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Co-evolution and Diversity in Evolutionary Game Theory : Stochastic Environment[1)]

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Abstract

This study examines the impacts of environmental variation on the game. Here, environmental variation corresponds to the fitness/payoff variation. In mathematical biology, if the fitness is "geometric mean," we know that the player chooses a Bet-Hedging strategy in the stochastic environment, and if it is "arithmetic average," the player does not. ([4, 5]) In addition, Selten [11] shows that no mixed equilibria, i.e., Bet-Hedging strategy, are evolutionarily stable when players can condition their strategies on their roles in a game. On the other hand, we know that Nash Equilibrium in the game with randomly disturbed payoffs is always mixed strategy ([3]). Thus, these results show a discrepancy, in spite of the similar model. This study examines and clarifies this discrepancy with Replicator equation.

1 Introduction

This study examines the strategic diversity in the stochastic environment. Here, environmental variation corresponds to the payoff/fitness variation. In mathematical biology, we know that the player chooses Bet-Hedging strategy in the stochastic environment. ([4, 5]) When the environment is varying, there exist various strategies in order to protect the species from extinction. This is owing to the shape of the fitness function. If the fitness is "geometric mean," the player chooses Bet-Hedging strategy. If it is "arithmetic average," the player does not. In game theory, we usually use the von Neumann-Morgenstern utility function, arithmetic average. If we apply these factors, there is no Bet-Hedging strategy in game theory.

In game theory, Harsanyi [3] examines this problem. This study proves that mixed strategy equilibrium is approximate to pure strategy equilibrium as a best response under each player's private information in the static framework, i.e., all mixed strategies are approximately a strict equilibrium[2)] and it is an evolutionarily stable strategy (ESS)[3)]. We say that mixed strategy equilibrium of G with this property is *approachable* under ε^*. This says that there are various strategies in the stochastic environment and this condition is ESS.

On the other hand, Selten [11] proves that the only strict equilibrium of the underlying game is an evolutionary stable one in the role-completed game. Mixed strategy equilibrium is not strict equilibrium because there exist strategies that obtain the same payoff. To begin with, this game with the variation payoff for an unknown role is related to Harsanyi [3]. Therefore, this game contradicts Harsanyi [3]'s result, because there are not various strategies. Thus, it is not easy to know which pair in the strategy becomes an ESS in the stochastic environment.

This paper extends the Harsanyi type game to the dynamical framework with the Replicator equation[4)],

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[2)] **Definition:** A Nash equilibrium $x \in \Theta$ is called *strict* if each component strategy x_i is the *unique* best reply to x, that is, if $\tilde{\beta}(x) = \{x\}$

[3)] **Definition:** $x \in \Delta$ is an *evolutionarily stable strategy* (ESS) if for every strategy $y \neq x$, there exists some $\bar{\varepsilon}_y \in (0,1)$ such that the following inequality holds for all $\varepsilon \in (0, \bar{\varepsilon}_y)$

$$u[x, \varepsilon y + (1-\varepsilon)x] > u[y, \varepsilon y + (1-\varepsilon)x],$$

where $\Delta = \{x \in R_+^k : \sum_{i \in K} x_i = 1\}$, $K = \{1, 2, \cdots, k\}$.

[4)] Replicator equation :

$$\frac{\dot{x}_i}{x_i} = \left((Ax)_i - x \cdot Ax\right), \quad i = 1, \cdots, n, \quad A : \textit{payoff matrix}.$$

means that if the player's payoff from the outcome i is greater than the expected utility $x \cdot Ax$, then the probability of the action i is higher than before.

and examines the impacts of environmental variation. Here, a payoff g_i is changed at random by the stochastic environment. Thus this study examines the following:

$$\frac{dx_i(t)}{dt} = x_i(t)\big(g_i(t) - \bar{g}(t)\big), \quad g_i(t) = g_i + \zeta(t).$$

Next, this study treats Selten type game as a game with multiple group structures and formulates and analyzes this game with Replicator equation. In this case, we can understand that the law, *Price's law* in evolutionary ecology has an important role.

Therefore, this study examines the two types of researches (Harsanyi type, Selten type) using the same method, the Replicator equation. We can understand this as follows: Mixed strategy is only affected by the stochastic environment; this is approachable under the variance σ. In this case, the stability changes in the symmetric two-person game, but it does not in the asymmetric two-person game. In the role-completed game, the group size variation is obeys the Replicator equation when we interpret that a game with multiple group structures is a game with role-completed players. Thus, this study examines the various strategies in the stochastic environment and obtains the relationship between these factors and asymmetric information.

This paper is organized as follows. In § 2, we review the related literatures. In § 3, we formulate and analyze the dynamical Harsanyi [3] model. In § 4, we formulate and analyze the dynamical Selten [11] model. In § 5, we present the conclusions and discuss future work.

2 Related Literatures

Harsanyi [3] shows that, in almost any game, the force of criticism is limited, since almost any mixed strategy Nash Equilibrium is close to a strict pure strategy equilibrium in any perturbation of the game in which the player's payoffs are subject to small random variations as in the following proposition:

Proposition 1. Fix a set of I players and strategy space S_i. For a set of payoffs $\{u_i(s)\}_{i\in F, s\in S}$ of Lebesgue measure 1, for all independent, twice-differentiable distributions p_i on $\Theta_i = [-1,1]^{\#S}$, any equilibrium of the payoffs u_i is the limit as $\varepsilon \to 0$ of a sequence of pure strategy equilibria of the perturbed payoffs $\tilde{u}_i$.

This research is often used for the interpretations of mixed strategy Nash Equilibrium. This interpretation is that mixed strategy is approximation of the pure strategy as the best response under each player's private information. We can seem to be choosing mixed strategy in spite of consciously choosing the pure strategy, each player i knowing the realization $\big(\varepsilon_i(a)\big)_{a\in A}$ of ε_i but not the realizations of the other players' random variables.

On the other hand, Selten [11] shows that a strict Nash Equilibrium of the underlying game is evolutionary stable in the role-completed game. Here, the opponents assume different roles like "owner" and "intruder" in a game. The roles may be defined by a combination of several variables like ownership and size. Information about the opponent's role may be incomplete.

Selten [11] obtains the following proposition:

Proposition 2. For any underlying game G, let Γ denote the associated extensive-form game in which the interacting individuals are allocated player positions. A behavior strategy $\bar{x}$ in Γ is evolutionarily stable if and only if $\bar{x}$ is a strict Nash Equilibrium of G.

Thus, in Harsanyi [3]'s type, in which the payoff varies randomly, mixed strategy is ESS. In Selten [11]'s type, in which the opponent player's role is unknown in the role-complete game, the pure strategy is ESS. We can understand that these results indicate a discrepancy.[5)]

[5)] Binmore and Samuelson [1] reconciles these results as well: Approximations of mixed equilibria have high invasion barriers, and hence are likely to persist, when payoff perturbations are relatively important and role identification is relatively noisy. When payoff perturbations are unimportant and role identification is precise, approximations of mixed equilibria will have small invasion barriers and are unlikely to persist.

3 Dynamical Harsanyi [3] Model

Here we examine a game using two different models with the Replicator equation. First, a game of Harsanyi's type extends the dynamics with the Replicator equation.

Let $G = \langle N, (A_i), (u_i)\rangle$ be a finite strategic game and let $\varepsilon = \big(\varepsilon_i(a)\big)_{i\in N, a\in A}$ be a collection of random variables with range $[-1, 1]$, where $\varepsilon_i = \big(\varepsilon_i(a)\big)_{i\in N, a\in A}$ has a continuously differentiable density function and an absolutely continuous distribution function, and the random vectors $(\varepsilon_i)_{i\in N}$, each player i knowing the realization $\big(\varepsilon_i(a)\big)_{a\in A}$ of ε_i but not the realizations of the other players' random variables. That is, consider the Bayesian game $G(\varepsilon)$ in which the set of states of nature is the set of all possible values of the realizations of ε, the (common) prior belief of each player is the probability distribution specified by ε, the signal function of player i at the outcome a and state ε is $u_i(a) + \varepsilon_i(a)$.

In this game, we obtain the Replicator equation as follows:

$$(1) \qquad \frac{dx_i(t)}{dt} = x_i(t)\big(g_i(t) - \bar{g}(t)\big), \quad g_i(t) = g_i + \zeta(t)$$

where $\zeta(t)$ is a dynamic function with the stochastic variation around g_i. In any equilibrium, we obtain the following proposition:

Proposition 3. A strategy distribution x is satisfied as follows:

$$P(x,t)dx = (2\pi\sigma^2 t)^{-1/2} \exp\Big[-\frac{\big(\log x - \log x^*(t)\big)^2}{2\sigma^2 t}\Big]\frac{dx}{x}.$$

Proof: See appendix.

This proposition gives the distribution of the strategy in a game. This distribution obeys the lognormal distribution. Therefore, we find that there exist various strategies depending on the size of the variance σ in equilibrium. We find that the purification theorem is realized as for the variance σ. I.e., this is approachable under the variance σ.

Remark 4. In the symmetric two-person game, the stability is changed by the perturbation; however, in the asymmetric two-person game, it is unchanged.

Proof: Omit.

4 Dynamical Selten [11] Model

Second, we formulate the game with group structure, with the players assigned a role, in order to analyze Selten [11]'s model with a dynamical system. Selten [11] assigns the role with probability $\frac{1}{2}$. Here we assume that a "role" is a "group." For example, this group is altruist, egoist, etc. We examine the variation of the group size/population.

There are $n(2 \le n < \infty)$ group structures, let f_i be each group's population share. If we let π_i be group i's average payoff, the average payoff of the total population is $\bar{\pi} = \sum_{i=2}^{n} f_i\pi_i$. For notational convenience, let $\sum_i$ be $\sum_{i=2}^{n}$, $\sum_{i=1}^{n}$. If each group size is changed by the relative other group's payoff, the group i's population share is $f_i' = f_i\frac{\pi}{\bar{\pi}}$ in the next step. Let x_i be shared with some trait in group i. The population that shares this trait is $\bar{x} = \sum_i f_i x_i$.

In this game, the relational expression, Price's law, is realized (Proposition 5). This law describes the variation of a group's payoff.

Proposition 5. Group size and it's payoff in a game with group structure are as follows: Price's law : $\bar{\pi}\Delta\bar{x} = \mathrm{Cov}[\pi, x] + E[\pi\Delta x]$.

Proof: See appendix.

Thus a game with group structure obeys the Price's law. This law is similar to the Replicator equation.

Example 6. We examine the most simple symmetric two-person game with two groups (S,A) in the population.

I\II	H	D
H	a, a	$0, 0$
D	$0, 0$	b, b

Payoff Matrix

Evolutionary game theory assumes that each player plays a game with a randomly matched player. However, in this game, the ways to match include three types: [AA],[AS],[SS]. We can compute that the average payoff in each case is $\pi_{AA} = a$, $\pi_{AS} = 0$, $\pi_{SS} = b$. Let f_{AA}, f_{AS}, f_{SS} being each group's population share. If we let f be the player A's rate in the population, the probability of the matching pair AA is f^2, SS is $(1-f)^2$, AS is $1 - f_{AA} - f_{SS} = 1 - f^2 - (1-f)^2 = 2f(1-f)$. The player's rate with trait A in each group is $x_{AA} = 1$, $x_{AS} = \frac{1}{2}$, $x_{SS} = 0$ respectively. On the other hand, the expected utility of the player with trait A in a pair of AS in the next step is 0, because the ratio of these players is $\frac{1}{2}$, and its' expected utility is 0. Thus, the ratio of this player is not changed.

Here, we examine this game with Price's law $\left(\bar{\pi}\Delta\bar{x} = \text{Cov}[\pi, x] + E[\pi\Delta x]\right)$. We obtain the expected value, second term in Price's law:

$$E[\pi, \Delta x] = \sum_{i \in \{AA,AS,SS\}} f_i \pi_i \Delta x_i = 0.$$

And we obtain as follows:

$$\bar{x} = \sum_{i \in \{AA,AS,SS\}} f_i \pi_i = f$$

We obtain the covariance, the first term in Price's law:

$$\sum_{i \in \{AA,AS,SS\}} f_i \bar{\pi}(x_i - \bar{x}) = 0.$$

As mentioned above, we obtain the covariance between a payoff and the population share in the group:

$$\text{Cov}\,[\pi, x] = \sum_{i \in \{AA,AS,SS\}} f_i(\pi_i - \bar{\pi})(x_i - \bar{x}) = f(1-f)\{f(a+b) - b\}.$$

This equation is similar to the Replicator equation as before.

5 Conclusion and Remarks

Thus, we construct and analyze the Harsanyi [3] type game and the Selten [11] type game with the Replicator equation. The results are consistent with the static cases, respectively, and we obtain the following: We find that the game with the varying payoff is approachable under the variance σ (Dynamical Harsanyi [3] Model). We find that there are not various strategies in the game with incomplete information (Dynamical Selten [11] Model). Whether there are various strategies or not depends on the fitness function's shape in related literatures [4, 5]. We find that there are various strategies or not depending on the asymmetric information.

If we treat the observation noise in the global game [2, 9] as assortative matching, we can construct a dynamical global game. We can obtain a similar result; this game's equilibrium is pure strategy with the observation noise (Kikkawa [7]).

Appendix

Proof of Proposition 3.: If we transform (1), we obtain the following:

$$\log\frac{x_i(t)}{x(0)} - g_i t + \int_0^t g(t)dt = \sum_{k=1}^{n} \xi_k,$$

where time step t divided $n\tau$, τ is the short time scale, n is integer. Let the integral of each short interval be $\xi_k = \int_{(k-1)\tau}^{k\tau} \zeta(t)dt$. $\xi_k(k = 1, 2, \cdots, n)$ are n-tuples random variable with the mean value 0. In this case, central limit theorem is realized.

Theorem A.1. (central limit theorem) Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables with finite mean m and finite a non-zero variance $\sigma^2 < \infty$ and let $S_n = X_1 + X_2 + \cdots + X_n$. Then

$$\frac{S_n - nm}{\sqrt{n\sigma^2}} \to N(0,1) \text{ as } n \to \infty.$$

If we apply the above theorem to the random variables $\xi_1, \xi_2, \cdots$, let $\sigma^2\tau$ be ξ_k's variance, $x^*(t) = x_0 \exp\left[(\tilde{g} - g_i)t\right]$ $\tilde{g} = \frac{1}{t}\int_0^t \bar{g}(t)$, $\sum \sigma_i^2 = n\sigma^2\tau = \sigma^2 t$, $dX_n = \frac{dX}{X}$. The distribution of the strategy obeys the following at time t as $n \to \infty$, $\tau \to 0$,

$$P(x,t)dx = (2\pi\sigma^2 t)^{-1/2} \exp\Big[-\frac{\big(\log x - \log x^*(t)\big)^2}{2\sigma^2 t}\Big]\frac{dx}{x}.$$

This is called *lognormal distribution.*

□

Remark A.2. If we let x be e^y, we can obtain the expected value of the strategy distribution as follows.

$$\begin{aligned}\langle x\rangle &= \int_0^\infty xP(x)dx = (2\pi\sigma^2 t)^{-1/2}\int_{-\infty}^{\infty} \exp\Big[-\frac{(y-y^*)^2}{2\sigma^2 t}\Big]dx \\ &= x_0 \exp\left[\tilde{g} - g_i\right]t \exp[\sigma^2 t/2]\end{aligned}$$

This equation is the non-stochastic environmental case times $\exp[\sigma^2 t/2]$.

Proof of Proposition 5.: Let π_i' and x_i' be the next step group i's payoff and the player's rate, with a trait in group i. We can obtain $\bar{x}' = \sum_i f_i' \cdot x_i'$. If we assume $\Delta x_i = x_i' - x_i$, we obtain as follows:

$$\begin{aligned}\bar{x}' - \bar{x} &= \sum_i f_i' x_i' - \sum_i f_i x_i = \sum_i f_i \frac{\pi_i}{\bar{\pi}} x_i' - \sum_i f_i x_i \\ &= \sum_i f_i \frac{\pi_i}{\bar{\pi}}(x_i + \Delta x_i) - \sum_i f_i x_i = \sum_i f_i\Big(\frac{\pi_i}{\bar{\pi}} - 1\Big)x_i + \sum_i f_i \frac{\pi_i}{\bar{\pi}}\Delta x_i\end{aligned}$$

If we assume $\Delta\bar{x} = \bar{x}' - \bar{x}$ and multiply the above equation by $\bar{\pi}$, we obtain the as follows:

$$\text{(A.1)} \qquad \bar{\pi}\Delta\bar{x} = \sum_i f_i(\pi - \bar{\pi})x_i + \sum_i f_i \pi_i \Delta x_i$$

The right side's second term is $\pi\Delta x$'s expected value, $E[\pi\Delta x]$. The covariance between π and x is $\text{Cov}[\pi, x] = \sum_i f_i(\pi_i - \bar{\pi})(x_i - \bar{x})$. Because of $\sum_i f_i(\pi_i - \bar{\pi})\bar{x} = 0$, the right side's first term in (A.1) is equivalent to $\text{Cov}[\pi, x]$. We can transform (A.1) as follows:

$$\bar{\pi}\Delta\bar{x} = \text{Cov}[\pi, x] + E[\pi\Delta x].$$

□

Remark A.3. Thus, Price's law is equivalent to the Replicator equation.[6)]

[6)] Page and Nowak [10] shows this is equivalent to the Replicator-Mutator equation ($\dot{x}_i = \sum_{j=1}^{n} x_j f_j(\mathbf{x}) q_{ji} - x_i f$), more generally than Replicator equation. The Replicator-Mutator equation is frequently used as a Learning Equation. In the

Proof: Omit.

Remark A.4. We can derive Fisher's second theorem, "The trait's average variation rate is equivalent to the covariance between trait and fitness" from Price's law. In evolutionary game theoretic framework, we can interpret "Population variation in group is equivalent to the covariance between "population size in group and payoff (fitness)".

We abridge the right hand side's second term in Price's law, $\dot{E}(p)$ =Cov (f, p). This is called *Fisher's second Theorem.*

Proof: Omit.

Remark A.5. If h is replaced by g in the second theorem, we obtain the first theorem.

Proof: Omit.

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Replicator equation, a strategy's share increases (decrease) when a strategy's payoff is more (less) than the average one. However, in Replicator-Mutator the probability of a strategy's increasing (decreasing) is dependent on the value of q_{ji}. I.e., this differential equation is taken account of learning's unsuccessful.