Traveling wave solutions to a malignant tumor invasion model
(悪性腫瘍の浸潤に関連するモデルの進行波解)

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1. Introduction

This report is an announcement of the results on traveling wave solutions (TWSs) to a system of partial differential equations related to malignant tumor invasion

$$\begin{cases} u_t = -(uc_x)_x + u(1-u), \\ c_t = -uc^2, \end{cases} \qquad x \in \mathbf{R}, \quad t > 0, \qquad (1)$$

which is numerically studied by Perumpanani et al. [3]. Here, $u(x,t)$ and $c(x,t)$ stand for the concentration of tumor cells and of connective tissue at position x and time t, respectively. Roughly speaking, tumor invasion is a phenomenon which a malignant tumor spreads partially while degrading contiguous healthy tissue. For the details of physiological and pathological backgrounds of tumor invasion phenomena, refer the reader to [4], [2], [3] and the references therein. Note that in the system (1) tumor cell diffusion is ignored. The constant states of the system (1) are $(1,0)$ and $(0,\hat{C})$ with arbitrary constant $\hat{C}$, so that we assume that $(u,c) = (1,0)$ and $(0,\hat{C})$ corresponds to a malignant tumor (as $x \to -\infty$) and to healthy tissue (as $x \to \infty$) with some positive constant $\hat{C}$, respectively. We think that it is important for us to study TWSs connecting $(1,0)$ and $(0,\hat{C})$ as the first step to understand a mathematical tumor invasion model (1).

Marchant, Norbury and Perumpanani have considered whether non-smooth TWSs (including discontinuous ones) to (1) exist or not mainly with the use of the numerical methods in [2]. Their numerical simulations have been performed to get solution orbits to the system of ordinary differential equations (ODEs) which TWSs must satisfy (see (3) below) arriving at a specific point H called the hole in the wall (see (4) below). Their assertions are summarized as follows:

(i) although the combination of the orbit from $(1,0)$ to H and that from H to $(0,\hat{C})$ is not differentiable at H, it is allowed to be a weak solution to the system (3) of ODEs, accordingly a TWS to (1),

(ii) orbits passing through H can have shock structures and cause jumps satisfying jump conditions (see e.g., Smoller [6]), so that they are weak solutions to (3) and also accepted as TWSs.

We claim in this report that we can rigorously support the numerical results in [3], specifically the existence of smooth TWSs to (1). See Section 3 below. On the other hand, concerning non-smooth waves, we need some delicate investigations in order to support the numerical studies in [2] analytically, and we cannot accomplish a rigorous theoretical analysis for such waves as (i) and (ii) described above at present.

Precise arguments will be given in the forthcoming paper.

2. Traveling waves

Let $\sigma > 0$ be a wave speed and introduce an independent variable $z = x - \sigma t$. We will obtain solutions to (1) of the form

$$(u, c)(x, t) = (U, C)(x - \sigma t) = (U, C)(z).$$

Then, (1) becomes

$$-\sigma U' = -(UC')' + U(1 - U), \qquad -\sigma C' = -UC^2, \tag{2}$$

for $z \in \mathbf{R}$, where $' = d/dz$. Hence we have $C' = UC^2/\sigma$. Substituting this relation to (2), we get the following system of ODEs:

$$\begin{cases} U' = \dfrac{-\frac{2}{\sigma^2} U^3 C^3 + U(1 - U)}{\frac{2}{\sigma} U C^2 - \sigma}, \\ C' = \dfrac{1}{\sigma} U C^2, \end{cases} \qquad z \in \mathbf{R}. \tag{3}$$

Here we state our first result. The function $(U, C)(z)$ obtained by the following Theorem 1 is a TWS to the system (1).

Theorem 1. *Take $\sigma > 0$ and fix it. Then, there exists a positive constant $\overline{C}(\sigma)$ such that (3) possesses a smooth nonnegative solution (U, C) for every $\hat{C} \in (0, \overline{C}(\sigma))$ satisfying*

$$\begin{cases} (U, C)(z) \to (1, 0) \text{ as } z \to -\infty, \\ (U, C)(z) \to (0, \hat{C}) \text{ as } z \to \infty. \end{cases}$$

The constant $\overline{C}(\sigma)$ satisfies $\overline{C}(\sigma) > (\sigma^2 + \sqrt{8\sigma^2 + \sigma^4})/4$.

3. Phase plane analysis (Sketch of a proof of Theorem 1)

Note that (3) has equilibrium points $(U, C) = (1, 0)$ and $(0, \hat{C})$ for any constant $\hat{C}$. The linearized matrix for (3) at $(1, 0)$ has eigenvalues 0 and $1/\sigma$ and the corresponding eigenvectors are $[0, 1]^t$ and $[1, 0]^t$, respectively. Also, at $(0, \hat{C})$ the linearized matrix for (3) has eigenvalues 0 and $-1/\sigma$ and the corresponding eigenvectors are $[0, 1]^t$ and $[1, -\hat{C}^2]^t$, respectively. Accordingly, we look for an orbit which is tangential to $[0, 1]^t$ and $[1, -\hat{C}^2]^t$ at $(1, 0)$ and $(0, \hat{C})$, respectively.

Put $P(U, C) = \frac{2}{\sigma} U C^2 - \sigma$ and $Q(U, C) = -\frac{2}{\sigma^2} U^2 C^3 + 1 - U$. See Figure 1 for an orbit which we will construct and the curves $P(U, C) = 0$ and $Q(U, C) = 0$, and verify the directions of the orbits defined by (3). As Pettet et al. [5] have stated, it is easily seen that (3) has singularities on the curve $P(U, C) = 0$ called the "wall of sigularities". Two curves $P(U, C) = 0$ and $Q(U, C) = 0$ crosses each other at only one point $H(U_H, C_H)$ called the "hole in the wall" (see [5]) where $U' = U \cdot Q(U, C)/P(U, C)$ is indefinite. Here,

$$U_H = \frac{8}{(\sigma + \sqrt{8 + \sigma^2})^2}, \qquad C_H = \frac{\sigma^2 + \sqrt{8\sigma^2 + \sigma^4}}{4}. \tag{4}$$

See Figure 1 again. Among the solution orbits going to the wall, only the ones arriving at the hole H may be continuated as a solution to (3).

It suffices to construct an invariant region on the lower left part of the phase plane (Figure 2). We make use of the following auxiliary system (cf. [5]):

$$\begin{cases} U' = \dfrac{2}{\sigma^2} U^3 C^3 - U(1 - U), \\ C' = -\dfrac{1}{\sigma} U C^2 \left(\dfrac{2}{\sigma} U C^2 - \sigma \right), \end{cases} \quad z \in \mathbf{R}. \tag{5}$$

This auxiliary system (5) is a non-singular type and its equilibrium points are $(1, 0)$, $(0, \hat{C})$ which are those of (3), and (U_H, C_H) which is the hole in the wall of (3). Remark that the orbits for (3) become those for (5) with the same or reverse directions each other and that their directions coincide on the lower left part of the phase plane (see Figures 2 and 3).

If we linearize (5) at H, then we get

$$\frac{d}{dz} \begin{bmatrix} U \\ C \end{bmatrix} = A_H \begin{bmatrix} U \\ C \end{bmatrix} = \begin{bmatrix} -U_H + 2 & 3U_H^2 \\ -\sigma^2/(2U_H) & 2U_H - 2 \end{bmatrix} \begin{bmatrix} U \\ C \end{bmatrix}.$$

We can easily see that H is a saddle point for (5). There exists an orbit from $(1, 0)$ to H and one from H to $(0, \hat{C})$ on the phase plane for the auxiliary system (5). These two orbits, U-axis and C-axis determine the invariant region which we desire. Compare Figures 2 and 3.

Now, we can find a finite z_H such that $(U, C)(z_H) = (U_H, C_H)$ for the solution $(U, C)(z)$ to (3). This implies that the solution orbits for (3) arriving at the hole H must pass through H while keeping C^1 (see Figure 4). In other words, the combination of the orbit from $(1, 0)$ to H and that from H to $(0, \hat{C})$ does not have differentiability at H, so that it cannot be a solution to (3) in the usual sense.

Finally in this section, we can represent $\overline{C}(\sigma)$ implicitly. Integrating $(-C^{-1})' = (1/\sigma)U$ yields $\overline{C}(\sigma) = \dfrac{C_H}{1 - \frac{C_H}{\sigma} \int_{z_H}^{\infty} U(\tau) d\tau}$ with the use of the solution $(U, C)(z)$ for (3) which goes to $(0, \overline{C}(\sigma))$ as $z \to \infty$ passing through H.

4. Asymptotic properties of the orbit

We consider the asymptotic relations of the solution orbit (U, C) obtained by Theorem 1 near the equilibrium points. First, we make use of the center manifold theory (see e.g., Carr [1]) and show the following result on the orbit near $(1, 0)$.

Theorem 2. *The solution orbit* (U, C) *to* (3) *on the phase plane has the following relation near* $(1, 0)$:

$$1 - U = \frac{2}{\sigma^2} C^3 + \frac{6}{\sigma^2} C^4 + O(C^5).$$

Next, we give the relation of the solution orbit near $(0, \hat{C})$.

Theorem 3. *Near* $(0, \hat{C})$,

$$U = \frac{1}{\hat{C}^2}(\hat{C} - C) + \frac{-\sigma^2 + 2\hat{C}\sigma^2 + 2\hat{C}^2}{2\hat{C}^4 \sigma^2}(\hat{C} - C)^2 + O(|\hat{C} - C|^3).$$

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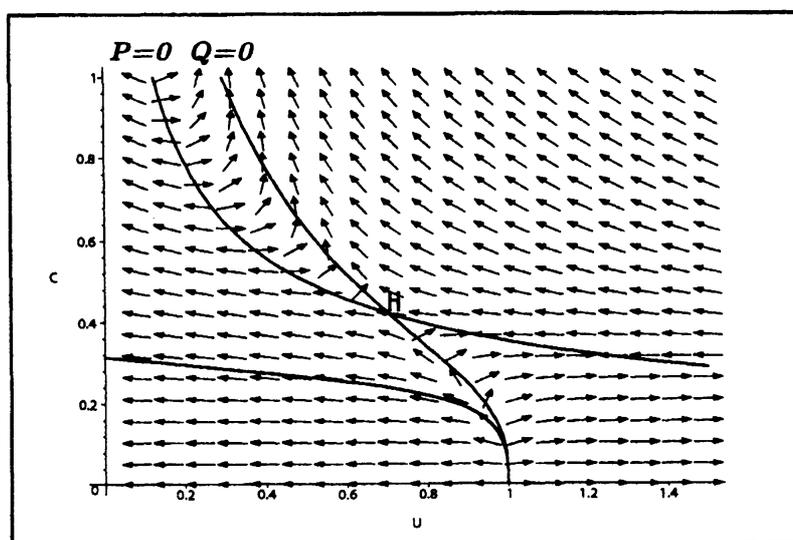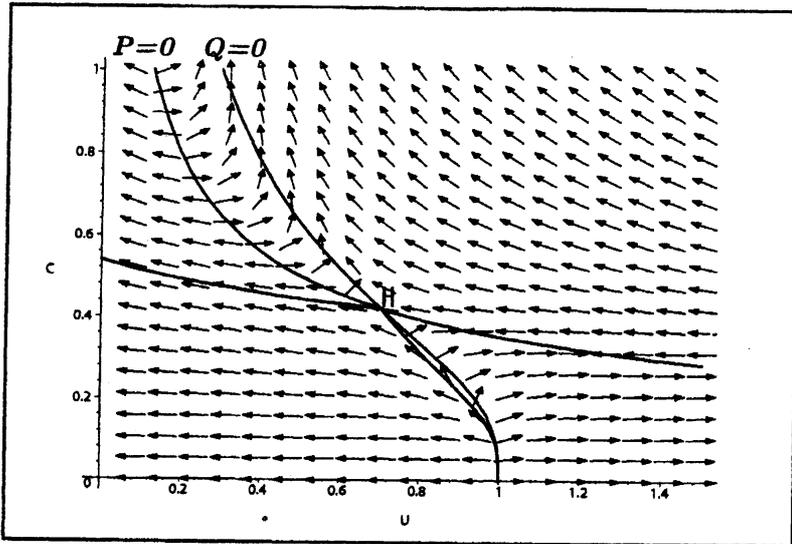[6] J. Smoller, *Shock Waves and Reaction-Diffusion Equations, 2nd ed.*, Springer Verlag, New York, Berlin, Heidelberg, 1994.
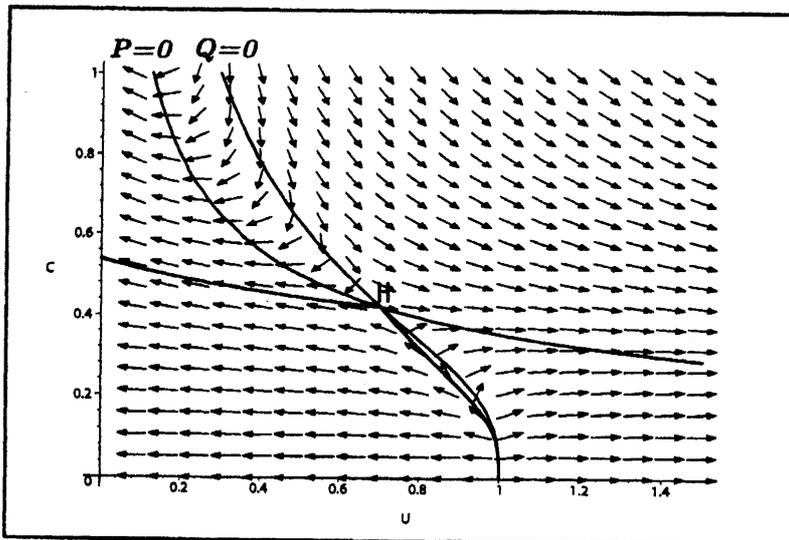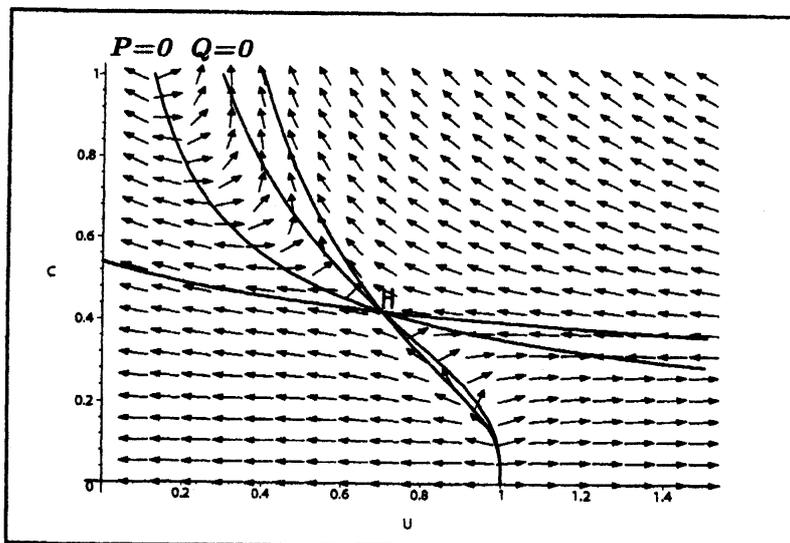

Figure 1

Figure 2

Figure 3

Figure 4