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The side parameter for punctured torus groups

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1 Introduction

Let T be the once-punctured torus. The Teichmüller space, $\mathcal{T}$, of T with Teichmüller distance d_T is canonically isometric to the hyperbolic plane $\mathbb{H}^2$. A *punctured torus group* is a Kleinian group which is freely generated by two elements with parabolic commutator. A *marked punctured torus group* is a faithful representation of $\pi_1(T)$ onto a punctured torus group. For any marked punctured torus group ρ, set $\Gamma_\rho = \rho(\pi_1(T))$. Then $M_\rho = \mathbb{H}^3/\Gamma_\rho$ is homeomorphic to $T \times \mathbb{R}$. Associated to each end (relative to the main cusp) of M_ρ, the *end invariant*, $\lambda^\pm(\rho)$, of ρ is defined to be the marked conformal structure at infinity if the end is geometrically finite and the projective measured lamination at infinity if the end is geometrically infinite (see Section 2). Let $\mathcal{P}$ be the space of conjugacy classes of marked punctured torus groups and set $\mathcal{E} = \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$. It can be proved that the end invariant map $\boldsymbol{\lambda} = (\lambda^-, \lambda^+) : \mathcal{P} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$ is well-defined. By Bers' simultaneous uniformization theorem, the restriction of $\boldsymbol{\lambda}$ to the subspace $\mathcal{QF} \subset \mathcal{P}$ of the quasifuchsian representations for T is a homeomorphism onto $\mathbb{H}^2 \times \mathbb{H}^2$. Minsky [9] proved that the end invariant map $\boldsymbol{\lambda} = (\lambda^-, \lambda^+) : \mathcal{P} \to \mathcal{E}$ is bijective, and the inverse of the map is continuous. Moreover, it was proved that $\mathcal{P}$ is equal to the closure of $\mathcal{QF}$. In contrast, Anderson and Canary [4] constructed such a sequence $\{\rho_n\}$ in $\mathcal{QF}$ which converges in $\mathcal{P}$ whose image $\{\boldsymbol{\lambda}(\rho_n)\}$ under the end invariant map diverges in $\mathcal{E}$.

In the famous unfinished manuscript [7], Jorgensen characterized the combinatorial structure of the Ford domain of Γ_ρ for $\rho \in \mathcal{QF}$, where he introduced a map $\boldsymbol{\nu} = (\nu^-, \nu^+) : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$, called the *side parameter*, by using the combinatorial structure. Though the complete proof is not written in his paper, it can be proved that the side parameter map is actually a homeomorphism (see [3] for a complete proof). The side parameter map

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can be extended to $\boldsymbol{\nu} : \mathcal{P} \to \mathcal{E}$ by applying Jorgensen's geometric continuity method to the strong limits of quasifuchsian representations.

The main result of this paper is the comparison of the two maps $\boldsymbol{\lambda} : \mathcal{P} \to \mathcal{E}$ and $\boldsymbol{\nu} : \mathcal{P} \to \mathcal{E}$ with the common domain and range.

Theorem 1.1. *The map $\boldsymbol{\nu} : \mathcal{P} \to \mathcal{E}$ is bijective. Moreover, the composition $\boldsymbol{\nu} \circ \boldsymbol{\lambda}^{-1} : \mathcal{E} \to \mathcal{E}$ is a homeomorphism.*

As an immediate consequence to Theorem 1.1, the Anderson-Canary sequence $\{\rho_n\}$ also has a divergent image $\{\boldsymbol{\nu}(\rho_n)\}$ in $\mathcal{E}$ under the side parameter map.

2 Punctured torus groups

Definition 2.1. Let $\rho_0 : \pi_1(T) \to PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$ be the holonomy representation of a complete hyperbolic structure on the punctured torus of finite area.

- A representation $\rho : \pi_1(T) \to PSL(2,\mathbb{C})$ is a *quasiconformal deformation* of ρ_0 if there is a quasiconformal homeomorphism $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\rho = w \circ \rho_0 \circ w^{-1}$, i.e., $\rho(g) = w \circ \rho_0(g) \circ w^{-1}$ for any $g \in \pi_1(T)$.
- The *quasifuchsian space* $\mathcal{QF}$ of the punctured torus is the space of conjugacy classes of quasiconformal deformations of ρ_0. We regard $\mathcal{QF}$ as a subspace of the space $\mathcal{X}$ of type-preserving $PSL(2,\mathbb{C})$-representations of $\pi_1(T)$.
- We denote the closure of $\mathcal{QF}$ in $\mathcal{X}$ by $\overline{\mathcal{QF}}$.

Definition 2.2.

- A *marked punctured torus group* is a faithful and discrete representation of $\pi_1(T)$ into $PSL(2,\mathbb{C})$ which sends the peripheral elements to parabolics.
- The space of marked punctured torus groups, denoted by $\mathcal{P}$, is the conjugacy classes of marked punctured torus groups. We regard $\mathcal{P}$ as a subspace of $\mathcal{X}$.

Proposition 2.3. *Let ρ be an arbitrary element of $\mathcal{P}$. Then the quotient manifold $\mathbb{H}^3/\operatorname{Im}\rho$ is homeomorphic to the product space $T \times (-1,1)$. The domain of discontinuity of the Kleinian group $\operatorname{Im}\rho$ is the disjoint union of two $(\operatorname{Im}\rho)$-invariant subsets $\Omega^\pm$ which correspond to the "ends" $e^- = T \times (-1,-1+\delta)$ and $e^+ = T \times (1-\delta,1)$ of $T \times (-1,1)$ respectively, and each Ω^ϵ ($\epsilon = \pm$) satisfies one of the following conditions.*

(i) Ω^ϵ *is homeomorphic to the open disk, and* $\Omega^\epsilon/\operatorname{Im}\rho$ *is homeomorphic to* T.

(ii) Ω^ϵ *is the countable union of open disks, and* $\Omega^\epsilon/\operatorname{Im}\rho$ *is homeomorphic to the thrice-punctured sphere.*

(iii) Ω^ϵ *is empty.*

Definition 2.4. The end satisfying one of the conditions (i) and (ii) of Proposition 2.3 is said to be *geometrically finite*, and one satisfying the condition (iii) is said to be *geometrically infinite.*

Definition 2.5. For every $\rho \in \mathcal{P}$, the *end invariant* $\lambda^\epsilon(\rho)$ of each end e^ϵ of $\mathbb{H}^3/\operatorname{Im}\rho$ is defined to be a point of the Thurston compactification, canonically identified with $\overline{\mathbb{H}^2}$, of the Teichmüller space of T as follows. Let Ω^ϵ be the subset of the domain of discontinuity of $\operatorname{Im}\rho$ corresponding to the end e^ϵ.

(i) If Ω^ϵ is homeomorphic to the open disk, then $\lambda^\epsilon(\rho) \in \mathbb{H}^2$ is the marked conformal structure on T defined by $\Omega^\epsilon/\operatorname{Im}\rho$.

(ii) If Ω^ϵ is the countable union of open disks, then $\lambda^\epsilon(\rho) \in \partial\mathbb{H}^2$ is the marked conformal structure on T with nodes defined by $\Omega^\epsilon/\operatorname{Im}\rho$.

(iii) If Ω^ϵ is empty, then there is a sequence of closed geodesics in $\mathbb{H}^3/\operatorname{Im}\rho$ which exits the end e^ϵ. $\lambda^\epsilon(\rho) \in \partial\mathbb{H}^2$ is defined to be the limit of the sequence.

Theorem 2.6 (Minsky [9]). *The end invariant map* $\boldsymbol{\lambda} = (\lambda^-, \lambda^+) : \mathcal{P} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \operatorname{diag}(\partial\mathbb{H}^2)$ *is a bijection and its inverse is a continuous map. Moreover,* $\mathcal{P}$ *is equal to* $\overline{\mathcal{QF}}$.

3 Jorgensen theory

In this section, we briefly review the work of Jorgensen [7] on the characterization of combinatorial structures of punctured torus groups. (See [3] for a complete proof of Jorgensen's results for quasifuchsian punctured torus groups.)

Definition 3.1. For a Kleinian group Γ, let Γ_∞ be the stabilizer of ∞ in Γ. The *Ford domain* of Γ in $\mathbb{C}$ (resp. $\mathbb{H}^3$), denoted by $P(\Gamma)$ (resp. $Ph(\Gamma)$), is defined by

$$P(\Gamma) = \bigcap_{\gamma \in \Gamma - \Gamma_\infty} E(\gamma), \qquad Ph(\Gamma) = \bigcap_{\gamma \in \Gamma - \Gamma_\infty} Eh(\gamma).$$

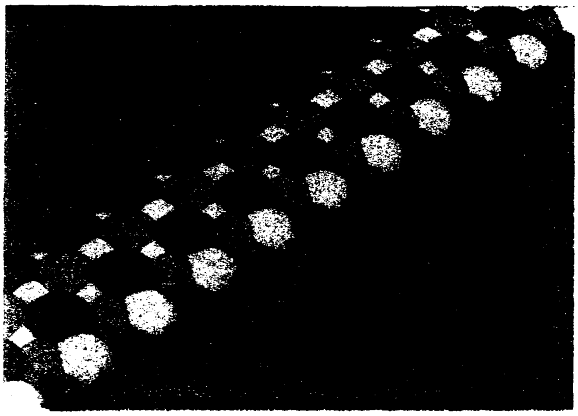
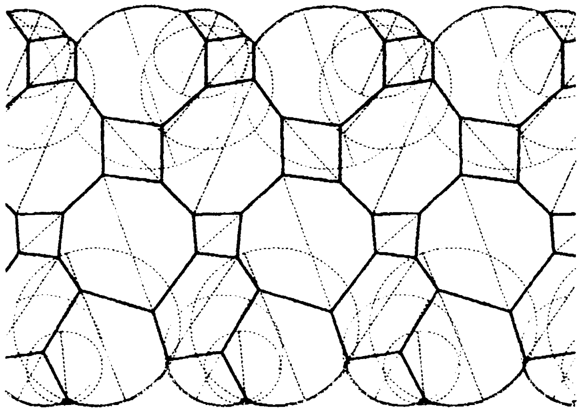

Figure 1: Ford domain of a generic quasifuchsian punctured torus group

Here, $E(\gamma)$ (resp. $Eh(\gamma)$) denotes the exterior of the isometric circle (resp. isometric hemisphere) of γ.

Remark 3.2. The Ford domain $P(\Gamma)$ (resp. $Ph(\Gamma)$) is not a fundamental domain for the action of Γ on $\Omega(\Gamma)$ (resp. $\mathbb{H}^3$), whenever Γ_∞ is nontrivial, where $\Omega(\Gamma)$ is the domain of discontinuity of Γ on which Γ acts discontinuously. Even in the case, the intersection of the Ford domain and a fundamental domain for Γ_∞ is actually a fundamental domain for Γ.

In what follows, for any $\rho \in \mathcal{P}$, we denote $P(\operatorname{Im}\rho)$ (resp. $Ph(\operatorname{Im}\rho)$) by $P(\rho)$ (resp. $Ph(\rho)$) for simplicity.

Example 3.3. The Ford domain of a generic quasifuchsian punctured torus group looks like Figure 1. Its combinatorial structure is described by using the "side parameter" defined in Definition 3.10. The upper and lower boundary components in the right figure define two spines of T. By following $\partial Ph(\rho)$ from the lower component to the upper, one finds the sequence of Whitehead moves connecting the two spines (cf. [3]).

Fix a framing $\{\alpha, \beta\} \subset H_1(T)$ and a peripheral element K of $\pi_1(T)$.

Definition 3.4. We call a pair of elements, (A, B), of $\pi_1(T)$ a *generator pair* if A and B generates $\pi_1(T)$ and satisfies $ABA^{-1}B^{-1} = K$. For such a pair, A (resp. B) is called a *left* (resp. *right*) *generator*, or simply a *generator*.

Remark 3.5. The situation may be more clear if we introduce the notion of *elliptic generator triple*, for which we need to extend the group $\pi_1(T)$ to the fundamental group of the orbifold obtained as the quotient space of T by the hyperelliptic involution (cf. [3]).

One can see that every generator in the above sense has a simple closed curve in T as a representative.

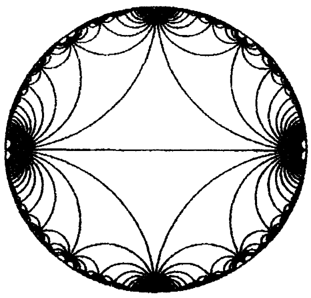

Figure 2: Farey triangulation

Definition 3.6. For each generator X which represents an element $p\alpha+q\beta \in H_1(T)$, the *slope*, $s(X)$, of X is defined by $p/q \in \widehat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$.

Definition 3.7. The Farey triangulation of $\mathbb{H}^2$ is an ideal triangulation consisting of the ideal triangles $\{\gamma\sigma_0 \,|\, \gamma \in PSL(2,\mathbb{Z})\}$, where σ_0 is the ideal triangle with vertices $\infty, 0, 1 \in \partial\mathbb{H}^2$ (Figure 2).

Lemma 3.8. *The following holds.*

1. *For any generator pair (A, B), the slopes of A, AB and B span an ideal triangle in the Farey triangulation.*

2. *For any ideal edge (resp. ideal triangle) σ in the Farey triangulation, there is a generator pair (A, B) such that the slopes of A and B (resp. A, AB and B) span σ.*

Theorem 3.9. *For any $\rho \in \mathcal{QF}$, $P(\rho) \subset \mathbb{C}$ consists of precisely two connected components $P^{\pm}(\rho)$, where $P^{-}(\rho)$ (resp. $P^{+}(\rho)$) is the component which is lower (resp. higher) than the other in $\mathbb{C}$. For each $\epsilon \in \{-,+\}$, there is a sequence $\{A_j^\epsilon\}$ of generators of $\pi_1(T)$ such that $\partial P^\epsilon(\rho)$ is the union of circular edges e_j^ϵ ($j \in \mathbb{Z}$) with the following property.*

(i) *For any $j, k \in \mathbb{Z}$, it follows that $s(A_{j+3k}^\epsilon) = s(A_j^\epsilon)$, and the three slopes $s(A_0^\epsilon), s(A_1^\epsilon), s(A_2^\epsilon)$ span a triangle σ^ϵ of $\mathcal{D}$.*

(ii) *For any $j \in \mathbb{Z}$, e_j^ϵ is contained in $I(\rho(A_j^\epsilon))$.*

(iii) *If we denote by θ_j^ϵ the half of the angle of e_j^ϵ in $I(\rho(A_j^\epsilon))$, then*

$$\theta_0^\epsilon + \theta_1^\epsilon + \theta_2^\epsilon = \pi/2.$$

Definition 3.10 (side parameter). For any $\rho \in \mathcal{QF}$, we define the two points $\nu^{\pm}(\rho)$ in $\mathbb{H}^2$ as follows. For each $\epsilon \in \{-,+\}$, let σ^{ϵ} be the triangle in $\mathcal{D}$ determined by Theorem 3.9. Then $\nu^{\epsilon}(\rho)$ is the point in the triangle σ^{ϵ} with barycentric coordinate $(\theta_0^{\epsilon}, \theta_1^{\epsilon}, \theta_2^{\epsilon})$. The point $\boldsymbol{\nu}(\rho) = (\nu^-(\rho), \nu^+(\rho)) \in \mathbb{H}^2 \times \mathbb{H}^2$ is called the *side parameter* of ρ.

Theorem 3.11. (1) *For any $\rho \in \mathcal{QF}$, the combinatorial structure of $Ph(\rho)$ is described by using $\boldsymbol{\nu}(\rho)$.*

(2) *The map $\boldsymbol{\nu} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ is a homeomorphism.*

The following theorem gives an extension of the side parameter to $\mathcal{P}$. (See [1] for an outline.)

Theorem 3.12. *The map $\boldsymbol{\nu} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ is extended to a map $\boldsymbol{\nu} = (\nu^-, \nu^+) : \mathcal{P} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial \mathbb{H}^2)$ with the following property.*

(1) *For any $\rho \in \mathcal{P}$, the combinatorial structure of $Ph(\rho)$ is described by using $\boldsymbol{\nu}(\rho)$.*

(2) *The map $\boldsymbol{\nu}$ is surjective, and it is continuous in the strong topology on $\mathcal{P}$.*

(3) *For each $\epsilon = \pm$, $\nu^{\epsilon}(\rho) \in \partial \mathbb{H}^2$ if and only if $\lambda^{\epsilon}(\rho) \in \partial \mathbb{H}^2$. Moreover, under the mutually equivalent conditions, it follows that $\nu^{\epsilon}(\rho) = \lambda^{\epsilon}(\rho)$.*

Since the fundamental group of a punctured torus bundle contains the fundamental group of the fiber surface as a normal subgroup, we obtain the following corollary, which is first proved by Lackenby [8] with a topological argument.

Corollary 3.13. *For any hyperbolic punctured torus bundle over the circle, the Ford domain of the image of the holonomy representation of the complete hyperbolic structure is dual to the "Jorgensen's triangulation" (cf. [6]).*

4 Comparison of $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$

In [2], some comparisons of the end invariant map $\boldsymbol{\lambda} : \mathcal{P} \to \mathcal{E}$ and the side parameter map $\boldsymbol{\nu} : \mathcal{P} \to \mathcal{E}$ is given (see Theorems 4.2 and 4.3 below). In order to give the statement of the comparison, we introduce a notation which describes a $PSL(2, \mathbb{Z})$-invariant family of horodisks in $\mathbb{H}^2$.

Notation 4.1. (a) Let $H_\infty(0)$ be the horodisk in $\mathbb{H}^2$ with the following property: (i) The center of $H_\infty(0)$ is equal to ∞. (ii) The interiors of $H_\infty(0)$ and $\gamma H_\infty(0)$ are disjoint for any $\gamma \in PSL(2,\mathbb{Z})$ which does not stabilize $H_\infty(0)$. (iii) For some $\gamma \in PSL(2,\mathbb{Z})$ which does not stabilize $H_\infty(0)$, $H_\infty(0)$ and $\gamma H_\infty(0)$ intersect.

(b) For any $t \in \mathbb{R}$, we define a horodisk $H_\infty(t)$ by the following property: (i) The center of $H_\infty(t)$ is equal to ∞. (ii) The distance between $\partial H_\infty(0)$ and $\partial H_\infty(t)$ is equal to $|t|$. (iii) If $t < 0$, then $H_\infty(t) \subset H_\infty(0)$, otherwise $H_\infty(t) \supset H_\infty(0)$.

(c) For any $s \in \widehat{\mathbb{Q}}$, pick $\gamma \in PSL(2,\mathbb{Z})$ so that $\gamma(\infty) = s$. Then, for any $t \in \mathbb{R}$, we set $H_s(t) = \gamma H_\infty(t)$.

(d) For any $t \in \mathbb{R}$, we set $\mathcal{H}(t) = \{H_s(t) \mid s \in \widehat{\mathbb{Q}}\}$.

Theorem 4.2 ([2]). *There exist two universal constants $\delta_a \in \mathbb{R}$ and $\delta_b \geq 0$ with the following property.*

(i) If $\rho \in \mathcal{QF}$ satisfies $\nu^\epsilon(\rho) \in H_s(\delta_a)$ for some $s \in \widehat{\mathbb{Q}}$ and $\epsilon \in \{-,+\}$, then the simple loop on T with slope s has the smallest extremal length among the simple loops on T with respect to the conformal structure $\lambda^\epsilon(\rho)$.

(ii) If $\rho \in \mathcal{QF}$ satisfies $\nu^\epsilon(\rho) \notin \cup\mathcal{H}(\delta_a)$ for some $\epsilon \in \{-,+\}$, then the hyperbolic distance between $\nu^\epsilon(\rho)$ and $\lambda^\epsilon(\rho)$ is at most δ_b.

We shall denote the Weil-Petersson distance on $\mathcal{T}$ by d_{WP} and regard it as an exotic distance on the hyperbolic plane $\mathbb{H}^2$. By combining Theorem 1.1 and Brock's result [5, Theorem 1.1], we obtain the following theorem.

Theorem 4.3 ([2]). *There exists a positive number δ_c such that for any $\rho \in \mathcal{QF}$ and for any $\epsilon \in \{-,+\}$, the Weil-Petersson distance $d_{WP}(\nu^\epsilon(\rho), \lambda^\epsilon(\rho))$ between the side parameter and the end invariant for the ϵ-side is bounded above by δ_c.*

We can obtain the following stronger Propositions 4.4–4.6 by the argument of [2], which are essential for the proof of Theorem 1.1.

Proposition 4.4. *For any $t \in \mathbb{R}$, there is $u = u(t) \in \mathbb{R}$ with the following property. If $\rho \in \mathcal{P}$ satisfies $\nu^\epsilon(\rho) \in H_s(u)$ for some $s \in \widehat{\mathbb{Q}}$ and $\epsilon \in \{-,+\}$, then $\lambda^\epsilon(\rho) \in H_s(t)$.*

Proposition 4.5. *For any $t \in \mathbb{R}$, there is $u = u(t) > 0$ with the following property. If $\rho \in \mathcal{P}$ satisfies $\nu^\epsilon(\rho) \in \mathbb{H}^2 - \mathcal{H}(t)$, then $d(\nu^\epsilon(\rho), \lambda^\epsilon(\rho)) \leq u$.*

Proposition 4.6. *For any $t \in \mathbb{R}$, there is $u = u(t) \in \mathbb{R}$ with the following property. If $\rho \in \mathcal{P}$ satisfies $\lambda^\epsilon(\rho) \in H_s(u)$ for some $s \in \widehat{\mathbb{Q}}$ and $\epsilon \in \{-,+\}$, then $\nu^\epsilon(\rho) \in H_s(t)$.*

5 Outline of the proof of Main Theorem

In what follows we denote the composition $\boldsymbol{\nu} \circ \boldsymbol{\lambda}^{-1}$ by Φ. We split the proof of Theorem 1.1 into the following 3 steps:

Step 1. The map $\Phi : \mathcal{E} \to \mathcal{E}$ is continuous.

Step 2. The map $\Phi : \mathcal{E} \to \mathcal{E}$ is injective.

Step 3. The map $\Phi : \mathcal{E} \to \mathcal{E}$ is a homeomorphism.

5.1 Continuity of $\Phi : \mathcal{E} \to \mathcal{E}$

Proposition 5.1. *The map $\Phi : \mathcal{E} \to \mathcal{E}$ is continuous.*

Let $\{\boldsymbol{\lambda}_n\}$ be a sequence in $\mathcal{E}$ which converges to $\boldsymbol{\lambda}_\infty \in \mathcal{E}$. We set $\boldsymbol{\lambda}_n = (\lambda_n^-, \lambda_n^+)$ $(n \in \mathbb{N})$, $\boldsymbol{\lambda}_\infty = (\lambda_\infty^-, \lambda_\infty^+)$, $\boldsymbol{\nu}_n = (\nu_n^-, \nu_n^+) = \Phi(\boldsymbol{\lambda}_n)$ and $\boldsymbol{\nu}_\infty = (\nu_\infty^-, \nu_\infty^+) = \Phi(\boldsymbol{\lambda}_\infty)$. In order to prove Proposition 5.1, we show that any subsequence of $\{\boldsymbol{\nu}_n\}$ contains a subsequence converging to some point in $\mathcal{E}$ which is independent of the choice of subsequence.

The following lemma is proved by using the Propositions 4.5 and 4.6.

Lemma 5.2. *Suppose that $\lambda_\infty^\epsilon \in \partial\mathbb{H}^2$. Then $\lim_{n\to\infty} \nu_n^\epsilon = \nu_\infty^\epsilon$.*

The following lemma is proved by using Propositions 4.4 and 4.5.

Lemma 5.3. *Suppose that all λ_n^ϵ is contained in $\mathbb{H}^2$ and that $\lambda_\infty^\epsilon \in \mathbb{H}^2$. Then there is a compact set in $\mathbb{H}^2$ containing all ν_n^ϵ.*

Idea of the proof of Proposition 5.1. By Lemma 5.2, the case which essentially remains to prove is that a component of $\boldsymbol{\lambda}_\infty$ is contained in $\mathbb{H}^2$ and the other is in $\partial\mathbb{H}^2$. In this case, we can prove the continuity by using Lemma 5.3 and the openness of the "Jorgensen's algorithm". □

5.2 Injectivity of $\Phi : \mathcal{E} \to \mathcal{E}$

First, by following the argument of [3, Chapter 9], we can prove the following proposition.

Proposition 5.4. *For any* $\lambda \in \widehat{\mathbb{Q}}$, *both* $\Phi|_{\{\lambda\}\times\mathbb{H}^2} : \{\lambda\} \times \mathbb{H}^2 \to \mathcal{E}$ *and* $\Phi|_{\mathbb{H}^2\times\{\lambda\}} : \mathbb{H}^2 \times \{\lambda\} \to \mathcal{E}$ *are injective.*

Proposition 5.5. *The map* $\Phi : \mathcal{E} \to \mathcal{E}$ *is injective.*

The proof of this proposition requires Proposition 5.6 below.

Proposition 5.6. *Let* $\boldsymbol{\lambda}_0 = (\lambda_0^-, \lambda_0^+)$ *be a parameter in* $\mathcal{E}$ *such that one of* $\lambda_0^\pm$ *is contained in* $\mathbb{H}^2$ *and the other is in* $\partial\mathbb{H}^2$. *For any neighborhood* N *of* $\boldsymbol{\lambda}_0$ *in* $\mathcal{E}$, *the image* $\Phi(N)$ *contains a neighborhood of* $\Phi(\boldsymbol{\lambda}_0)$ *in* $\partial\mathcal{E}$.

The proof of this proposition uses an argument similar to the one of [3, Chapter 9], however, it also uses the theory of quasiconformal deformation of geometrically infinite representations due to Ahlfors-Bers-Sullivan.

Proof of Proposition 5.5 using Proposition 5.6. Recall that the map $\boldsymbol{\lambda} : \mathcal{P} \to \mathcal{E}$ is bijective. Pick any $\boldsymbol{\nu}_0 = (\nu_0^-, \nu_0^+) \in \mathcal{E}$. If both $\nu_0^\pm$ are contained in $\mathbb{H}^2$, then $\Phi^{-1}(\boldsymbol{\nu}_0)$ is a singleton because the restriction $\boldsymbol{\nu}|_{\mathcal{QF}} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ is bijective. If both $\nu_0^\pm$ are contained in $\partial\mathbb{H}^2$, then $\Phi^{-1}(\boldsymbol{\nu}_0)$ is the singleton consisting of $\boldsymbol{\nu}_0$ by Theorem 3.12. Suppose that one of $\nu_0^\pm$ is contained in $\mathbb{H}^2$ and the other is in $\widehat{\mathbb{Q}}$. We assume that $\nu_0^- \in \mathbb{H}^2$. Then $\Phi^{-1}(\boldsymbol{\nu}_0)$ is contained in $\mathbb{H}^2 \times \{\nu_0^+\}$ by Theorem 3.12. Thus $\Phi^{-1}(\boldsymbol{\nu}_0)$ is a singleton by Proposition 5.4.

Finally, suppose that one of $\nu_0^\pm$ is contained in $\mathbb{H}^2$ and the other is in $\partial\mathbb{H}^2 - \widehat{\mathbb{Q}}$. We assume that $\nu_0^- \in \mathbb{H}^2$. Suppose moreover that $\Phi^{-1}(\boldsymbol{\nu}_0)$ contains distinct points $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$. Then both $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ satisfy the assumption of Proposition 5.6 by Theorem 3.12. Since $\mathcal{E}$ is a Hausdorff space, there are disjoint open sets N_1 and N_2 containing $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ respectively. By Proposition 5.6, each $\Phi(N_1)$ and $\Phi(N_2)$ contains a neighborhood of the common point $\Phi(\boldsymbol{\lambda}_1) = \boldsymbol{\nu}_0 = \Phi(\boldsymbol{\lambda}_2)$ in $\partial(\mathcal{E})$. In particular, there is $\nu_*^+ \in \widehat{\mathbb{Q}}$ such that the point (ν_0^-, ν_*^+) is contained in the intersection $\Phi(N_1) \cap \Phi(N_2)$. Since N_1 and N_2 are disjoint, the inverse image of (ν_0^-, ν_*^+) contains at least two points. This is a contradiction. $\square$

5.3 $\Phi : \mathcal{E} \to \mathcal{E}$ is a homeomorphism

Proposition 5.7. $\Phi : \mathcal{E} \to \mathcal{E}$ *is a homeomorphism.*

This proposition follows immediately from Lemmas 5.8 and 5.9 below.

Lemma 5.8. *Let $f : X \to X$ be a continuous bijection of a "topological space" X onto itself. Suppose that f satisfies the following condition. For any convergent sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_j}\}$ such that the sequence $\{f^{-1}(x_{n_j})\}$ converges in X. Then $f^{-1} : X \to X$ is also continuous.*

This lemma is proved by a general argument.

Lemma 5.9. *For any convergent sequence $\{\boldsymbol{\nu}_n\}$ in $\mathcal{E}$, there is a subsequence, denoted by the same symbol, such that the sequence $\{\boldsymbol{\lambda} \circ \boldsymbol{\nu}^{-1}(\boldsymbol{\nu}_n)\}$ converges in $\mathcal{E}$.*

This lemma is proved by using Propositions 4.4 and 4.5.

References

[1] H. Akiyoshi, "On the Ford domains of once-punctured torus groups", *Hyperbolic spaces and related topics*, RIMS, Kyoto, Kokyuroku **1104** (1999), 109–121.

[2] H. Akiyoshi, "End invariants and Jorgensen's angle invariants of punctured torus group", *Perspectives of Hyperbolic Spaces II*, RIMS, Kyoto, Kokyuroku **1387** (2004), 59–69.

[3] H. Akiyoshi, M. Sakuma, M. Wada, and Y. Yamashita, "Punctured torus groups and 2-bridge knot groups (I)", Lecture Notes in Mathematics, 1909. *Springer, Berlin*, (2007).

[4] J. Anderson and R. Canary, "Algebraic limits of Kleinian groups which rearrange the pages of a book", *Invent. Math.* **126** (1996), 205–214.

[5] J. F. Brock, "The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores", *J. Amer. Math. Soc.* **16** (2003), *no. 3*, 495–535.

[6] W. Floyd and A. Hatcher, "Incompressible surfaces in punctured torus bundles", *Topology Appl.*, **13** (1982), 263–282.

[7] T. Jorgensen, "On pairs of punctured tori", in *Kleinian Groups and Hyperbolic 3-Manifolds*, Y. Komori, V. Markovic & C. Series (Eds.), London Mathematical Society Lecture Notes **299**, Cambridge University Press, (2003).

[8] M. Lackenby, "The canonical decomposition of once-punctured torus bundles", *Comment. Math. Helv.*, **78** (2003), no. 2, 363–384.

[9] Y. Minsky, "The classification of punctured torus groups", *Ann. of Math.*, **149** (1999), 559–626.

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