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Monodromy of Painlevé VI Equation Around Classical Special Solutions*

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Abstract

A global structure of the sixth Painlevé equation is described by its nonlinear monodromy map along a loop, and it is interesting to investigate its dynamical properties around classical special solutions, that is, around Gauss hypergeometric function solutions. In a generic situation one sees that the monodromy map admits a horseshoe and thus exhibits a chaotic behavior in any small neighborhood of the classical solutions.

1 Introduction

This is a report of a work [11] in progress concerning the monodromy of the sixth Painlevé equation and the associated dynamical system created by a monodromy map.

In general, a total understanding of the Painlevé equation would be achieved by the scheme in Table 1, in which some typical issues in various scales are listed, from microlocal to macroscopic levels. Recent works by the author and his coworkers are mainly concerned with global-to-macroscopic structures of the Painlevé equation. Usually, some properties of this equation have been studied from the viewpoint of isomonodromic deformations, but this approach is often too local in many respects. One should take more global points of view.

A global structure of the Painlevé equation is represented by the nonlinear monodromy map (of a single turn along a given loop). A clear picture of this part is made by establishing a very precise Riemann-Hilbert correspondence based on a suitable moduli theory in algebraic geometry. An even more global (namely, macroscopic) structure of the equation is represented by the iterations of the monodromy map, that is, by infinitely many turns of the loop. Dynamical systems theory and ergodic theory come into context at this stage.

In the linear case of Gauss hypergeometric equation, the monodromy map of a single turn and its iterations of infinitely many turns make no essential difference, since the former is only a linear map and the dominant effect of the latter is controlled by the spectral data

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microlocal	local	global	mcroscopic
instanton solutions	differential equation itself	monodromy (a single turn)	monodromy (∞-many turns)
exact WKB analysis	isomonodromic deformation	Riemann-Hilbert correspondence	dynamical system ergodic theory

Table 1: A total understanding of Painlevé equation

of the former, namely, by the largest eigenvalue and its eigenspace. In the nonlinear case of Painlevé equation, there exists a large gap between the single turn and the infinitely many turns, due to the "nonlinear effect" of Painlevé equation. The analysis of the latter requires advanced methods from dynamical systems theory and ergodic theory. But this leads to the new feature of a chaotic dynamical system, which never exists in Gauss equation and which makes the global structure of Painlevé equation much more interesting than that of Gauss equation. We are interested in such an aspect of Painlevé equation.

The main focus of this paper is on a chaotic nature of Painlevé equation around classical special solutions, that is, around Gauss hypergeometric function solutions (or in other words, Riccati solutions). The Riccati solutions are parametrized by a curve called the Riccati curve. In this paper we announce the following result: In any small neighborhood of the Riccati curve the nonlinear monodromy map admits a Smale horseshoe and thus exhibits a very complicated dynamical behavior, for almost all loops and for almost all parameters for which Painlevé equation admits Riccati solutions. See Result 4 for the precise statement.

2 The Sixth Painlevé Equation

The sixth Painlevé equation $P_{VI}(\kappa)$ is a Hamiltonian system

$$\frac{dq}{dz} = \frac{\partial H(\kappa)}{\partial p}, \qquad \frac{dp}{dz} = -\frac{\partial H(\kappa)}{\partial q}, \tag{1}$$

with a complex time variable $z \in Z := \mathbb{P}^1 - \{0, 1, \infty\}$ and unknown functions $q = q(z)$ and $p = p(z)$, depending on complex parameters κ in the four-dimensional affine space

$$\mathcal{K} := \{\, \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5_\kappa \,:\, 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \,\},$$

where the Hamiltonian $H(\kappa) = H(q, p, z; \kappa)$ is given by

$$z(z-1)H(\kappa) = (q_0 q_z q_1)p^2 - \{\kappa_1 q_1 q_z + (\kappa_2 - 1)q_0 q_1 + \kappa_3 q_0 q_z\}p + \kappa_0(\kappa_0 + \kappa_4)q_z,$$

with $q_\nu := q - \nu$ for $\nu \in \{0, z, 1\}$. Note that $P_{VI}(\kappa)$ fails to make sense at $z = 0, 1, \infty$. These points are called the fixed singular points of the Painlevé equation $P_{VI}(\kappa)$.

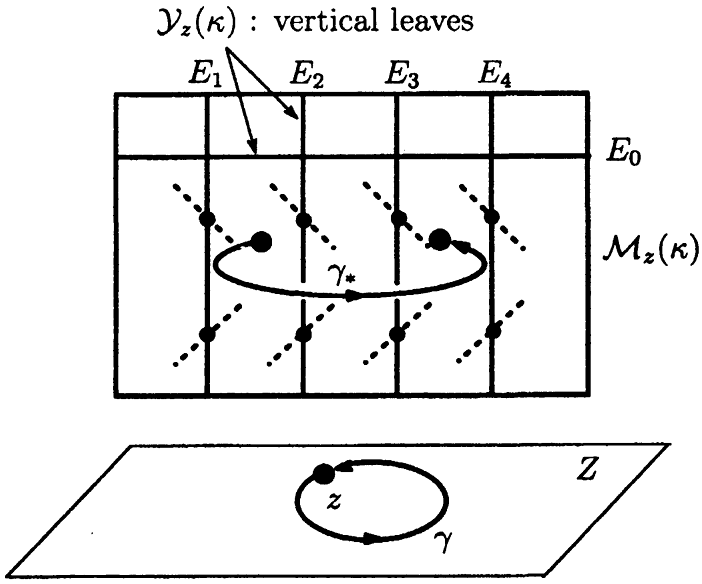

Figure 1: Monodromy map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ along a loop $\gamma \in \pi_1(Z, z)$.

3 Moduli Theory

Let $\mathcal{M}_z(\kappa)$ be the set of all meromorphic solution germs to $\mathrm{P_{VI}}(\kappa)$ at a base point $z \in Z$. The set $\mathcal{M}_z(\kappa)$ can be realized as the moduli space of (certain) stable parabolic connections, so that it can be equiped with the structure of a smooth quasi-projective rational complex surface [6, 7, 8], where a stable parabolic connection is a rank-two vector bundle over $\mathbb{P}^1$ together with a Fuchsian connection having four regular singular points and a parabolic structure that satisfies a sort of stability condition in geometric invariant theory.

Moreover there exists a natural compactification of the moduli space

$$\mathcal{M}_z(\kappa) \hookrightarrow \overline{\mathcal{M}}_z(\kappa),$$

where $\overline{\mathcal{M}}_z(\kappa)$ is the moduli space of stable parabolic phi-connections. Here, roughly speaking, a stable parabolic phi-connection "$\nabla = \phi \otimes d + A$" is a variant of stable parabolic connection allowing a "matrix-valued Planck constant" ϕ, called a phi-field (that may be degenerate or semi-classical). The compactified modulis space $\overline{\mathcal{M}}_z(\kappa)$ has a unique anti-canonical effective divisor $\mathcal{Y}_z(\kappa)$, which has the irreducible decomposition

$$\mathcal{Y}_z(\kappa) = 2E_0 + E_1 + E_2 + E_3 + E_4. \tag{2}$$

The objects on $\mathcal{Y}_z(\kappa)$ are exactly those with degenerate phi-field ϕ, where the coefficients of the irreducible decomposition (2) stand for the ranks of degeneracy of ϕ. Thus one has

$$\mathcal{M}_z(\kappa) = \overline{\mathcal{M}}_z(\kappa) - \mathcal{Y}_z(\kappa),$$

and there exists a holomorphic two-form $\omega_z(\kappa)$ on $\mathcal{M}_z(\kappa)$, meromorphic on $\overline{\mathcal{M}}_z(\kappa)$ with pole divisor $\mathcal{Y}_z(\kappa)$. It is unique up to constant multiples and yields a natural holomorphic area-form on the moduli space $\mathcal{M}_z(\kappa)$.

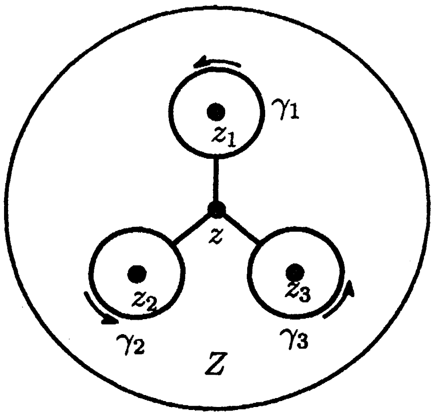

Figure 2: Three basic loops in $\pi_1(Z, z)$, where $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$.

4 Nonlinear Monodromy

It is known that $P_{VI}(\kappa)$ enjoys the Painlevé property, that is, any solution germ $Q \in \mathcal{M}_z(\kappa)$ can be continued analytically along any loop $\gamma \in \pi_1(Z, z)$ as a meromorphic function. Thanks to this property, the monodromy map along the loop γ,

$$\gamma_* : \mathcal{M}_z(\kappa) \to \mathcal{M}_z(\kappa), \qquad Q \mapsto \gamma_* Q, \tag{3}$$

is well defined, where $\gamma_* Q$ is the result of the analytic continuation (see Figure 1). It is a holomorphic automorphism of $\mathcal{M}_z(\kappa)$ preserving the holomorphic area-form $\omega_z(\kappa)$.

We are interested in the dynamics of the monodromy map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ along a given loop $\gamma \in \pi_1(Z, z)$. The fundamental group $\pi_1(Z, z)$ is represented as

$$\pi_1(Z, z) = \langle\, \gamma_1, \gamma_2, \gamma_3 \,|\, \gamma_1\gamma_2\gamma_3 = 1 \,\rangle,$$

where γ_i $(i = 1, 2, 3)$ are the basic loops as in Figure 2, with $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$.

Definition 1 A loop $\gamma \in \pi_1(Z, z)$ is said to be *elementary* if γ is conjugate to the loop γ_i^m for some $i \in \{1, 2, 3\}$ and $m \in \mathbb{Z}$, namely, if it makes a finite number of turns around only one of the three fixed singular points. Otherwise, γ is said to be *non-elementary*.

The dynamics along an elementary loop is relatively simpler [10, 13] and we are more interested in the dynamics along a non-elementary loop.

5 Riccati Curves

For particular parameters κ of codimension one in $\mathcal{K}$, there exist particular solutions to $P_{VI}(\kappa)$ that can be expressed in terms of Gauss hypergeometric functions. They are known as Riccati solutions, as they appear as solutions to the Riccati equation associated with a Gauss equation. Let $\mathcal{E}_z(\kappa)$ be the set of all Riccati solution germs to $P_{VI}(\kappa)$ at the base point z. It is known that $\mathcal{E}_z(\kappa)$ is an algebraic set in $\mathcal{M}_z(\kappa)$, each irreducible component of which must be a (-2)-curve in $\mathcal{M}_z(\kappa)$, that is, a curve isomorphic to $\mathbb{P}^1$ and of self-intersection

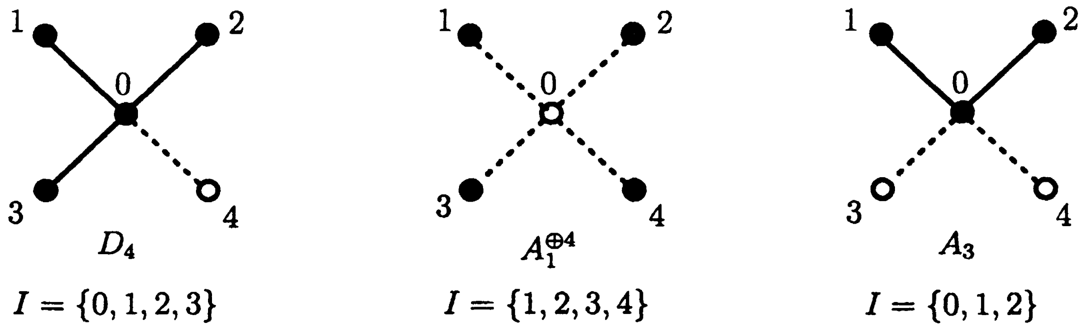

Figure 3: Some strata and their abstract Dynkin types

number -2. Conversely, any (-2)-curve in $\mathcal{M}_z(\kappa)$ is an irreducible component of $\mathcal{E}_z(\kappa)$. For this reason a (-2)-curve is called a Riccati curve. We can think of the dual graph of $\mathcal{E}_z(\kappa)$ which encodes the intersection relations among the Riccati curves in $\mathcal{M}_z(\kappa)$.

6 Affine Weyl Groups

The configuration of Riccati curves in $\mathcal{M}_z(\kappa)$ can most clearly be described in terms of some affine Weyl group structures and an associated stratification on $\mathcal{K}$ (see Lemma 2). Consider the (complex) inner product on $\mathcal{K}$ induced from the standard Euclidean inner product on $\mathbb{C}^4_\kappa$ through the forgetful isomorphism $\mathcal{K} \to \mathbb{C}^4_\kappa$, $\kappa \mapsto (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. For each $i \in \{0, 1, 2, 3, 4\}$ let $w_i : \mathcal{K} \circlearrowleft$ be the orthogonal reflection in the affine hyperplane $H_i := \{\kappa \in \mathcal{K} \,:\, \kappa_i = 0\}$. These five reflections generate an affine Weyl group of type $D_4^{(1)}$,

$$W(D_4^{(1)}) = \langle w_0, w_1, w_2, w_3, w_4 \rangle \curvearrowright \mathcal{K}.$$

Denote the nodes of the Dynkin diagram $D_4^{(1)}$ by $\{0, 1, 2, 3, 4\}$, where 0 represents the central node. The automorphism group of the Dynkin diagram $D_4^{(1)}$ is the symmetric group S_4 of degree 4 permuting $\{1, 2, 3, 4\}$ while fixing the central node 0. The semi-direct product

$$W(F_4^{(1)}) := W(D_4^{(1)}) \rtimes S_4 \curvearrowright \mathcal{K}$$

is an affine Weyl group of type $F_4^{(1)}$, which is the full symmetry group of Painlevé VI.

7 Stratification

There exists a natural stratification of $\mathcal{K}$, namely, the one by proper subdiagrams of the Dynkin diagram $D_4^{(1)}$, which we shall now describe. Let $\mathcal{I} := \{I \subset \{0, 1, 2, 3, 4\}\}/S_4$ be the set of all proper subsets of $\{0, 1, 2, 3, 4\}$, including the empty set $\emptyset$, up to the action of S_4. Note that each element of $\mathcal{I}$ represents the abstract Dynkin type of a proper subdiagram of $D_4^{(1)}$. For each $[I] \in \mathcal{I}$ with $I \subset \{0, 1, 2, 3, 4\}$ we put

$$\begin{aligned} \overline{\mathcal{K}}([I]) &= \text{the } W(F_4^{(1)})\text{-translates of the affine subspace } H_I := \bigcap_{i \in I} H_i, \\ \mathcal{K}([I]) &= \overline{\mathcal{K}}([I]) - \bigcup_{|J| = |I|+1} \overline{\mathcal{K}}([J]), \quad \text{where } |I| \text{ denotes the cardinality of } I, \end{aligned}$$

$$\begin{array}{ccccccccc} \emptyset & \longrightarrow & A_1 & \longrightarrow & A_1^{\oplus 2} & \longrightarrow & A_1^{\oplus 3} & \longrightarrow & A_1^{\oplus 4} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A_2 & \longrightarrow & A_3 & \longrightarrow & D_4 & & \end{array}$$

Figure 4: Adjacency relations among the strata

The sets $\mathcal{K}(*)$ with $* \in \mathcal{I}$ define a stratification of $\mathcal{K}$. For $I = \emptyset$ one has the big open stratum $\mathcal{K}(\emptyset)$ and some other strata are given in Figure 3. The adjacency relations among the strata are depicted in Figure 4, where $* \to **$ indicates that $\mathcal{K}(**)$ is in the closure of $\mathcal{K}(*)$.

Lemma 2 *If $\kappa \in \mathcal{K}(*)$ with $* \in \mathcal{I}$, then the dual graph of $\mathcal{E}_z(\kappa) \subset \mathcal{M}_z(\kappa)$ is the Dynkin graph of type $*$. In particular $\mathcal{M}_z(\kappa)$ contains no Riccati curve precisely when $\kappa \in \mathcal{K}(\emptyset)$.*

8 Dynamics around a Riccati Curve

Assume that $\kappa \in \mathcal{K}(A_1)$ for simplicity. Recall that H_0 is the hyperplane in $\mathcal{K}$ defined by the equation $\kappa_0 = 0$, namely, by $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1$. Let $H_0^{\times}$ denote the set of all points lying on H_0 but not on any other $D_4^{(1)}$ reflection hyperplane, that is,

$$\kappa_i = m, \qquad \kappa_1 \pm \kappa_2 \pm \kappa_3 \pm \kappa_4 = 2m + 1 \qquad (i \in \{1,2,3,4\},\ m \in \mathbb{Z}).$$

Then any point $\kappa \in \mathcal{K}(A_1)$ can be sent to a point in $H_0^{\times}$ by applying a suitable transformation in $W(D_4^{(1)})$. Thus we may assume that $\kappa \in H_0^{\times}$ from the beginning.

If $\kappa \in H_0^{\times}$ then $\mathcal{M}_z(\kappa)$ contains a unique Riccati curve $\mathcal{E}_z(\kappa) \cong \mathbb{P}^1$. The Riccati solutions paramatrized by $\mathcal{E}_z(\kappa)$ are described as follows. The second equation of system (1) has the null solution $p \equiv 0$. Substituting this into the first equation yields the Riccati equation

$$z(z-1)q' + \kappa_1 q_1 q_z + (\kappa_2 - 1) q_0 q_1 + \kappa_3 q_0 q_z = 0,$$

which is linearized to the Gauss hypergeometric equation

$$z(1-z)f'' + \{(1 - \kappa_3 - \kappa_4) - (\kappa_2 - \kappa_4 + 1)z\} f' + \kappa_2 \kappa_4 f = 0, \tag{4}$$

via the change of dependent variable $q = \dfrac{z(1-z)}{\kappa_4} \dfrac{d}{dz} \log\{(1-z)^{-\kappa_4} f\}$. The Riccati curve $\mathcal{E}_z(\kappa)$ is just the projective space (line) associated with the solution space of equation (4).

Given a loop $\gamma \in \pi_1(Z, z)$, the nonlinear monodromy map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ restricts to an automorphism $\gamma_* : \mathcal{E}_z(\kappa) \circlearrowleft$ of the Riccati curve. It is just a Möbius transformation, arising as the projective monodromy map along γ of the hypergeometric equation (4), and thus the dynamics on $\mathcal{E}_z(\kappa)$ is very simple. Now the following problem naturally occurs to us.

Problem 3 How does the dynamics look like in a small neighborhood of $\mathcal{E}_z(\kappa)$?

As to this problem, we will see that it is very complicated, actually, chaotic in any small neighborhood of $\mathcal{E}_z(\kappa)$ for almost all parameters κ, provided that γ is a non-elementary loop.

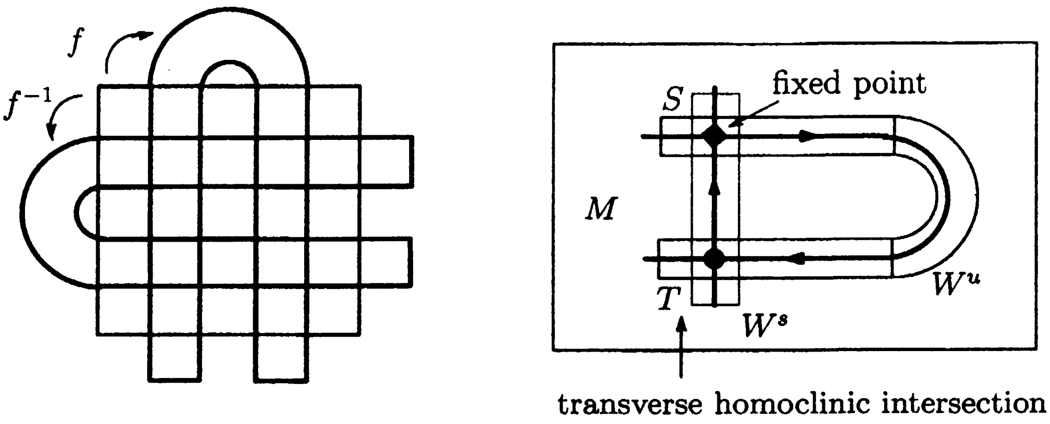

Figure 5: Horseshoe: Smale's geometric model (left) and homoclinic intersection (right).

9 Smale Horseshoe

A homeomorphism $f : M \circlearrowleft$ of a topological space M is said to admit a *horseshoe* if there exist an f-invariant Cantor subset $J \subset M$ and a homeomorphism $J \to \Sigma$ that transfers $f : J \circlearrowleft$ to the standard symbolic dynamics $\sigma : \Sigma \circlearrowleft$, where $\Sigma := \{0, 1\}^{\mathbf{Z}}$ is the topological space of bi-infinite sequences of 0's and 1's, and σ is the shift map on Σ. This abstract sense of horseshoe can be realized by Smale's famous geometric model of a horseshoe-like figure (see Figure 5, left) [19, 18]. The existence of a horseshoe gives evidence of chaos such as the positivity of topological entropy and the exponential growth of the number of periodic points as the period tends to infinity, and so on.

When $f : M \circlearrowleft$ is a diffeomorphism of a differentiable manifold M, the existence of a horseshoe is usually established through the existence of a transverse homoclinic intersection of stable and unstable manifolds (see Figure 5, right) [20, 18]. This scenario will be applied to the Painlevé dynamics in a neighborhood of a Riccati curve.

10 Main Result

Let $\gamma \in \pi_1(Z, z)$ be a non-elementary loop and assume that $\kappa \in H_0^{\times}$ as in Section 8. If the Möbius transformation $\gamma_* : \mathcal{E}_z(\kappa) \circlearrowleft$ is hyperbolic, then it admits exactly two fixed points, one of which, say P, is expanding at dilation rate $\mu = \mu(\gamma)$ and the other, say Q, is attracting at dilation rate μ^{-1} for some $|\mu| > 1$. Notice that P and Q are saddle fixed points at dilation rates $\mu^{\pm 1}$ of the map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$, since this map is area-preserving with respect to the area form $\omega_z(\kappa)$. Thus one can speak of the stable curve W^s through P and the unstable curve W^u through Q of the map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$. Here we remark that $\mathcal{E}_z(\kappa)$ is the unstable curve through P and at the same time the stable curve through Q. In order to assure the presence of a horseshoe, it is important to ask when W^s and W^u have a transverse intersection (see Figure 6). An answer to this question is given by the following.

Result 4 *For any non-elementary loop $\gamma \in \pi_1(Z, z)$ there exists a nontrivial entire function $\phi_\gamma : H_0 \to \mathbb{C}$ such that if $\kappa \in H_0^{\times} \cap \phi_\gamma^{-1}(\mathbb{C} \setminus [-1, 1])$, then*

(1) *the Möbius transformation $\gamma_* : \mathcal{E}_z(\kappa) \circlearrowleft$ is hyperbolic;*

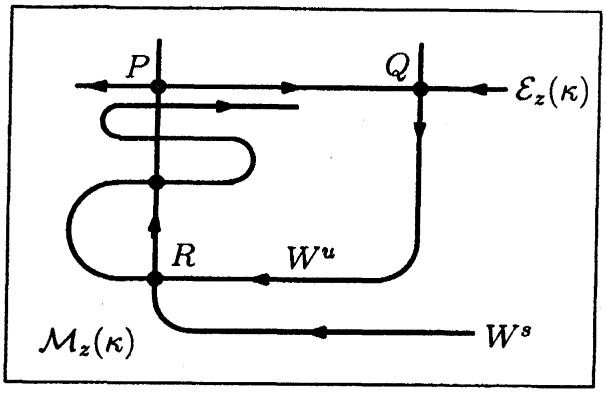

Figure 6: transverse intersection of the stable and unstable curves

(2) *the stable and unstable curves* W^s *and* W^u *have a transverse intersection; and*

(3) *there exists an* $N \in \mathbb{N}$ *such that* $\gamma_*^N : \mathcal{M}_z(\kappa) \circlearrowleft$ *admits a Smale horseshoe in any small neighborhood of the Riccati curve* $\mathcal{E}_z(\kappa)$, *where* N *depends on the neighborhood chosen.*

Here ϕ_γ *being nontrivial means that it is not a constant function with value in* $[-1,\ 1]$. *The function* $\phi_\gamma(\kappa)$ *is computable once the loop* γ *is given explicitly.*

This result may fail if $\kappa \in H_0^\times \cap \phi_\gamma^{-1}([-1,\ 1])$, but this exceptional subset is very tiny, being at most of real codimension one in $H_0^\times$, since ϕ_γ is a nontrivial entire function. In this sense the result holds for almost all parameters $\kappa \in H_0^\times$.

Example 5 We illustrate the function $\phi_\gamma(\kappa)$ for two loops.

(1) An eight-figured loop $\mathfrak{e}_{ij}$ is a loop conjugate to the loop $\gamma_i\gamma_j^{-1}$ for a cyclic permutation (i,j,k) of $(1,2,3)$ as in Figure 7 (left). If γ is an eight-figured loop $\mathfrak{e}_{ij}$, then

$$\phi_\gamma(\kappa) = \cos\pi(\kappa_i - \kappa_k) - \cos\pi(\kappa_i + \kappa_k) - \cos\pi(\kappa_j - \kappa_4).$$

(2) A Pochhammer loop $\wp_{ij}$ is a loop conjugate to $[\gamma_i, \gamma_j^{-1}] = \gamma_i\gamma_j^{-1}\gamma_i^{-1}\gamma_j$ for a cyclic permutation (i,j,k) of $(1,2,3)$ as in Figure 7 (right). If γ is a Pochhammer loop $\wp_{ij}$,

$$\begin{aligned}\phi_\gamma(\kappa) &= 2 - \cos 2\pi\kappa_1 - \cos 2\pi\kappa_2 - \cos 2\pi\kappa_3 - \cos 2\pi\kappa_4 \\ &\quad + \cos 2\pi(\kappa_1+\kappa_2) + \cos 2\pi(\kappa_2+\kappa_3) + \cos 2\pi(\kappa_3+\kappa_1).\end{aligned}$$

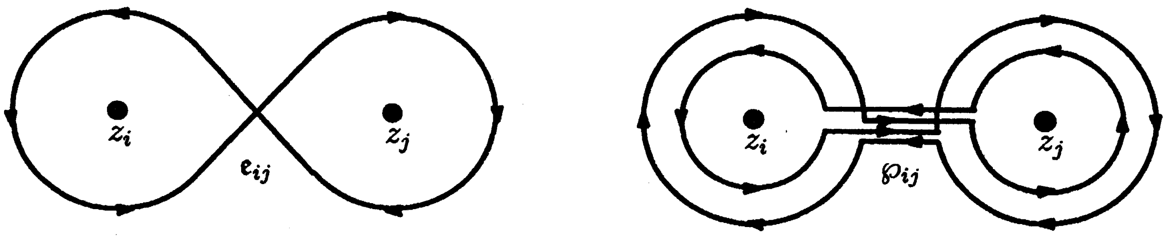

Figure 7: An eight-figured loop (left) and a Pochhammer loop (right).

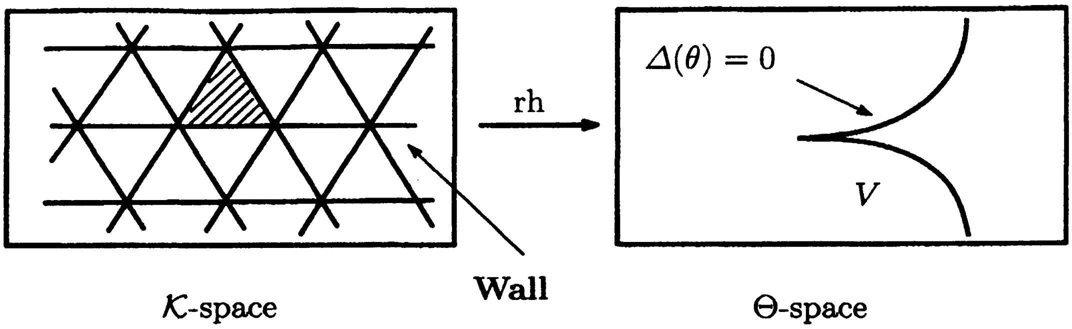

Figure 8: The Riemann-Hilbert correspondence in the parameter level

So far we have restricted our attention to the stratum $\mathcal{K}(A_1)$ for the sake of simplicity. There are similar results for the other strata. Result 4 will be shown in [11].

11 Riemann-Hilbert Correspondence

Result 4 is established, not directly on the moduli space $\mathcal{M}_z(\kappa)$, but by passing to a character variety $\mathcal{S}(\theta)$ through the Riemann-Hilbert correspondence [6, 7, 8, 10],

$$\mathrm{RH}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \mathcal{S}(\theta), \quad Q \mapsto \rho, \qquad \text{with} \quad \theta = \mathrm{rh}(\kappa). \tag{5}$$

Here the character varieties for Painlevé VI can be realized as a four-parameter family of complex affine cubic surfaces $\mathcal{S}(\theta)$ parametrized by $\theta \in \Theta := \mathbb{C}^4_\theta$ and $\mathrm{rh} : \mathcal{K} \to \Theta$ is a holomorphic map that is a branched $W(D_4^{(1)})$-covering ramifying along **Wall** (the union of all reflection hyperplanes) and mapping it onto the discriminant locus $V := \{\theta \in \Theta : \Delta(\theta) = 0\}$ of the cubics (see Figure 8). A fundamental fact for the map (5) is the following.

Theorem 6 ([6, 7, 8]) *If $\kappa \in \mathcal{K}(*)$ then the character variety $\mathcal{S}(\theta)$ with $\theta = \mathrm{rh}(\kappa)$ has simple singularities of Dynkin type $*$ and the Riemann-Hilbert correspondence* (5) *is a proper surjective holomorphic map that is an analytic minimal resolution of singularities.*

Take an *algebraic* minimal desingularization $\varphi : \widetilde{\mathcal{S}}(\theta) \to \mathcal{S}(\theta)$. Then the Riemann-Hilbert correspondence (5) uniquely lifts to a biholomorphism $\widetilde{\mathrm{RH}}_{z,\kappa} : \mathcal{M}_z(\kappa) \to \widetilde{\mathcal{S}}(\theta)$ such that

$$\begin{array}{ccc}
\mathcal{M}_z(\kappa) & \xrightarrow{\widetilde{\mathrm{RH}}_{z,\kappa}} & \widetilde{\mathcal{S}}(\theta) \\
\| & & \downarrow{\scriptstyle\varphi} \\
\mathcal{M}_z(\kappa) & \xrightarrow{\mathrm{RH}_{z,\kappa}} & \mathcal{S}(\theta)
\end{array}$$

is commutative. The lifted Riemann-Hilbert correspondence $\widetilde{\mathrm{RH}}_{z,\kappa}$ maps the Riccati locus $\mathcal{E}_z(\kappa) \subset \mathcal{M}_z(\kappa)$ isomorphically onto the exceptional set $\mathcal{E}(\theta) \subset \widetilde{\mathcal{S}}(\theta)$ of the algebraic resolution φ. The cubic surface $\mathcal{S}(\theta)$ has a natural area-form, that is, the Poincaré residue

$$\omega(\theta) = \frac{dx_1 \wedge dx_2 \wedge dx_3}{d_x f(x, \theta)},$$

where $x = (x_1, x_2, x_3)$ is the standard coordinates of $\mathbb{C}_x^3$ and $f(x, \theta) = 0$ is the defining equation of the surface $\mathcal{S}(\theta)$ in $\mathbb{C}_x^3$. The Poincaré residue $\omega(\theta)$ lifts to a holomorphic area-form $\tilde{\omega}(\theta) := \varphi^* \omega(\theta)$ on $\widetilde{\mathcal{S}}(\theta)$, with respect to which the biholomorphism $\widetilde{\mathrm{RH}}_{z,\kappa}$ is area-preserving [9]. The monodromy map $\gamma_* : (\mathcal{M}_z(\kappa), \omega_z(\kappa)) \circlearrowleft$ is strictly conjugated to an automorphism $\sigma : (\widetilde{\mathcal{S}}(\theta), \tilde{\omega}(\theta)) \circlearrowleft$, which in turn can be extended to a birational map on the natural compactification of $\widetilde{\mathcal{S}}(\theta)$. We then apply the ergodic theory of birational maps on compact surfaces [1, 2, 5, 4] to the last map in order to establish our main result.

12 Ergodic Theory

Let $\gamma \in \pi_1(Z, z)$ be a non-elementary loop. For the monodromy map $\gamma_* : \mathcal{M}_z(\kappa) \circlearrowleft$ the "recurrent" dynamics takes place only away from infinity, where the vertical leaves $\mathcal{Y}_z(\kappa)$ are thought of as the points at infinity in $\mathcal{M}_z(\kappa)$. Namely the non-wandering set $\Omega_\gamma(\kappa)$ of γ_* is compact in $\mathcal{M}_z(\kappa)$. Under the interations of γ_*, the trajectory of each initial point $Q \in \mathcal{M}_z(\kappa) \setminus \Omega_\gamma(\kappa)$ tends to infinity$\mathcal{Y}_z(\kappa)$ very rapidly.

The topological entropy $h_{\mathrm{top}}(\gamma)$ of the map $\gamma_* : \Omega_\gamma(\kappa) \circlearrowleft$ is positive, being represented as

$$h_{\mathrm{top}}(\gamma) = \log \lambda(\gamma), \qquad \lambda(\gamma) \geq 3 + 2\sqrt{2},$$

where $\lambda(\gamma)$ is a number called the dynamical degree of γ, which depends on γ but is independent of κ. There exists a unique γ_*-invariant probability measure $\mu_\gamma = \mu_\gamma(\kappa)$, with its support in $\Omega_\gamma(\kappa)$, that is mixing, hyperbolic of saddle type, and of maximal entropy. There are positive $(1,1)$-currents $\mu_\gamma^\pm$ on $\mathcal{M}_z(\kappa)$, called the stable and unstable currents, such that $\gamma_*^{\pm 1} \mu_\gamma^\pm = \lambda(\gamma) \mu_\gamma^\pm$ and the probability measure μ_γ is given by the wedge product

$$\mu_\gamma = \mu_\gamma^+ \wedge \mu_\gamma^-, \tag{6}$$

where the currents $\mu_\gamma^\pm$ have continuous potentials so that the wedge product is well defined. The saddle periodic points of γ_* are dense in $\operatorname{supp} \mu_\gamma$ and the measure is also represented as

$$\mu_\gamma = \lim_{n \to \infty} \frac{1}{\lambda(\gamma)^n} \sum_p \delta_p \qquad \text{(weak limit)},$$

where the sum is taken over all saddle points of period n and δ_p is the Dirac mass at p.

Let $D^s \subset W^s$ be a stable disk centered at $P \in \mathcal{E}_z(\kappa)$ (see Figures 6 and 9). Similarly let $D^u \subset W^u$ be an unstable disk centered at $Q \in \mathcal{E}_z(\kappa)$. Then there exist positive constants $c^\pm > 0$ such that one has weak convergence of currents

$$\lim_{n \to \infty} \frac{1}{\lambda(\gamma)^n} [\gamma_*^{\mp n} D^{s/u}] = c^\pm \mu_\gamma^\pm,$$

where $[D]$ denotes the current of integration defined by $\langle [D], v \rangle := \int_D v$ for a test form v. Thus the wedge product in (6) represents the geometric intersections of the stable and unstable curves $W^{s/u}$. Then some geometric structures of the invariant measure μ_γ lead to the existence of a transverse intersection of $W^{s/u}$.

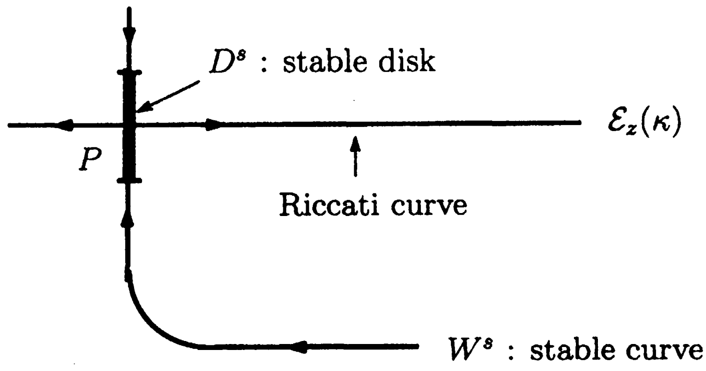

Figure 9: A stable disk D^s

13 Concluding Remark

There are two classes of classical special solutions to the sixth Painlevé equation; one is the class of Riccati solutions discussed in this paper and the other is that of algebraic solutions (see e.g. [12, 14, 17, 21]). Here a solution of the first class can be characterized in terms of a compact one-dimensional algebraic subset (a curve) in $\mathcal{M}_z(\kappa)$ invariant by the nonlinear monodromy map along every loop, while a solution of the second class can be characterized by a compact zero-dimensional algebraic subset (a set of finite points) enjoying the same invariance property [10]. Perhaps the method in this paper could also be applied to a solution of the second class in order to reveal the presence of chaos around it.

A closely related topic is the non-integrability test for a Hamiltonian system in terms of differential Galois theory developed in [16], with an application to the second Painlevé equation around a rational solution [15]. We hope that our dynamical approach would lead to a deeper result as to the "complexity" of Painlevé equations.

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