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The moduli space of equilateral hexagons

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Abstract

Here we clarify the topology of the moduli space of equilateral hexagons. The detail argment is discussed in author's paper, "The moduli space of equilateral hexagons".

1   Introduction

By an n-gon, we mean a polygon with n edges in the complex line $\mathbb{C}$. Here we do not assume that an n-gon is a simple polygon. It could be wild. For instance, edges may intersect each other, may coincide, or even may be degenerate. We only assume that not all edges are degenerate simultaneously.

An n-gon can be identified simply with an ordered set of n points. The moduli space $\mathcal{P}_n$ of n-gons, where two n-gons are identified if they are orientation preservingly similar with marking, is naturally homeomorphic to a complex projective space of complex dimension $n - 2$ by assigning edge vectors. There is a natural map of $\mathcal{P}_n$ to $\mathbb{RP}^n$ by assigning the ratio of edge lengths to each n-gon. The fiber is the moduli space of n-gons with prescribed ratio of edge lengths, and its real dimension is at most $n - 3 = 2(n - 2) - (n - 1)$.

Here is a history of the study on the topology of the moduli space of n-gons with a fixed ratio of edge lengths. In [2], Havel showed that the moduli space of equilateral pentagons is homeomorphic to a closed surface of genus 4 using computational techniques with distance geometry methods. Kamiyama re-discovered this fact by using Morse theory in [4]. More systematically to these works, Kapovich and Millson developed their study on the topology of moduli spaces of n-gons when the edge lengths are not necessarily equal to each other but preserve a ratio of edge lengths, and obtained several generic conclusions for the case $n = 4, 5, 6$ in [6], with an extensive list of related works. We come back about genericity later, but should like to mention at

this stage that equilateral case is generic when $n = 5$ in fact, but not when $n = 6$.

Assigning the direction of edges to each n-gon after Gauss map, we obtain a marked n point configuration on the unit circle up to rotations. Since our n-gon may have edges with the same direction, the configuration assigned could have collision of points. Thus the assignment defines a map from the moduli space of n-gon with a fixed ratio of edge lengths to the moduli space of marked n point configurations on the unit circle up to conformal transformations. However, some point configuration does not come from any n-gon. Moreover, the target space is not Hausdorff. Hence it is natural to exclude some of marked point configurations which cause non-Hausdorff property. For this purpose, Kapovich and Millson imposed the "semi-stability" condition on the ratio of edge lengths in Definition 5 of [6] which rejects degenerate n-gons, where an n-gon is said to be *degenerate* if it is contained in an euclidean line. Equilateral pentagons never be degenerate but some equilateral hexagons are so.

The purpose of this paper is to clarify the topology of the moduli space of equilateral hexagons, including degenerate ones. We will denote it by X throughout this paper. There are ten degenerate hexagons. If we exclude these exceptions, the moduli space turns out to have bijective correspondence to the moduli space X' of semi-stable marked 6 point configurations on the circle, which was known to be homeomorphic to a hyperbolic 3-manifold with ten cusps by Kojima, Nishi and Yamashita in [7]. Hence X is known, a priori, to be homeomorphic to the union of X' with ten points.

The goal is to show that X is actually equal to the space obtained from X' by compactifying each end with a point. This identification will be a conclusion of our completely independent description of X. In fact, using new and elementary view for X, we prove that it has a very natural cell decomposition by 16 copies of a polyhedron with 10 faces. Our method also provides a new proof of hyperbolization for X' via Koebe-Andreev-Thurston's theorem [5], [1], [9],.

The organization of this paper is as follows. In §2, we define the moduli space of hexagons and briefly describe the basic idea for the analysis. We divide X into 16 parts and construct a uniform combinatorial model H_3 for the parts, in §3. Then we find the gluing rule of parts in §4, and clarify the topology of neighborhoods of edges and vertices of X in §5. §6 is to conclude the results obtained up to §5.

2 Definitions and the basic idea

Let Y be the space of equilateral hexagons in $\mathbb{C}$ and X the moduli space of marked equilateral hexagons. More concretely, we set

$$Y = \{p = (p_1, p_2, \cdots, p_6) \in \mathbb{C}^6 \mid |p_i - p_{i+1}| = 1 \text{ for all } i \bmod 6\},$$
$$X = Y/\sim$$

where $p \sim q$ if and only if p is congruent to q orientation preservingly, i.e. there exist $v, w \in \mathbb{C}$ with $|v| = 1$ such that $p_i = vq_i + w$ for all i.

We use $p \in Y$ to indicate also its equivalence class $p \in X$ confusingly. For example, $\{p \in X \mid |p_1 - p_3| = 1\}$ means $\{(p_1, \cdots, p_6) \in Y \mid |p_1 - p_3| = 1\}/\sim$. Note that the condition "$|p_1 - p_3| = 1$" here is invariant under congruent transformations and independent of a choice of representatives of $p \in X$.

Definition 1. For each equilateral hexagon $p = (p_1, p_2, \cdots, p_6) \in Y$ or X, we call the triangle spanned by p_1, p_3, p_5 the *inner triangle* of p, and denote it by $\triangle(p)$.

Denote by $x, y, z : X \to \mathbb{R}$ the maps which assign to each $p \in Y$ the length of each side of $\triangle(p)$. More precisely, $x(p) = |p_3 - p_1|$, $y(p) = |p_5 - p_3|$ and $z(p) = |p_1 - p_5|$. Since equivalent hexagons have congruent inner triangles, $x(p)$, $y(p)$ and $z(p)$ make sense.
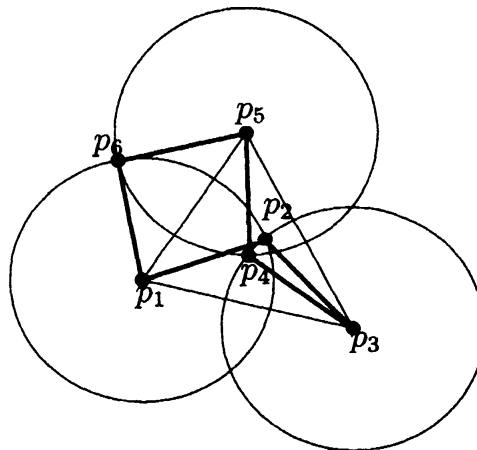

Fig. 1. Hexagon $p = (p_1, p_2, \cdots, p_6)$ and its inner triangle $\triangle(p)$

Definition 2. Let

$$f : X \to \mathbb{R}^3$$

be a continuous map assigning $(x(p), y(p), z(p))$ to each $p \in X$.

Then, the image of X by f is equal to

$$H = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x, y, z \le 2,\ x \le y + z,\ y \le z + x,\ z \le x + y\}$$

Denote the interior of H by H° and set Q to be $f^{-1}(H^\circ)$. In other words,

$$Q = \{p \in X \mid 0 < x(p), y(p), z(p) < 2, \triangle(p) \text{ has non-zero area}\}.$$

It is obvious that Q is open and dense in X.

Fix a hexagon $p \in Q$. There are exactly 8 hexagons in Q whose inner triangles are marked congruent to $\triangle(p)$ (including p itself). Thus by taking the orientation of inner triangles into account, there are 16 hexagons which share the congruent class of $\triangle(p)$.
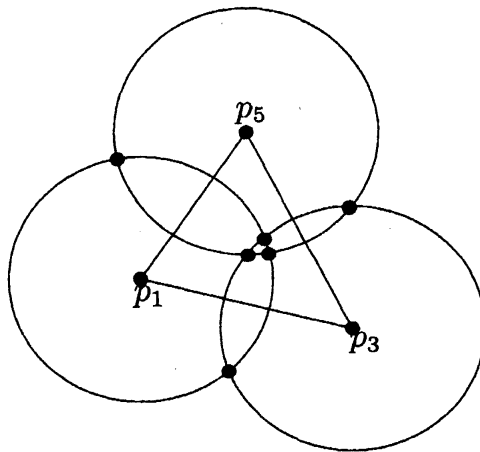

Fig. 2. candidates for p_2, p_4 and p_6

In particular, Q is homeomorphic to a disjoint union of 16 copies of H°, and $f|_Q : Q \to H^\circ$ is a 16-fold trivial covering.

The basic idea for attaining our goal is to study the closure of a component of Q in X carefully, and to find a compact model polyhedron H_3 with 10 faces by skinning the boundary to H°, so that X can be obtained by gluing 16 copies of H_3 with the canonical gluing rule. The construction of H_3 will be done by blowing up H along some edges and vertices of in the next section.

3 Construction of a model polyhedra H_3

For each $p = (p_1, p_2, \cdots, p_6) \in Q$ with nondegenerate inner triangle $\triangle(p)$, let s_1 be either $+$ or $-$ according to whether the orientation of $\triangle(p)$ coincides

with that of $\mathbb{C}$ or not. s_1 may be explicitly expressed in terms of the sign of the imaginary part of $(p_5 - p_1)/(p_3 - p_1)$. We call s_1 the sign of $\triangle(p)$.

Also, let s_2 (s_3 and s_4 resepectively) be either $+$ or $-$ according to whether p_2 (p_4 and p_6) is located on the right hand side or left hand side of the line from p_1 (p_3 and p_5) towards p_3 (p_5 and p_1). Again s_2 may be identified with the sign of the imaginary part of $(p_2 - p_1)/(p_3 - p_1)$ and so on. We call s_2, s_3 and s_4 the sign of p_2, p_4 and p_6 respectively.

s_1, s_2, s_3, s_4 may be regarded as maps from Q to $\{\pm\}$. Notice moreover that some of $s_i(p)$'s make sense even when p is not a member of Q, and we will use this terminology also for p not in Q whenever it makes sense.

To each $s \in \{\pm\}^4$, set $H^\circ(s)$ to be equal to $\{p \in X \mid (s_1(p), s_2(p), s_3(p), s_4(p)) = s\}$ and $H(s)$ the closure of $H^\circ(s)$ in X. $H^\circ(s)$ is open in X and $H^\circ(s) \cap H^\circ(s') = \emptyset$ if $s \neq s'$. Moreover, Q is equal to the union $\cup_{s \in \{\pm\}^4} H^\circ(s)$.

Choose a subspace of $\mathbb{R}^3$,

$$H_0 = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x, y, z \leq 2, \ x \leq y + z, \ y \leq z + x, \ z \leq x + y\}.$$

$H^\circ \subset H_0 \subset H$ and H_0 is obtained from H by removing four vertices and three edges, see Figure 3.
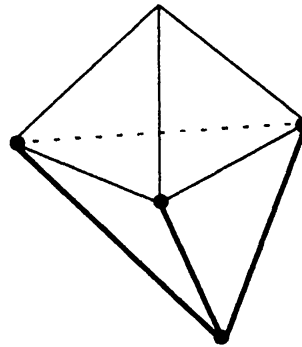

Fig. 3. H_0 (thick edges and vertices are not included)

Now, by looking at H_0, we choose 5 parts of $H(s)$ with overlaps as follows;

$$W_0(s) = \{p \in H(s) \mid 0 < x(p) \leq 2, 0 < y(p) \leq 2, 0 < z(p) \leq 2\},$$
$$W_1(s) = \{p \in H(s) \mid 0 \leq x(p) < 2, 0 < y(p) \leq 2, 0 < z(p) \leq 2\},$$
$$W_2(s) = \{p \in H(s) \mid 0 < x(p) \leq 2, 0 \leq y(p) < 2, 0 < z(p) \leq 2\},$$
$$W_3(s) = \{p \in H(s) \mid 0 < x(p) \leq 2, 0 < y(p) \leq 2, 0 \leq z(p) < 2\} \text{ and}$$
$$W_4(s) = \{p \in H(s) \mid 0 \leq x(p) < 2, 0 \leq y(p) < 2, 0 \leq z(p) < 2\}.$$

Lemma 1. $H(s) = \cup_{i=0}^4 W_i(s)$ for each $s \in \{\pm\}^4$.

Details can be found in [3]. Note that $W_i(s)$'s are open in $H(s)$ because x, y and z are continuous maps.

Definition 3. Denote the restriction of f to $W_0(s)$ by $f_s : W_0(s) \to \mathbb{R}^3$.

We also let half open edges E_1, E_2, E_3 and a vertex E_4 in H be $\{(0, t, t) \in \mathbb{R}^3 \mid 0 < t \le 2\}$, $\{(t, 0, t) \in \mathbb{R}^3 \mid 0 < t \le 2\}$, $\{(t, t, 0) \in \mathbb{R}^3 \mid 0 < t \le 2\}$ and $\{(0, 0, 0)\}$ respectively. It is clear that $H = H_0 \sqcup (\sqcup_{i=1}^4 E_i)$, $\operatorname{Im} f_s = H_0$, $f(W_i(s) - W_0(s)) = E_i$ for $i = 1, 2, 3$ and $f(W_4(s) - \cup_{i=0}^3 W_i(s)) = E_4$.

Lemma 2. $W_0(s)$ *is homeomorphic to H_0 by f_s for each $s \in \{\pm\}^4$.*

Details can be found in [3].

Recall that f_s is the restriction of f, $H(s) \supset W_0(s)$ and $H(s) \supset W_1(s)$. f is injective on W_0 in fact, but no longer injective on $W_1(s)$. Thus $W_1(s) - W_0(s) = f^{-1}(E_1) \cap H(s)$ will be the space to analyze for finding an appropriate blowing up of H along E_1. To see this, remember that $H_0 = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x, y, z \le 2, x \le y + z, y \le z + x, z \le x + y\}$. By setting $w = (y - z)/x$, we transform H_0 homeomorphically onto

$$H_0' = \{(x, w, z) \in \mathbb{R}^3 \mid 0 < x, \; xw + z, z \le 2, \; x \le wx + 2z, \; -1 \le w \le 1\}.$$

Partially skinning H_0', we define H_1 explicitly by

$$H_1 = \{(x, w, z) \in \mathbb{R}^3 \mid 0 \le x \le 2, \; 0 < xw + z, \; z \le 2, \; x \le wx + 2z, \; -1 \le w \le 1\}$$
$$= H_0 \cup \{(0, w, z) \in \mathbb{R}^3 \mid -1 \le w \le 1, 0 < z \le 2\}$$

H_1 is a noncompact polyhedron-like space surrounded by 2 ruled faces and 5 flat faces.
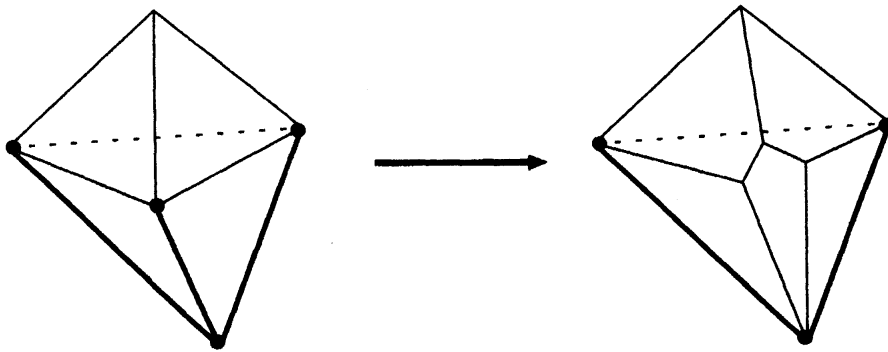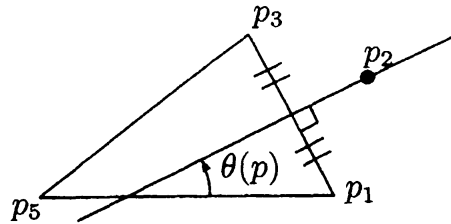

Fig. 4. H_0 and H_1 (thick edges and vertices are not included)

To see H_1 more precise, let us denote the argument between $p_1 - p_5$ and the oriented bisector $p_2 - \frac{p_1 + p_3}{2}$ by $\theta(p)$.

Fig. 5. $\theta(p)$

We will show by subsequent 3 lemmas that H_1 is a blowing up of $H_0 \cup E_1$ along E_1, where the inverse image of E_1 is parametrized by $\theta \times E_1$.

Definition 4. Let
$$g_s : W_0(s) \cup W_1(s) \to H_1$$
be a map defined by setting $g_s(p) = (x(p), w(p), z(p))$, where

$$w(p) = \begin{cases} w_0(p) = (y(p) - z(p))/x(p) & \text{if } p \in W_0(s), \\ w_1(p) = \sin(\theta(p)) & \text{if } p \in W_1(s) - W_0(s). \end{cases}$$

Lemma 3. g_s is a homeomorphism between $W_0(s) \cup W_1(s)$ and H_1.

Details can be found in [3].

Moreover, since E_2 and E_3 are disjoint from E_1 and the union $\cup_{i=1}^3 E_i$ has cyclic symmetry of order 3, blowing up $H_0 \cup \cup_{i=1}^3 E_i$ along E_1, E_2 and E_3 simultaneously in a similar manner to the above, we obtain a polyhedron-like space H_2, so that

Corollary 1. H_2 is homeomorphic to $W_0(s) \cup W_1(s) \cup W_2(s) \cup W_3(s)$.

To see how we blow up H at $E_4 = \{(0,0,0)\}$, let Θ be the space of angles, which is the quotient space of $[0, 2\pi]$ by identifying 0 and 2π. Θ^2 is thus a torus, and will be the union of blown up surfaces of the origin in H. It corresponds to the torus formed by hexagons with completely degenerate inner triangle.

Definition 5. Letting

$$\phi(p) = \arg\left(\frac{p_4 - (p_3 + p_5)/2}{p_2 - (p_1 + p_3)/2}\right) \quad \text{and} \quad \psi(p) = \arg\left(\frac{p_6 - (p_5 + p_1)/2}{p_4 - (p_3 + p_5)/2}\right),$$

we define a continuous map

$$h_s : W_4(s) \to \Theta^2$$

by setting $h_s(p) = (\phi(p), \psi(p))$.

Here, ϕ measures the argument of oriented bisectors of p_3p_5 and p_1p_3 which ranges a half in Θ. Similarly ψ measures the argument of oriented bisectors of p_5p_1 and p_3p_5 which ranges a half in Θ.
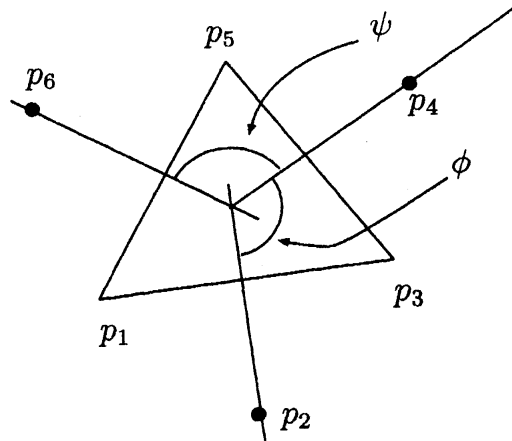

Fig. 6. $p \in W_4(s)$ where $s = (+. - .-, -)$

Notice that (ϕ, ψ) is invariant under similar changes of $\triangle(p)$, and therefore the values stay constant along the curve in $H(s)$ corresponding to a ray through E_4 in $H_0 \cup E_4$.

It is easy to see for example that if $s = (+, -, -, -)$, then the image of h_s is a triangular region $T = \{(\phi, \psi) \in \Theta^2 \mid 0 \leq \phi, \psi \leq \pi, \pi \leq \phi + \psi\}$ on Θ^2, see Figure 7. Three vertices $(0, \pi)$, $(\pi, 0)$, (π, π) of T correspond to rays on the face of H_0 defined by the triangle equalities. In this case, every rays on each face are mapped to a point. Three edges of T correspond to the boundary of the faces of H_2 appeared by the blow up along edges E_i ($i = 1, 2, 3$). An interior point of T corresponds to a ray lying in the interior of H_0. Moreover, the restriction of h_s to $W_4(s) \cap f^{-1}(E_4)$ is a homeomorphism onto T.
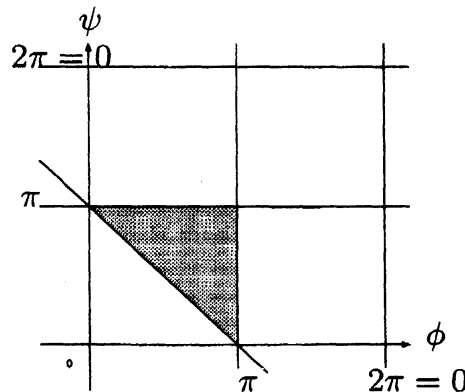

Fig. 7. $T = h_s(W_4(s))$ when $s = (+, -, -, -)$

Then, by looking at the case when $s = (+, -, -, -)$, we can construct a compact polyhedron-like space $H_3 = H_2 \cup T$ such that the projection $\rho :$ $H_3 \to H$ is continuous and $\rho^{-1}(E_4) = T$. In other words, H_3 is a blowing up of H along $\cup_{i=1}^4 E_i$ and is homeomorphic to $H(s)$.
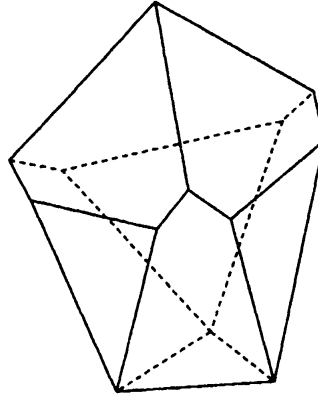

Fig. 8. H_3

Since a canonical homeomorphism from $H(s)$ to $H(s')$ induced by the change of signs extends to their closures in X as a canonical homeomorphism, we have

Corollary 2. H_3 *is homeomorphic to* $H(s) = \cup_{i=0}^4 W_i(s)$ *for each* $s \in \{\pm\}^4$.

4   Gluing rule along faces

Recall that H_3 has 10 faces. Let us denote the corresponding faces of $H(s)$ for each $s \in \{\pm\}^4$ by

(1) $F_0(s) = Cl(\{p \in W_0(s) \mid x(p) = 2\})$
(2) $F_1(s) = Cl(\{p \in W_0(s) \mid y(p) = 2\})$
(3) $F_2(s) = Cl(\{p \in W_0(s) \mid z(p) = 2\})$
(4) $F_3(s) = Cl(\{p \in W_0(s) \mid x(p) + y(p) = z(p)\})$
(5) $F_4(s) = Cl(\{p \in W_0(s) \mid y(p) + z(p) = x(p)\})$
(6) $F_5(s) = Cl(\{p \in W_0(s) \mid z(p) + x(p) = y(p)\})$
(7) $F_6(s) = Cl(\{p \in W_1(s) \mid x(p) = 0\})$
(8) $F_7(s) = Cl(\{p \in W_2(s) \mid y(p) = 0\})$
(9) $F_8(s) = Cl(\{p \in W_3(s) \mid z(p) = 0\})$
(10) $F_9(s) = Cl(\{p \in W_4(s) \mid x(p) = y(p) = z(p) = 0\})$

where $Cl(Z)$ is the closure of the interior points of Z in the boundary of $H(s)$. See Figure 9.
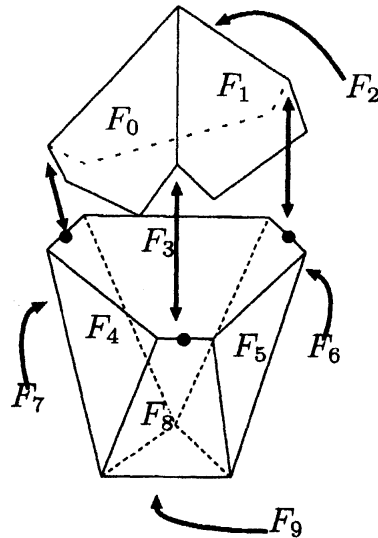
Fig. 9. Faces

For each $s \in \{\pm\}^4$, we want to know which components are adjacent to $H(s)$ and how they are gluing to $H(s)$. As we see in the following, for each $i \in \{0, \cdots, 9\}$, there exists $s' \in \{\pm\}^4$ such that $s \neq s'$ and $F_i(s) = F_i(s')$. In other words, gluing is cellular with respect to the polyhedral structure of $H(s)$'s.

Fix $s = (s_1, s_2, s_3, s_4)$. $F_0(s_1, s_2, s_3, s_4)$ consists of $p \in H(s)$ with $x(p) = 2$. Such p is also a member of $F_0(s_1, -s_2, s_3, s_4)$ and vise versa as in Figure 10, showing typical hexagons in $H(+, -, -, -)$, $F_0(+, -, -, -) = F_0(+, +, -, -)$ and $H(+, +, -, -)$. Thus, $F_0(s_1, s_2, s_3, s_4) = F_0(s_0, -s_2, s_3, s_4)$. Similarly, $F_1(s_1, s_2, s_3, s_4) = F_1(s_1, s_2, -s_3, s_4)$, and $F_2(s_1, s_2, s_3, s_4) = F_2(s_1, s_2, s_3, -s_4)$.
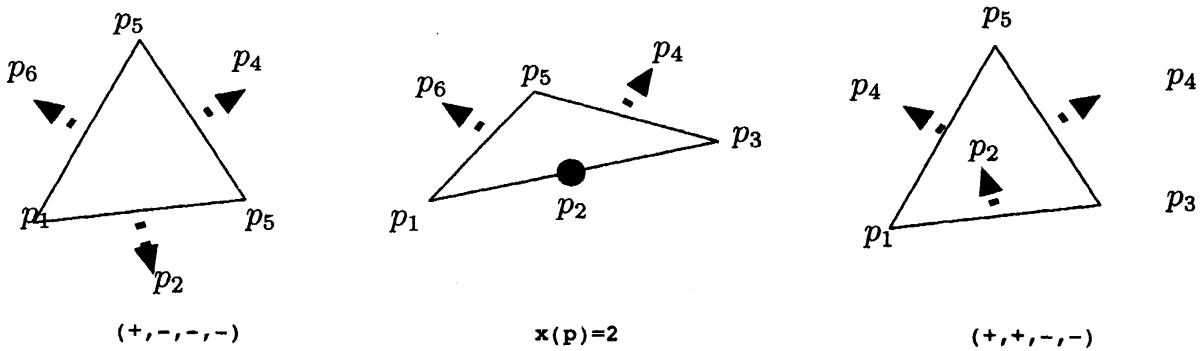

Fig. 10. F_0

Similarly, we can see following rules:

- $F_3(s_1, s_2, s_3, s_4) = F_3(-s_1, s_2, s_3, s_4)$
- $F_4(s_1, s_2, s_3, s_4) = F_4(-s_1, s_2, s_3, s_4)$

- $F_5(s_1, s_2, s_3, s_4) = F_5(-s_1, s_2, s_3, s_4)$
- $F_6(s_1, s_2, s_3, s_4) = F_6(-s_1, -s_2, s_3, s_4)$
- $F_7(s_1, s_2, s_3, s_4) = F_7(-s_1, s_2, -s_3, s_4)$
- $F_8(s_1, s_2, s_3, s_4) = F_8(-s_1, s_2, s_3, -s_4)$
- $F_9(s_1, s_2, s_3, s_4) = F_9(s_1, -s_2, -s_3, -s_4)$

We thus have understood how X is divided into $H(s)'s$. In other words, we can construct the space homeomorphic to X by gluing 16 copies of H_3 by looking at the above correspondence.

5 Links of edges and vertices

Now, we want to understand a topological structure of X, or equivalently the union of 16 copies of H_3 indexed by $s \in \{\pm\}^4$ according to the gluing rule established in the previous section, around edges and vertices.

By symmetry, we only need to check structures about the edges A, B, C, D, E and vertices a, b, c, d indicated in Figure 11 in $H(+, -, -, -)$.
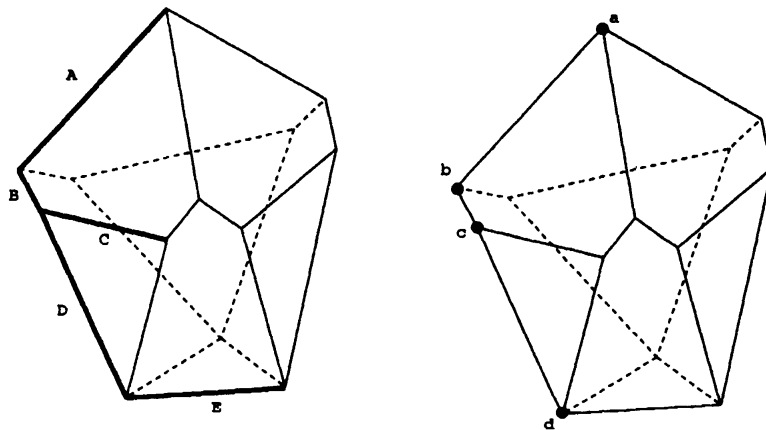

Fig. 11. Edges and Vertices to be Checked

5.1 Edges

The edge A consists of the interior points of $F_0(+, -, -, -) \cap F_2(+, -, -, -)$. If we choose p from A, we can move s_2 and s_4 freely by moving p_2 and p_6 a little as in Figure 12, Thus four $H(s)$'s share p and the link of A is a circle as in Figure 13.
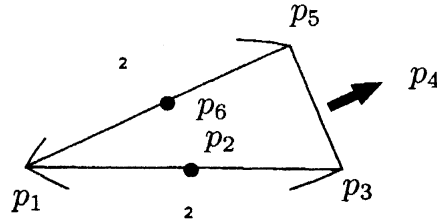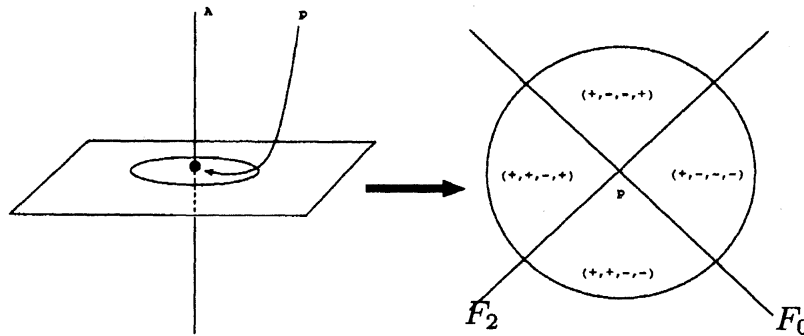
Fig. 12. $p \in A$

Fig. 13. A and its neighborhood

Similarly, we can verify that eight $H(s)$'s share B, four $H(s)$'s share C, four $H(s)$'s share D and four $H(s)$'s share E. Moreover, each edge has a link homeomorphic to a circle.

5.2 Vertices

We can easily verify that eight $H(s)$'s share a, eight $H(s)$'s share b and sixteen $H(s)$'s share c.

To count how many $H(s)$'s share the last vertex d, we let $h : f^{-1}(E_4) \to \Theta^2$ be a map defined by $h(p) = (\phi(p), \psi(p))$, where

$$\phi(p) = \arg\left(\frac{p_4 - (p_3 + p_5)/2}{p_2 - (p_1 + p_3)/2}\right), \quad \psi(p) = \arg\left(\frac{p_6 - (p_5 + p_1)/2}{p_2 - (p_1 + p_3)/2}\right).$$

The restriction of h to $H(s) \cap f^{-1}(E_4)$ is equal to the restriction of $h_s : W_4(s) \to \Theta^2$ to $W_4(s) \cap f^{-1}(E_4)$. For each $s \in \{\pm\}^4$, $h(H(s) \cap f^{-1}(E_4))$ is a triangle on Θ^2 as in Figure 14 and 6 triangles share a vertex. This means 12 $H(s)$'s share d.
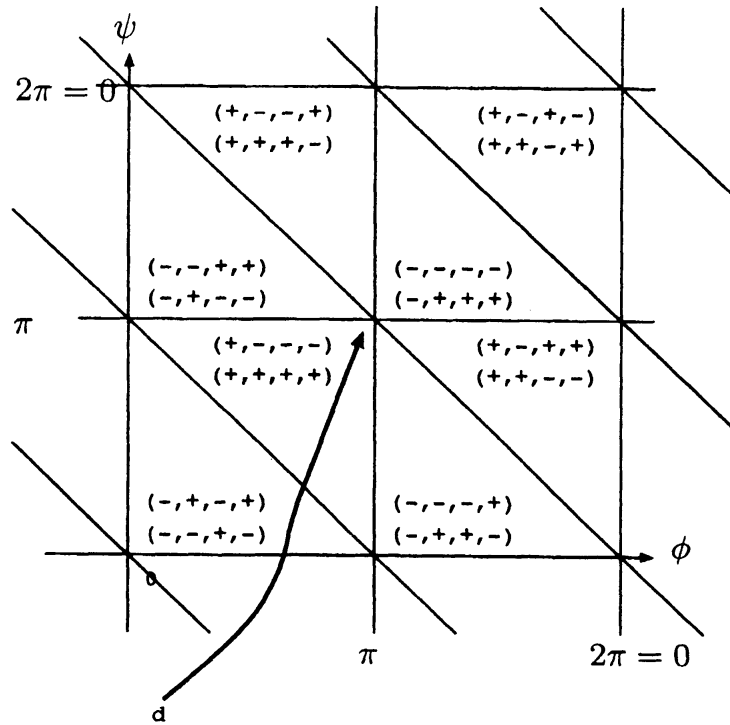
Fig. 14. $h(f^{-1}(E_4))$

Now we know how many $H(s)$'s share an edge and how many $H(s)$'s share a vertex. Using these information we can calculate the Euler characteristic of the link L_v of a vertex v, denoted by $\chi(L_v)$, so that $\chi(L_a) = 2 = \chi(L_c)$ and $\chi(L_b) = 0 = \chi(L_d)$. It is routine and easy to see that these link are connected and orientable and hence L_a, L_c are homeomorphic to a sphere and L_b, L_d a torus.

Since the link of a point that is not a vertex in $X = \cup_{s \in \{\pm\}^4} H(s)$ is homeomorphic to a sphere, all points whose link is homeomorphic to a torus are the vertices arising from type b and d. There are 10 such hexagons and each of them is contained in a straight line. Remember

Definition 6. A polygon is *degenerate* if it is contained in a straight line,

which is due to Kapovich and Millson in [6]. Since there are exactly 10 degenerate hexagons in X, a hexagon in X is degenerate if and only if it is of type b or d. Thus, we can conclude from the argument above that

Lemma 4. *The link of a degenerate equilateral hexagon is homeomorphic to a torus. The link of a nondegenerate equilateral hexagon is homeomorphic to a sphere.*

6 Results

Here we would like to conclude the results obtained in the previous sections. Recall that X' is the space of nondegenerate equilateral hexagons.

Theorem 1. *The moduli space X of equilateral hexagon is not a manifold though the moduli space X' of nondegenerate equilateral hexagons is a manifold. The non manifold points consists of 10 degenerate hexagons. The link of each degenerate hexagon is homeomorphic to a torus.*

Let us denote $X' \cap H(s)$ by $H'(s)$ for each $s \in \{\pm\}^4$. $H'(s)$ can be identified with H_3 minus vertices which correspond to of type b and d. Assigning to each edge the number of $H'(s)$'s which share it, we obtain an orbifold structure on each $H'(s)$. Then, by applying the theorem of Koebe-Andreev-Thurston [5], [1], [9], we find a unique hyperbolic structure on $H'(s)$. The gluing rule provides isometries between faces, and we have

Theorem 2. *The moduli space X' of non degenerate equilateral hexagons admits a hyperbolic structure of finite volume.*

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