and at $v'$ of $g'$ for $g, g' \in H_\infty$. Then, we say that $X$ is parallel up to $X'$, and $X'$ down to $X$ when the following hold.

(i) $X$ and $X'$ are isotopic, i.e., they are the same as domains on $S$.
(ii) $T(X, g) = \emptyset$ and $I(X', g') = \emptyset$.
(iii) In the case when $X$ is an annulus, there is no matching pair $h^-, h^+$ with $\bar{D}(h^\pm)$ containing a core curve of $X$ such that either $(g, v) \swarrow h^-$ or $h^+ \searrow (g', v')$.

When we need not to specify the positions of $X$ and $X'$, we simply say that $X$ and $X'$ are parallel. We also call the geodesic $h$ above the geodesic *realising the parallelism* between $X$ and $X'$.

(6) Suppose that $X$ is a component domain with $\xi(X) \neq 3$ at the last vertex $v$ of $a$ such that $I(X, a) \neq \emptyset$, that $X'$ is a component domain at the first vertex $v'$ of a geodesic $a' \in H_\infty$ such that $T(X', a') \neq \emptyset$, and that $X$ and $X'$ are parallel. Then, one and only one of the following holds.
  (a) There is a unique finite tight geodesic $b \in H_\infty$ supported on $X$ and $X'$ with $(a, v) \overset{d}{\swarrow} b \overset{d}{\searrow} (a', v')$.
  (b) There is a unique pair of geodesic rays $b^-$ and $b^+$, called the lower ray and the upper ray of a matching pair also in this case, such that $b^-$ is supported on $X$ and $(a, v) \overset{d}{\swarrow} b^-$ and $b^+$ is supported on $X'$ and $b^+ \overset{d}{\searrow} (a', v')$. As in (5), for any simplices $s^- \in b^-$ and $s^+ \in b^+$, we have $d_X(s^-, s^+) \geq 3$, and also $d_X(I(b^-), T(b^+)) \geq 3$.

(7) Suppose that there is either a finite geodesic or a matching pair of geodesic rays is supported on $X$ and $X'$ which are component domains of distinct simplices $v$ at $b$ and $v'$ at $f$. Then $X$ and $X'$ are parallel.

We say that a component domain $X$ is a *terminal domain* if there is a descending sequence $(g_n, w_n) \overset{d}{\swarrow} (g_{n-1}, w_{n-1}) \overset{d}{\swarrow} \cdots \overset{d}{\swarrow} (g_1, w_1) \overset{d}{\swarrow} X$ in $H_\infty$ such that $w_j$ is the last vertex of $g_j$ for $j = 1, \ldots, n$, the geodesic $g_n$ is the main geodesic, and $T(X, g_1) = \emptyset$. Similarly, we say that $X$ is an *initial domain* if there is a descending sequence $X \overset{d}{\searrow} (g_1, w_1) \overset{d}{\searrow} \cdots \overset{d}{\searrow} (g_n, w_n)$ in $H_\infty$ such that $w_j$ is the first vertex of $g_j$ for $j = 1, \ldots, n$, the geodesic $g_n$ is the main geodesic, and $I(X, g_1) = \emptyset$.

(8) Suppose that $b^-$ is a lower ray supported on a component domain $X$ at a simplex $v$ of $a$, which is not a terminal domain, and that $a \overset{d}{\swarrow} b^-$. Then there exists a unique downward ray

$b^+$ such that $b^-$ and $b^+$ constitute a matching pair. Similarly, suppose that $b^+$ is a downward ray supported on a component domain $X'$ at a simplex $v'$ of $a'$, which is not an initial domain, and that $b^+ \overset{d}{\searrow} a$. Then there exists a unique upward ray $b^-$ such that $b^-$ and $b^+$ constitute a matching pair.

(9) A geodesic ray $g$ which is neither the main geodesic nor a part of a matching pair, which we call *solitary*, must be either a downward one satisfying $g \searrow (g_{H_\infty}, v)$ for the first vertex $v$ of $g_{H_\infty}$ and $I(g) \in \mathcal{EL}(D(g))$ or an upward one satisfying $(g_{H_\infty}, w)$ for the last vertex $w$ of $g_{H_\infty}$ and $T(g) \in \mathcal{EL}(D(g))$. A component domain $Y$ of the last (resp. the first) vertex of $g_{H_\infty}$ supports a solitary ray $g$ if and only if $T(H_\infty)|Y$ (resp. $I(H_\infty)|Y$) is contained in $\mathcal{EL}(Y)$. In this case, we have $T(g) = T(H_\infty)|Y$ (resp. $I(g) = I(H_\infty)|Y$).

## 3. Slices in a bug-infested hierarchy

To define the geometric convergence of hierarchies to an bug-infested hierarchy, we need to fix some slice in the main geodesic for each hierarchy as a basepoint and allow each domain to be twisted when viewed from the base point.

We first recall the definition of (saturated) slice from Minsky [8], which can be used also for a bug-infested hierarchy. In this paper, when we refer to a slice, we always assume that it is *saturated* in the sense of [7].

**Definition 3.1** (Slice). A *slice* $\sigma$ of a bug-infested hierarchy $H_\infty$ is a non-empty set of pairs $(g, v)$ for a geodesic $g$ in $H_\infty$ and a simplex $v$ on $g$ with the following properties.

(1) A geodesic appears in pairs of $\sigma$ at most once.
(2) If $(g, v)$ is a pair contained in $\sigma$ and $g$ is not the main geodesic of $H_\infty$, then there is $(g', v') \in \sigma$ such that $D(g)$ is parallel to a component domain of $v'$.
(3) If $(g, v)$ is contained in $\sigma$, for any component domain $D$ of $v$ supporting a geodesic in $H_\infty$, there is $(g', v') \in \sigma$ such that $D(g')$ is parallel to $D$.

For a simplex $v \in g_{H_\infty}$, we define the *bottom slice* at $v$ to be a slice $\sigma$ containing $(g_H, v)$ such that for any $(g, w) \in \sigma$, the geodesic $g$ is either a segment or an upward ray and $w$ is the first vertex of $g$.

Note that by the definition of bug-infested hierarchies, if $v$ is not the first vertex of $g_{H_\infty}$, the bottom slice at $v$ always exists. From now on, when we consider the bottom slice at some simplex of $g_{H_\infty}$, we assume the simplex is not the first vertex. Since we assumed in Definition 2.1 that $g_{H_\infty}$ contains at least two simplices, $g_{H_\infty}$ always contains a simplex which has bottom slice.

We shall define the twisting operation below which is necessary for the definition of the geometric convergence of hierarchies. For specifying where we should twist, we need to define some terms concerning the position of a component domain in a bug-infested hierarchy first.

**Definition 3.2.** Let $H$ be a bug-infested hierarchy and $\sigma$ a slice of $H$. A geodesic $g$ in $H$, its domain $D(g)$ and its simplices are said to be situated *before* $\sigma$ when one of the following two conditions holds.

(1) There are a geodesic $g'$ such that $g \tilde{\searrow} (g', w)$ and a simplex $v$ on $g'$ with $w < v$ and $(g', v) \in \sigma$.
(2) There is a downward ray $h$ such that $g \tilde{\searrow} (h, w)$ such that the upward ray $h'$, which is the other half of $h$, has a simplex $w'$ with $(h', w') \in \sigma$.

Similarly we define $g$, $D(g)$ and the simplices on $g$ to be situated *after* $\sigma$ when one of the following two conditions holds.

(1) There are a geodesic $g'$ such that $(g', w) \tilde{\swarrow} g$ and a simplex $v$ on $g'$ with $w > v$ and $(g', v) \in \sigma$.
(2) There is a lower ray ray $h$ of a matching pair such that $(h, w) \tilde{\swarrow} g$ such that the upper ray $h'$, which is the other half of $h$, has a simplex $w'$ with $(h', w') \in \sigma$.

We say that $g$ is *hanging* on $\sigma$ if it is situated neither before nor after $\sigma$.

**Definition 3.3.** We say that a simplex $v$ on a geodesic $g$ is situated before a slice $\sigma$ when either $g$ is situated before $\sigma$ or $g$ is hanging on $\sigma$ and contains $(g, p)$ with $v < p$. Similarly, we define the condition that $v$ is situated after $\sigma$ by reversing the directions of the subordinacy and the order.

3.1. **$\delta$-distance, descending paths and twisting systems.** We next define the distance between a slice of $H$ and a simplex $w$ contained in some geodesic in $H$.

**Definition 3.4** ($\delta$-distance). Let $H$ be a bug-infested hierarchy, and $w$ a simplex on a geodesic $g \in H$. We first consider the case when $g$ is situated after $\sigma$. We consider a sequence of simplices $w_0 = w, w_1, \dots, w_k$ such that

(1) $w_j$ is a vertex in $g_j \in H$ for each $j = 0, \dots k$, and for $j < k$ there is no simplex $u_j$ with $(g_j, u_j) \in \sigma$,
(2) there is a simplex $u_k$ such that $(g_k, u_k) \in \sigma$, and

(3) for each $j = 0, \dots, k-1$, we have $(g_{j+1}, w_{j+1}) \overset{d}{\swarrow} g'_j \simeq g_j$.

Let $v_0^j$ be the first vertex of $g'_j$ and $v_\infty^j$ the last vertex of $g_j$. Then we define $\delta_j$ to be $\min\{d_{D(g'_j)}(w_j, v_0^j), d_{D(g_j)}(w_j, v_\infty^j)\} + \epsilon_j + 1$, where the distance between simplices belonging to different rays in the matching pair is set to be $\infty$ and $\epsilon_i$ is set to be 0 if the vertex, $v_0^j$ or $v_\infty^j$ realising the minimum is equal to $I(g_j)$ or $T(g_j)$, and to be 1 otherwise. We say that $w_j$ lies in the first half if the minimum is attained by $d_{D(g'_j)}(w_j, v_0^j)$ including the case when the two terms in min are equal, and in the second half otherwise. We need to introduce another term $\eta_j$ which is defined to be 0 when both $w_j$ and $w_{j+1}$ lie in the same half, the first or the second, and to be 1 if they lie in different halves. Let $v$ be the simplex of $g_H$ such that $(g_H, v) \in \sigma$. Then by summing up distances, we let $\delta(w, \sigma) = \sum_{j=0}^{k-1} \delta_j + \eta_j + d_S(w_k, u_k)$.

Similarly, we define $\delta$ in the case when $g$ is situated before $\sigma$ as follows. We consider a sequence of vertices $w_0 = w, w_1, \dots, w_k$ such that

(4) $w_j$ is a vertex in $g_j \in h$, and for $j < k$ there is no simplex $u_j$ with $(g_j, u_j) \in \sigma$,

(5) there is $u_k$ such that $(g_k, u_k) \in \sigma$, and

(6) for each $j = 0, \dots, k-1$, we have $g_j \simeq g'_j \overset{d}{\searrow} (g_{j+1}, w_{j+1})$.

Then we let $\delta_j = \min\{d_{D(g_j)}(w_j, v_0^j), d_{D(g'_j)}(w_j, v_\infty^j)\} + 1 + \epsilon_j$, where $v_0^j$ is the first vertex and $v_\infty^j$ is the last vertex of $g'_j$ and $\epsilon_j$ is defined in the same way as above. Having defined $\delta_j$, the rest of the definition is the same as the previous case.

Finally, suppose that $g$ is hanging on $\sigma$. Then there is $v$ on $g$ such that $(g, v) \in \sigma$ by Lemma ??. We set $\delta(w, \sigma)$ to be $d_{D(g)}(v, w)$.

This definition of the distance may appear asymmetric. We adopt this definition because of its convenience in defining twisting systems below and then the geometric convergence of hierarchies. It will turn out later in §?? that in fact this $\delta$-distance is within uniformly bounded error from a symmetric distance which should be used geometric convergence for model manifolds.

Let $Y$ be a component domain of a simplex in a hierarchy $h$ (an ordinary one, not bug-infested), which supports a geodesic $\gamma$, and $v$ a simplex of the main geodesic. We associate each $Y$ with a homeomorphism $f_Y : S \longrightarrow S$ fixing $S \setminus Y$ and call $\{f_Y\}$ a *twisting system* and each of its elements a *twisting map*. Twisting maps are necessary to control geometric convergence for geodesics in hierarchies whose lengths go to infinity. For geodesics other than these, the twisting maps will be defined to be the identity.

Suppose that $w$ is a simplex on a geodesic $g$ in a hierarchy $h$, and that $\sigma$ is a slice. We have a simplex $v'$ with $(g_m, v') \in \sigma$ and a descending sequence $g = g_0 \overset{d}{\searrow} (g_1, w_1) \overset{d}{\searrow} (g_2, w_2) \overset{d}{\searrow} \cdots \overset{d}{\searrow} (g_m, w_m)$ with $w_m < v'$ or $(g_m, w_m) \overset{d}{\swarrow} \cdots \overset{d}{\swarrow} (g_1, w_1) \overset{d}{\swarrow} g_0 = g$ with $v' < w_m$ as in Definition 3.4, where we regard $m$ as 0 if $g_m$ is hanging on $\sigma_v$. Since we are considering an ordinary hierarchy, we do not need to consider the relation $\simeq$ in this sequence. Note that this sequence is uniquely determined. We call this sequence *the descending sequence* from $w$ to $\sigma$. Let $v_0^j$ be the first vertex of $g_j$ and $v_{N_j}^j$ the last one, where $N_j$ is the length of $g_j$. For $j = 0, \ldots m-1$, we connect $w_j$ to one of $v_0^j$ and $v_{N_j}^j$ which is nearer a geodesic path, regarding $w$ as $w_0$ by a sub-geodesic $c_i$ on $g_i$. If the distances from $w_j$ to $v_0^j$ and $v_{N_j}^j$ are the same, then we connect $w_j$ to $v_0^j$ in the case when the descending sequence is forward and to $v_{N_j}^j$ otherwise. We define $c_m$ to be the geodesic path on $g_m$ connecting $w_m$ to $v'$. Denote the simplices of the path $c_j$ by $w_j = x_1^j, \ldots, x_{k_j}^j$. We concatenate these paths $c_0, \ldots, c_m$ first. Then in the case when $w_j$ lies in the first half, we interpose $I(g_j)$ just after $x_{k_j}^j$ if $\epsilon_j = 1$ and $T(g_j)$ before $x_1^{j+1}$ if $\eta_j = 1$. (Refer to Definition 3.4 for the definition of $\epsilon_j$ and $\eta_j$.) In the case when $w_j$ lies in the second half, we interpose $T(g_j)$ after $x_{k_j}^j$ if $\epsilon_j = 1$ and $I(g_j)$ if $\eta_j = 1$. We call a sequence of positions thus obtained a *descending path* from $w$ to $\sigma$ and the paths $c_0, \ldots, c_m$ its *constituent paths.* A constituent path is called *forward* or *backward* depending on it lies on a forward geodesic or a backward geodesic. We should note that the length of a descending path from $w$ to $\sigma$, which is the number of positions contained in the path minus 1, is equal to $\delta(w, \sigma)$ by our definition of descending path.

**Definition 3.5.** We say that the descending path from $w$ to $\sigma$ passes a component domain $T$ of $h$ if one of the following conditions is satisfied.

(a) $T$ is the support of some constituent path $g_j$ for which both $I(g_j)$ and $T(g_j)$ are in the descending path.
(b) There is subordinacy $(g_j, y_1) \swarrow T \searrow (g_j, y_2)$ with both $y_1$ and $y_2$ being among the simplices $x_1^j, x_2^j, \ldots, x_{k_j}^j$ of a constituent path such that $y_1 \neq x_1^j$ if the path is backward and $y_2 \neq x_1^j$ if it is forward.

(c) There is subordinacy $(g_j, I(g_j)) \swarrow T \searrow (g_j, x^j_{k_j})$ for the case when $x^j_{k_j}$ is the first vertex of $g_j$ or $(g_j, x^j_{k_j}) \swarrow T \searrow (g_j, T(g_j))$ for the case when $x^j_{k_j}$ is the last vertex of $g_j$.

(d) There is subordinacy $(g_{j_1}, y_1) \swarrow T \searrow (g_{j_2}, y_2)$ such that one of $y_1$ and $y_2$ is the first or last vertex of a constituent path and the other appears in another constituent path coming after that.

(e) In the case when $w$ is situated before $\sigma$, there is subordinacy $(g_{j_1}, y_1) \swarrow T \searrow (g_{j_2}, y_2)$ such that $y_1$ appears in a constituent path and $(g_{j_2}, z_2) \in \sigma$ for some $z_2 \leq y_2$. In the case when $w$ is situated before $\sigma$, there is the same subordinacy such that $y_2$ appears in a constituent path and $(g_{j_1}, z_1) \in \sigma$ for some $z_1 \geq y_1$.

Evidently, a component domain can satisfy at most one of the above four conditions.

We say that $T$ is outermost when in addition, the following hold.

(1) $f_T$ is not the identity.

(2) The domain $T$ is not subordinate to another domain $T'$ which the descending path passes with $f_{T'} \neq id$.

We say that $T$ is *subordinate* to the constituent path on $g_j$ in the cases (b) and (c), and to one of the two constituent paths related to $T$ which appears later in the descending path, the one either on $g_{j_1}$ or $g_{j_2}$ depending on which appears later in the case (d). In the case (e), we say that $T$ is forward subordinate to $\sigma$ and backward subordinate to the constituent path containing $y_1$ if $(g_{j_2}, y_2) \in \sigma$, and backward subordinate to $\sigma$ and forward subordinate to the constituent path containing $y_2$ if $(g_{j_1}, y_1) \in \sigma$.

We call the constituent path on which $y_1$ as above lies the *front leg* of $T$ and the one on which $y_2$ lies the *back leg* in the cases (b), (c) and (d). In the case (a), we define $I(g_j)$ to be the back leg and $T(g_j)$ the front leg.

Now we shall give an order among the domains that the descending path passes. Let $T_1, T_2$ be outermost domains which the descending path from $w$ to $\sigma$ passes, and suppose that they support geodesics $g_{T_1}, g_{T_2}$ respectively in $h$. We state the definition of the ordering under the assumption that $w$ is situated after $\sigma$, and explain how to modify it in the case when $w$ is situated before $\sigma$ after that.

**Definition 3.6** (Path-ordering). Let $\gamma$ be a descending path from $w$, which is situated after $\sigma$, to $\sigma$. We define $T_1$ to be greater than $T_2$ in the path ordering on $\gamma$ and write $T_1 \succ_\gamma T_2$ when the following hold.

(1) Both $T_1$ and $T_2$ are subordinate to the same forward constituent path and there is a time-ordering $T_1 \prec_t T_2$.
(2) Both $T_1$ and $T_2$ are subordinate to the same backward constituent path and there is a time-ordering $T_1 \succ_t T_2$.
(3) $T_1$ is subordinate to a constituent path appearing earlier (i.e. in a geodesic with a smaller subscript) in the descending path than the one to which $T_2$ is subordinate.
(4) Both $T_1$ and $T_2$ are subordinate to the same forward constituent path, and one of the following two conditions holds:
  (a) $T_1 \subset T_2$ and $T_1$ is forward subordinate to $(g_{T_2}, z_2)$ for the first vertex $z_2$ of $g_{T_2}$, whereas $T_1$ is not backward subordinate to $g_{T_2}$.
  (b) $T_1 \supset T_2$ and $T_2$ is backward subordinate to $(g_{T_1}, z_1)$ for the last vertex $z_1$ of $g_{T_1}$, whereas $T_2$ is not forward subordinate to $g_{T_1}$.
(5) Both $T_1$ and $T_2$ are subordinate to the same backward constituent path and one of the following two conditions holds:
  (a) $T_2 \subset T_1$ and $T_2$ is forward subordinate to $(g_{T_1}, z_1)$ for the first vertex $z_1$ of $g_{T_1}$, whereas $T_2$ is not backward subordinate to $g_{T_1}$.
  (b) $T_2 \supset T_1$ and $T_1$ is backward subordinate to $(g_{T_2}, z_2)$ for the last vertex $z_2$ of $g_{T_2}$, whereas $T_1$ is not forward subordinate to $g_{T_2}$.
(6) $T_2$ is backward subordinate to $\sigma$ and one of the following holds:
  (a) $T_2$ is forward subordinate to $(g_{T_1}, z_1)$ for the first vertex $z_1$.
  (b) $T_2 \prec_t T_1$.
(7) The front and back legs of $T_1$ are $I(g_j)$ and $T(g_j)$, and $T_2$ is subordinate to a constituent path appearing later than $I(g_j)$ and $T(g_j)$ or the front and back legs of $T_2$ are initial and terminal markings appearing later than $I(g_j)$ and $T(g_j)$.
(8) The front and back legs of $T_2$ are $I(g_j)$ and $T(g_j)$, and $T_1$ is subordinate to a constituent path appearing earlier than $I(g_j)$ and $T(g_j)$ or the front and back legs of $T_1$ are initial and terminal markings appearing earlier than $I(g_j)$ and $T(g_j)$.

In the case when $w$ is situated before $\sigma$, we need to change the condition (6) as follows.

(6)′ $T_2$ is forward subordinate to $\sigma$ and one of the following holds.
  (a) $T_2$ is backward subordinate to $(g_{T_1}, z_1)$ for the first vertex of $g_{T_1}$.
  (b) $T_1 \prec_t T_2$.

Now we return to our hierarchy $h$ with twisting system $\{f_Y\}$ for component domains $Y$ of $h$. Let $v$ be a base simplex on the main geodesic $g_h$, and $Y$ a component domain of a simplex $w$ which supports a geodesic $g \in h$. Recall that there is a bottom slice $\sigma_v$ for $v$. Consider all the outermost domains $S_1, \dots, S_{l_j}$ that the descending path, denoted by $\gamma$, from $w$ to $\sigma_v$ passes, and their twisting maps $f_{S_1}, \dots, f_{S_{l_j}}$ arrayed according to the path-ordering of the domains so that a smaller one comes before a greater one: if $S_k \prec_\gamma S_l$, then $k < l$. When two domains have no relation in path-ordering, by Lemma ??, their twisting maps commute each other and the order does not matter. We define the initial twisting of $Y$ as the composed map $f_{\text{init}}(g) = (f_S \circ f_{S_1}^{\epsilon_1} \circ \cdots \circ f_{S_l}^{\epsilon_l})|Y$, where $\epsilon_i$ is defined to be 1 when either $S_i$ is subordinate to a backward constituent path or is subordinate to no constituent path and supports a backward constituent path, and $-1$ otherwise. On the other hand, the terminal twisting $f_{\text{term}}(g)$ is defined to be $f_{\text{init}} \circ f_Y(g)$. We also use the symbols $f_{\text{init}}(Y)$ and $f_{\text{term}}(Y)$ putting the domain $Y$ of $g$ into the parenthesis instead of $g$. When we use these notations, we need not assume that $Y$ supports a geodesic. Since our choices of the descending path is unique, the initial and the terminal twistings are uniquely determined.

The initial and the terminal twistings of the main geodesic should be defined separately. Let $v_\sigma$ be the simplex of $g_h$ such that $(g_h, v_\sigma) \in \sigma$. We consider all the domains $T$ such that either we have $(g_h, w_1) \swarrow T \searrow (g_h, w_2)$ with $v_\sigma < w_1$ or $T$ is backward subordinate to $\sigma$, and denote the set of such domains by $\mathcal{T}_+$. Similarly, we define $\mathcal{T}_-$ to be the set of domains $T$ that either satisfy $(g_h, w_1) \swarrow T \searrow (g_h, w_2)$ with $v_\sigma > w_2$ or are forward subordinate to $\sigma$. Compose all the twistings $f_T$ with $T \in \mathcal{T}_+$ so that a smaller one in the order $\prec_\gamma$ comes before a greater one, letting $\gamma$ be the sub-geodesic of $g_h$ consisting of simplices coming after $v_\sigma$ with the orientation reversed, and postcompose $f_S$, then we get the terminal twisting for the main geodesic. Similarly, we get the initial twisting of the main geodesic by composing all the twistings $f_T^{-1}$ with $T \in \mathcal{T}_-$ in the order of $\prec_\gamma$ and then postcomposing $f_S$.

Next we define the *internal twisting* of a geodesic. Let $g$ be a geodesic in a hierarchy $h$ supported on $T$, and $v_0^g, \dots, v_{n_g}^g$ its simplices, where $n_g$ is the length of $g$. For each $v_j^g$, we can consider the path $v_j^g, v_{j-1}^g, \dots, v_0^g$ and regard it as a descending path. This is in fact a descending path from $v_j^g$ to a slice containing $(g, v_0^g)$ such that its restriction to $D(g)$ is the bottom slice at $v_0^g$. As before, we array all the outermost domains $Y_1, \dots, Y_p$ that the descending path passes in such a way that if $Y_s \prec_{\gamma_j} Y_t$, then $s < t$. (By Lemma ?? the order between domains without time-ordering does not matter.) We consider the composition $f_{\text{init}}(g) \circ f_{Y_1} \circ \cdots \circ f_{Y_p}$ and let it be $f(g)_j$. We define $f(g)_j$ to be the identity if there are no domains which the descending path passes. Now, we define the internal twisted geodesic $g^f$ to be a geodesic (segment or ray) whose $j$-th simplex is $f(g)_j(v_j^g)$. It is easy to check that this is really a geodesic, by observing the domains of the twisting maps composed to $f(g)_j$ to get $f(g)_{j+1}$ are all disjoint from $v_{j+1}$.

To deal with the case when the geodesics with basepoint at the terminal vertex converge to a geodesic ray, we need to introduce the internal twisting upside down. Let $\sigma_g^{ud}$ be a slice containing $(g, v_{n_g}^g)$ such that for every $(k, w) \in \sigma_g^{ud}$ with $D(k) \subset D(g)$, we have $k \searrow g$ and the simplex $w$ is the last vertex. For each $j$, we consider the descending path from $v_{n_g-j}^g$ to $\sigma_g^{ud}$. We define $\bar{f}(g)_j$ to be $f_{\text{term}}(g) \circ (f_{Y_1})^{-1} \circ \cdots \circ (f_{Y_p})^{-1}$, where $Y_1, \ldots Y_p$ are all the domains that the descending path passes, arrayed in such a way that if $Y_s \prec_{\bar{\gamma}_j} Y_t$, then $s < t$. Then we define the reversed internally twisted geodesic $\bar{g}^f$ to be a geodesic ray whose $j$-th simplex is $\bar{f}(g)_j(v_{n_g-j}^g)$.

To simplify symbols, *for a simplex $v$ of a geodesic $g$, we denote by $f(v)$ the image of $v$ under the internal twisting and by $\bar{f}(v)$ that under the internal twisting upside down.*

**Definition 3.7** (Geometric convergence of hierarchies). We say that hierarchies $H_i$ with base simplices $v_i$ at the main geodesics converge geometrically to a bug-infested hierarchy $H_\infty$ with a vertex $v_\infty \in g_{H_\infty}$ when there is a twisting system $\{f_Y^i\}$ for $H_i$ with the following properties.

First of all, we have $f_S^i(v_i) = v_\infty$ except for finitely many $i$.

Let $\sigma_\infty$ and $\sigma_i$ be the bottom slices in $H_\infty$ and $H_i$ containing $v_\infty$ and $v_i$ respectively. For any $K \in \mathbb{N}$, let $U_\infty(K)$ be the set of simplices of $H_\infty$ which either lie on geodesic segments whose first vertices are within the distance $K$ from the bottom slice $\sigma_\infty$ containing $v_\infty$ or are situated within the $\delta$-distance $K$ from $\sigma_0$ and lie on geodesic rays. Then there exist $i_0$ and a set $U_i(K)$ for $i \geq i_0$ including the $K$-neighbourhoods in $H_i$ of the bottom slice containing $v_i$ such that the following hold for any $i \geq i_0$.

(1) For any geodesic $\gamma$ in $H_i$ such that all the vertices of $\gamma$ are in $U_i(K)$, the twisting $f_{D(\gamma)}^i$ is the identity.
(2) Otherwise $\gamma \cap U_i(K)$ consists of two geodesic segments $\gamma_{\text{init}}$ and $\gamma_{\text{term}}$ such that the first vertex is lies in $\gamma_{\text{init}}$ and the last vertex lies in $\gamma_{\text{term}}$.

(3) For any geodesic $\gamma_i \in H_i$ with $\xi(D(\gamma)) \geq 4$ or $\xi(D(\gamma)) = 2$ and $w_{H_i}(D(\gamma)) < K$ such that $\gamma_i \cap U_i(K)$ is not empty, $f^i_{\text{init}}(\gamma)$ takes $D(\gamma)$ to a component domain $D'$ of $H_\infty$ supporting a geodesic segment or two geodesic rays such that if $(g, v) \overset{d}{\swarrow} D(\gamma)$ for the $j$-th simplex $v$, then $(g^{f^i}, f^i(g)_j(v)) \overset{d}{\swarrow} D'$ or $(\bar{g}^{f^i}, \bar{f}^i(g)_{|g|-j}(v)) \overset{d}{\swarrow} D'$, and if $D(\gamma) \overset{d}{\searrow} (g, v)$, then $D' \overset{d}{\searrow} (g^{f^i}, f^i(g)_j(v))$ or $D' \overset{d}{\searrow} (\bar{g}^{f^i}, \bar{f}^i(g)_{|g|-j}(v))$. In this situation, we denote $D' = \mathbf{f}^i(D(\gamma_i))$.

(4) If $U_i(K) \supset \gamma_i$, then $D' = \mathbf{f}^i(D(\gamma_i))$ supports a geodesic segment equal to $\gamma_i^{f^i}$.

(5) Suppose that $U_i(K) \cap \gamma_i$ consists of two geodesic segments $\gamma(i)_{\text{init}}$ and $\gamma(i)_{\text{term}}$ and that either $\xi(D(\gamma_i)) \geq 4$ or $\xi(D(\gamma_i)) = 2$ and $w_{H_i}(D(\gamma_i)) < K$. We define $\gamma(i)^{f^i}_{\text{init}}$ to be a geodesic subsegment in the internally twisted geodesic $\gamma(i)^{f^i}$ corresponding to the simplices contained in $\gamma(i)_{\text{init}}$. Similarly we define $\gamma(i)^{f_i}_{\text{term}}$ to be a subsegment in $\bar{\gamma}(i)^{f^i}$. Then $\gamma(i)^{f^i}_{\text{init}}$ is contained in the lower ray supported on $D'$ and $\gamma(i)^{f^i}_{\text{term}}$ in the upper ray supported on on $D'$, where $\bar{\gamma}$ denotes the geodesic obtained by reversing the direction of $\gamma$, both as subsegments beginning from the starting points of the rays.

(6) Let $w$ be a simplex of a geodesic $\gamma \in H_\infty$ contained in $U_\infty(K)$. Then for every large $i$, there exist a geodesic $g_i \in h_i$ and a simplex $w_i \in g_i$ such that $D(\gamma) = \mathbf{f}^i(D(g_i))$, and $f^i(w_i) = w$ if $\gamma$ is either a finite geodesic or a upward ray and $\bar{f}^i(w_i) = w$ if $g$ is a downward ray. Also, the descending sequence from $w_i$ to $\sigma_i$ corresponds to that of $w$ to $\sigma_\infty$ and at each direct subordination in the descending sequence of $w_i$, the simplex to which the previous term is subordinate is located at the same position of a geodesic (counting from the first vertex or the last vertex) as that of the corresponding simplex in the descending sequence of $w$.

In (3)-(5) above, we also say that the geodesics $\gamma_i$ in $H_i$ *correspond under twistings* to a geodesic or a matching pair of geodesics in $H_\infty$ to which they are mapped by the twistings in (4) or into which they are embedded in (5).

The main result of this note is the following:

**Theorem 3.8.** *Let $\{(H_i, v_i)\}$ be a sequence of internally complete hierarchies on $S$ with base simplices such that each of their main geodesics*

*$g_{H_\infty}$ has more than one simplices. Then there is a subsequence of $\{(H_i, v_i)\}$ which converges geometrically to an internally complete bug-infested hierarchy $H_\infty$ with a base simplex $v_\infty$.*

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