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Singular domains in higher dimensional complex dynamics

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This article aims to extend the fundamental Cremer theorem from the iteration theory of one complex variable to the setting of higher-dimensional dynamics over more general valued fields, not necessarily $\mathbb{C}$. This article is an announcement of the preprint [Oku2].

Projective spaces over valued fields. Let K be a commutative algebraically closed field which is complete and nondiscrete with respect to a non-trivial absolute value (or valuation) $|\cdot|$. This $|\cdot|$ is said to be *non-Archimedean* if $\forall z, \forall w \in K, |z-w| \leq \max\{|z|,|w|\}$. Otherwise, $|\cdot|$ is said to be *Archimedean* and K is then topologically isomorphic to $\mathbb{C}$ (with Hermitian norm). We extend $|\cdot|$ to K^{ℓ} ($\ell \in \mathbb{N}$) as the maximum norm $|Z| = |Z|_{\ell} = \max_{j=1,\dots,\ell}|z_j|$ for $Z = (z_1, \dots, z_{\ell})$. Let $\pi : K^{n+1}\setminus\{O\} \to \mathbb{P}^n(K)$ be the canonical projection and set $\ell(n) \in \mathbb{N}$ so that $\bigwedge^2 K^{n+1} \cong K^{\ell(n)}$. The *chordal distance* $[\cdot,\cdot]$ on $\mathbb{P}^n(K)$ is defined as

$$[z,w] := \frac{|Z \wedge W|_{\ell(n)}}{|Z|_{n+1}|W|_{n+1}},$$

where $Z \in \pi^{-1}(z), W \in \pi^{-1}(w)$ (cf. [KS]). For $z_0 \in \mathbb{P}^n(K)$ and $r > 0$, we consider the ball

$$\overline{B}(z_0, r) := \{z \in \mathbb{P}^n(K); [z, z_0] \leq r\}.$$

Nonlinearity of morphisms. Let $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ be a (finite) *morphism*, i.e., there is a homogeneous polynomial map $F : K^{n+1} \to K^{n+1}$ over K, which is called a *lift* of f, such that $F^{-1}(O) = \{O\}$ and satisfies

$$\pi \circ F = f \circ \pi.$$

The degree $d = \deg f$ is that of F as homogeneous polynomial map. As in the case of $K = \mathbb{C}$, the *Fatou set* $F(f)$ is the largest open set at each point of which the family $\{f^k; k \in \mathbb{N}\}$ is equicontinuous.

The *Julia set* $J(f)$ is defined by $\mathbb{P}^n(K) \setminus F(f)$. In non-Archimedean case, $J(f)$ may be empty even if $d \geq 2$. One of the main results is

Theorem 1 (nonlinearity of morphisms). *Let $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ be a morphism of degree $d \geq 1$. If there are a ball $\overline{B}(z_0, r) \subset \mathbb{P}^n(K)$ and a morphism $g : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ such that*

$$\liminf_{k \to \infty} \frac{1}{d^k} \log \sup_{\overline{B}(z_0,r)} [f^k, g] = -\infty,$$

then either f is linear or $J(f) = \emptyset$.

We give a few applications of Theorem 1.

Analytic linearization over a field *K*. Consider the K-algebra

$$\mathcal{O}_\ell \cong K\{X_1, \dots, X_\ell\} = \{f = \sum c_I X^I; \limsup_{|I| \to \infty} |c_I|^{1/|I|} =: r_f^{-1} < \infty\}$$

of all germs of analytic functions at the origin $O \in K^\ell$. Here $I = (i_1, \dots, i_\ell) \in \mathbb{Z}^\ell_{\geq 0}$ is a multi-index, $X_1^{i_1} \cdots X_\ell^{i_\ell}$ is denoted by X^I and we put $|I| := i_1 + \cdots + i_\ell$. For germ of analytic *map* $\phi = (f_1, \dots, f_n) \in (\mathcal{O}_n)^n$, we identify the linear part of $\phi - \phi(O)$ at O with

$$A_\phi := \left(\frac{\partial f_i}{\partial X_j}(O) \right)_{i,j=1,\dots,n} \in M(n, K) \cong \mathrm{End}(K^n).$$

We also denote the operator norm on $M(n, K)$ by $|\cdot|$.

A germ $\phi = (f_1, \dots, f_n) \in (\mathcal{O}_n)^n$ fixing O is (analytically) *linearizable* if there is $H \in (\mathcal{O}_n)^n$ fixing O such that $A_H = I_n$ (unit matrix) and H satisfies the *Schröder* (or Poincáre) equation

$$\phi \circ H = H \circ A_\phi.$$

From Siegel and Sternberg ([Sie], [Ste]) and its non-Archimedean version by Herman-Yoccoz [HY], ϕ is linearizable if A_ϕ is diagonalizable and its eigenvalues $\lambda_1, \dots, \lambda_n$ satisfy the *Diophantine* condition: there exist $C > 0$ and $\beta \geq 0$ such that for every $I \in \mathbb{Z}^n_{\geq 0}$ (multi-index) with $|I| \geq 1$,

$$|(\lambda_1, \dots, \lambda_n)^I - 1| \geq \frac{C}{|I|^\beta}.$$

On the other hand, consider an inverse of a coordinate chart

$$\sigma : K^n \ni (z_1, \dots, z_n) \mapsto (1 : z_1 : \cdots : z_n) \in \mathbb{P}^n(K).$$

When a morphism $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ fixes a point $z_0 \in \mathbb{P}^n(K)$, assuming that $z_0 = \sigma(O)$ without loss of generality, we say f to be *linearizable* at z_0 if the germ $\phi_f \in (\mathcal{O}_n)^n$ of the analytic map $\sigma^{-1} \circ f \circ \sigma : \overline{P}^n(O, r) \to K^n$ is linearizable. The following is regarded as a higher dimensional version of the Cremer condition [Cre, p. 157].

Theorem 2 (nonresonance). *Let $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ be a morphism of degree $d \geq 2$ which fixes $z_0 \in \mathbb{P}^n(K)$, and suppose that $J(f) \neq \emptyset$. If f is linearizable at z_0 and $|A_{\phi_f}| \leq 1$, then*

$$\liminf_{k \to \infty} \frac{1}{d^k} \log |(A_{\phi_f})^k - I_n| > -\infty.$$

If in addition A_{ϕ_f} is diagonalizable, then its eigenvalues $\lambda_1, \dots, \lambda_n$ satisfy

$$\liminf_{k \to \infty} \frac{1}{d^k} \log \max_{j=1,\dots,n} |\lambda_j^k - 1| > -\infty.$$

Singular domain over the field $\mathbb{C}$. Let $f : \mathbb{P}^n = \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n$ be a morphism, which is now holomorphic, of degree $d \geq 2$.

Each component D of $F(f)$, which is called a *Fatou component* of f, is Stein and Kobayashi hyperbolic [Ued1]. In particular, D is holomorphically separable and the biholomorphic automorphisms $\mathrm{Aut}(D)$ is a Lie group. When there is a sequence $(f^{k_j}) \subset \{f^k\}$ which converges to Id_D locally uniformly on D, we have $f^p(D) = D$ for some $p \in \mathbb{N}$ and moreover $f^p|D \in \mathrm{Aut}(D)$. Following Fatou [Fat, §28], we call such D a *singular domain* (un domaine *singulier*) of f. A singular domain is also called a *Siegel* domain or *rotation* domain. When $n = 1$, a singular domain D is either a Siegel disk or an Herman ring. When $n \geq 2$, a partial analogue is known: let G be the closed subgroup generated by $f^p|D$ in $\mathrm{Aut}(D)$, and G_0 the component of G containing Id_D. Then there is a Lie group isomorphism $G_0 \to \mathbb{T}^s$ for some $s \in [1, n]$, which maps $f^q|D$ for some $q \in \mathbb{N}$ to $(e^{2i\pi\alpha_1}, \dots, e^{2i\pi\alpha_s})$ for some $\alpha_1, \dots, \alpha_s \in \mathbb{R} \setminus \mathbb{Q}$ (see [FS1], [Ued2], [Mih]). In the maximal case of $s = n$, we say the singular domain D to be *of maximal type.*

A singular domain D of maximal type is exactly a generalization of one-dimensional Siegel disks and Herman rings: setting $\lambda_j := e^{2i\pi\alpha_j}$ ($j = 1, \dots, n$), we have by [BBD, Theorem 1] a biholomorphic homeomorphism Φ from a Reinhardt domain $U \subset \mathbb{C}^n$ to D such that the Schröder equation

$$f^q(\Phi(w_1, \dots, w_n)) = \Phi(\lambda_1 w_1, \dots, \lambda_n w_n) \quad \text{on } U$$

holds.

Theorem 3 (a priori bound). *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$. If a singular domain D of f is of maximal type, then under the same notation as in the above, D satisfies*

$$\lim_{k\to\infty} \frac{1}{d^{qk}} \log \max_{j=1,\dots,n} |\lambda_j^k - 1| = 0.$$

In the case of $n = 1$, every singular domain of f is of maximal type. In this case, Theorem 3 is essentially proved in [FS2, p. 169] by pluripotential theory, and in [Oku1, Main Theorem 3] by a Nevanlinna theoretical argument. Both proofs contain some one-dimensional arguments which are not easily extended to higher dimensions. Our proof of Theorem 3 is based on a proof of Theorem 1, which dispenses with pluripotential theory.

Finally, we give a *vanishing* result on the Valiron deficiency

$$\delta_V(\mathrm{Id}_{\mathbb{P}^n}, (f^k)) := \limsup_{k\to\infty} \frac{1}{d^k} \int_{\mathbb{P}^n} \log \frac{1}{[f^k, \mathrm{Id}]} d\omega_{FS}^{\wedge n}$$

(cf. [DO]). Here ω_{FS} denotes the Fubini-Study Kähler form on $\mathbb{P}^n$.

Theorem 4 (a vanishing theorem). *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree ≥ 2. If every singular domain of f is of maximal type, then*

$$\delta_V(\mathrm{Id}_{\mathbb{P}^n}, (f^k)) = 0.$$

We expect that the assertion of Theorem 4 still remains true with no maximality assumption on singular domains.

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