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Wave front propagation and the discriminant of a tame polynomial

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ABSTRACT. *In this note we present a description of a wave front starting from an algebraic hypersurface surface as a pull-back of the discriminantal loci of a tame polynomial by a polynomial mapping. As an application we give examples of wave fronts which define free/almost free divisors near the focal point.*

1 Preliminaries on the wave fronts

In this section we prepare fundamental notations and lemmata to develop our studies in further sections. Let us denote by $Y := \{(z,u) \in \mathbb{C}^{n+1}; F(z) + u = 0\}$ the complexified initial wave front set defined by a polynomial $F(z) \in \mathbb{R}[z_1, \cdots, z_n]$, $z = (z_1, \cdots, z_n)$. Of course the real initial wave front set is $Y \cap \mathbb{R}^{n+1}$.

Let us consider the traveling of the ray starting from a point $(z,u) \in Y$ along unit vectors perpendicular to the hypersurface tangent to Y at (z,u). It will reach at the point $(x_1, \cdots, x_{n+1})$

$$x_j = \pm t \frac{1}{|(d_z F(z), 1)|} \frac{\partial F(z)}{\partial z_j} + z_j, 1 \leq j \leq n,$$
$$x_{n+1} = \pm t \frac{1}{|(d_z F(z), 1)|} + u \ \text{ with } \ (z,u) \in Y, \tag{1.1}$$

at the moment t. Further on, we denote by $x' = (x_1, \cdots, x_n)$, $x = (x', x_{n+1})$. We see that (x,t) and (z,u) satisfying the relation (1.1) are located on the zero loci of two phase functions

$$\psi_{\pm}(x,t,z,u) = (\langle x' - z, d_z F(z) \rangle \ + (x_{n+1} - u)) \pm t |(d_z F(z), 1)|, \tag{1.2}$$

each of which corresponds to the backward $\psi_+(x,t,z,u)$ (resp. the forward $\psi_-(x,t,z,u)$) wave propagation. To simplify the argument, we will not distinguish forward and backward wave propagations in future. This leads us to introduce an unified phase function

$$\psi(x,t,z,u) := \psi_+(x,t,z,u) \cdot \psi_-(x,t,z,u)$$
$$= (\langle x' - z, d_z F(z) \rangle \ + (x_{n+1} + u))^2 - t^2 |(d_z F(z), 1)|^2, \tag{1.3}$$

Let us denote by W_t the wave front at time t with the initial wave front Y i.e. $Y = W_0$.

Lemma 1.1. *For $x \in W_t$, the point (x,t) belongs to the critical value set of the projection,*

$$\begin{array}{ccc} \{(z,u) \in Y : \psi(x,t,z,u) = 0\} & \rightarrow & \mathbb{C}^{n+2} \\ (x,t,z,u) & \mapsto & (x,t). \end{array}$$

We can understand this fact in several ways. Instead of purely geometrical interpretation, in our previous publication [9] we adopted investigation of the singular loci of the integral of type,

$$I(x,t) = \int_\gamma H(z,u)\left(\frac{1}{\psi_+(x,t,z,u)} + \frac{1}{\psi_-(x,t,z,u)}\right) dz \wedge du$$

for $\gamma \in H_n(Y)$ and $H(z,u) \in \mathcal{O}_{\mathbb{C}^{n+1}}$. The above integral ramifies around its singular loci W_t and by the general theory of the Gel'fand-Leray integrals (cf. [11]), W_t is contained in the critical value set mentioned in the Lemma 1.1.

According to the Lemma 1.1, The set $LW := \cup_{t\in\mathbb{C}} W_t \subset \mathbb{C}^{n+1}$ (the real part of it is the large wave front after Arnol'd [1] I, 22.1) can be interpreted as a subset of the discriminant of the function (called the phase function)

$$\Psi(x,t,z) := \left(\langle x' - z, d_z F(z)\rangle + x_{n+1} + F(z)\right)^2 - t^2(|d_z F(z)|^2 + 1) \tag{1.4}$$

for $x' = (x_1, \cdots, x_n)$. This is a set of (x,t) for which the algebraic variety

$$X_{x,t} := \{z \in \mathbb{C}^n : \Psi(x,t,z) = 0\}$$

has singular points.

Remark 1.1. *Masaru Hasegawa [7] and Toshizumi Fukui (Saitama University) study the wave front W_t as a discriminantal loci of the function,*

$$\Phi(x,t,z) = -\frac{1}{2}(|(x' - z, x_{n+1} + F(z))|^2 - t^2),$$

that measures the tangency of the sphere $\{(z, z_{n+1}) \in \mathbb{R}^{n+1} : |(z - x', z_{n+1} - x_{n+1})|^2 = t^2\}$ with the hypersurface $Y \cap \mathbb{R}^{n+1}$. In some cases, this approach allows us to get less complicated expression of the defining equation of LW in comparison with ours in Theorem 2.5.

We assume that the variety $X_{x,t}$ has at most isolated singular points for a point (x,t) of the space-time. Among those points, we choose a focal point $(x_0, t_0) \in \mathbb{C}^{n+2}$ i.e. the point where the maximum of the sum of all local Milnor numbers is attained. If we denote by $z^{(1)}, \cdots, z^{(k)}$ the singular points located on X_{x_0,t_0} and Milnor numbers corresponding to these points by $\mu(z^{(i)})$, $i = 1, \ldots, k$, the following inequality holds for the focal point

$$\text{sum of Milnor numbers of singular points on } X_{x,t} \leq \sum_{i=1}^{k} \mu(z^{(i)}),$$

for every $(x,t) \in \mathbb{C}^{n+2}$.

Assume that the quotient ring

$$\frac{\mathbb{C}[z]}{(d_z\Psi(x_0, t_0, z))\mathbb{C}[z]} \tag{1.5}$$

is a μ dimensional $\mathbb{C}$ vector space such that it admits a basis $\{e_1(z), \cdots, e_\mu(z)\}$ that contains a set of basis elements as follows,

$$e_1(z) = 1, e_{j+1}(z) = (z_j - z_j^{(i)}), 1 \leq j \leq n, \tag{1.6}$$

for a fixed $i \in [1, k]$. Here we remark that $\sum_{i=1}^{k} \mu(z^{(i)}) \leq \mu$. The denominator $(d_z\Psi(x_0, t_0, z))\mathbb{C}[z]$ of the expression (1.5) means the Jacobian ideal of the polynomial $\Psi(x_0, t_0, z)$.

Now we decompose the difference

$$\Psi(x, t, z) - \Psi(x_0, t_0, z) = \sum_{j=1}^{m} s_j(x, t)e_j(z)$$

by means a set of polynomials in z, $\{e_1(z), \cdots, e_\mu(z), e_{\mu+1}(z), \cdots, e_m(z)\}$ and a set of polynomials in (x, t),

$$\begin{array}{cccc} \iota : \mathbb{C}^{n+2} & \rightarrow & \mathbb{C}^m \\ (x, t) & \mapsto & \iota(x, t) := (s_1(x, t), \cdots, s_m(x, t)) \end{array} \tag{1.7}$$

thus defined.In this way we introduce a set of polynomials $\{e_{\mu+1}(z), \cdots, e_m(z)\}$ in addition to the basis of (1.5). We consider a polynomial $\varphi(z, s) \in \mathbb{C}[z, s]$ for $s = (s_1, \cdots, s_m)$ defined by

$$\varphi(z, s) = \Psi(x_0, t_0, z) + \sum_{j=1}^{m} s_j e_j(z). \tag{1.8}$$

Locally this is a versal (but not miniversal) deformation of the holomorphic function germ $\Psi(x_0, t_0, z)$ at $z = z^{(i)}$.

2 Discriminant of a tame polynomial

Definition 2.1. *The polynomial $f(z) \in \mathbb{C}[z]$ is called tame if there is a compact set U of the critical points of $f(z)$ such that $\|d_z f(z)\| = \sqrt{(d_z f(z), \overline{d_z f(z)})}$ is away from 0 for all $z \notin U$.*

In the sequel we use the notation $s' = (s_2, \cdots, s_m)$ and $s = (s_1, s')$.

Further on we impose the following conditions on $\varphi(z, s)$ introduced in (1.8). Assume that there exists an open set $0 \in V \subset \mathbb{C}^{m-1}$ such that

$$dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{(d_z\varphi(z, s))\mathbb{C}[z]} < \infty, \tag{2.1}$$

for every $s' \in V$ and $s_1 \in \mathbb{C}$. In addition to this, we assume that for every $s = (s_1, \cdots, s_{n+1}, 0, \cdots, 0) \in V$, the equality

$$dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{\left(d_z(\Psi(x_0, t_0, z) + \sum_{j=2}^{n+1} s_j e_j(z))\right) \mathbb{C}[z]} = \mu, \tag{2.1$'$}$$

holds.

Lemma 2.1. *Under the conditions* (1.5), (2.1), (2.1)$'$ *there exists a constructible subset* $\tilde{U} \subset V$, *such that* $\varphi(z, s)$ *is a tame polynomial for every* $s \in \mathbb{C} \times \tilde{U}$ *and*

$$dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{(d_z\varphi(z,s))\mathbb{C}[z]} = \mu,$$

for every $s \in \mathbb{C} \times U$.

Proof

By [3], Proposition 3.1, (2.1)$'$ yields the tameness of $\varphi(z, 0)$. After Proposition 3.2 of the same article, the set of s such that $\varphi(z, s)$ be tame is a constructible subset (i.e. locally closed set with respect to the Zariski topology) of the form $\mathbb{C} \times W$ for $W \subset V$. According to [3], Proposition 2.3, the set

$$T_n = \{s \in \mathbb{C} \times W : dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{(d_z\varphi(z,s))\mathbb{C}[z]} \leq n\},$$

is Zariski closed for every n. We can take $\mathbb{C} \times \tilde{U} = T_\mu \setminus T_{\mu-1}$. **Q.E.D.**

Assumption I

(i) By shrinking $\tilde{U}$ if necessary, we assume that a constructible set $U \subset \tilde{U}$ can be given locally by holomorphic functions $(s_{\nu+1}, \cdots, s_m)$ on the coordinate space with variables $(s_2, \cdots, s_\nu)$, $\nu \geq \mu$.

(ii) The image of the mapping ι of a neighbourhood of (x_0, t_0) is contained in $\mathbb{C} \times U$. In other words,

$$\iota(\mathbb{C}^{n+2}, (x_0, t_0)) \subset (\mathbb{C} \times U, \iota(x_0, t_0)).$$

For a fixed $\tilde{s}' = (\tilde{s}_2, \cdots, \tilde{s}_m) \in U$ and the constructible subset $U \subset V$ of the Assumption I,(i) we see that $\varphi(z, s_1, \tilde{s}')$ is a tame polynomial for all $s_1 \in \mathbb{C}$. For such $\varphi(z, s_1, \tilde{s}')$, we define the following modules,

$$\mathcal{P}_\varphi(\tilde{s}') := \frac{\Omega^{n-1}_{\mathbb{C}^n}}{d_z\varphi(z, s_1, \tilde{s}') \wedge \Omega^{n-2}_{\mathbb{C}^n} + d\Omega^{n-2}_{\mathbb{C}^n}}, \tag{2.2}$$

$$\mathcal{B}_\varphi(\tilde{s}') := \frac{\Omega^{n}_{\mathbb{C}^n}}{d_z\varphi(z, s_1, \tilde{s}') \wedge d\Omega^{n-2}_{\mathbb{C}^n}}. \tag{2.3}$$

the module $\mathcal{B}_\varphi(\tilde{s}')$ is called an algebraic Brieskorn lattice. In considerig the holomorphic forms multiplied by $\varphi(z, s_1, \tilde{s}')$ be zero in (2.2), (2.3) we can treat two modules as $\mathbb{C}[s_1]$ modules.

These modules contain the essential informations on the topology of the variety

$$Z_{(s_1,\tilde{s}')} = \{z \in \mathbb{C}^n : \varphi(z, s_1, \tilde{s}') = 0\}. \tag{2.4}$$

Let us denote by $D_\varphi \subset \mathbb{C} \times U$ the discriminantal loci of the polynomial $\varphi(z, s)$ i.e.

$$D_\varphi := \{s \in \mathbb{C} \times U : \exists z \in Z_s,\ s.t.\ d_z\varphi(z,s) = \vec{0}\}. \tag{2.5}$$

Theorem 2.2. *For a fixed* $\tilde{s}' = (\tilde{s}_2, \cdots, \tilde{s}_m) \in U$, *both* $\mathcal{P}_\varphi(\tilde{s}')$ *and* $\mathcal{B}_\varphi(\tilde{s}')$ *are free* $\mathbb{C}[s_1]$ *modules of rank* μ.

Proof First we show the statement on $\mathcal{B}_\varphi(\tilde{s}')$. After [5], Theorem 0.5, the algebraic Brieskorn lattice $\mathcal{B}_\varphi(\tilde{s}')$ is isomorphic to a free $\mathbb{C}[s_1]$ module of finite rank (so called the Brieskorn-Deligne lattice). The absence of the vanishing cycles at infinity for $\varphi(z, s_1, \tilde{s}')$ ensures this isomorphism.

On the other hand, for $(\tilde{s}_1, \tilde{s}') \in \mathbb{C} \times U$, the Corollary 0.2 of the same article tells us the following equality.

$$dim\ Coker(s_1 - \tilde{s}_1 | \mathcal{B}_\varphi(\tilde{s}'))$$
$$= dim\ H_{n-1}(Z_{(\tilde{s}_1,\tilde{s}')}) + \text{sum of Milnor numbers of singular points on } Z_{(\tilde{s}_1,\tilde{s}')}.$$

For $(\tilde{s}_1, \tilde{s}') \in \mathbb{C} \times U \setminus D_\varphi$, the right hand side of the above equality equals

$$\sum_{s_1 : Z_{(s_1,\tilde{s}')} \text{ singular}} \text{sum of Milnor numbers of singular points on } Z_{(s_1,\tilde{s}')}$$

by [3], Theorem 1.2.

Now we show that $\mathcal{B}_\varphi(\tilde{s}')$ is isomorphic to $\mathcal{P}_\varphi(\tilde{s}')$.

We show the bijectivity of the mapping $d : \mathcal{P}_\varphi(\tilde{s}') \longrightarrow \mathcal{B}_\varphi(\tilde{s}')$. To see the injectivity, we remark that the condition $d(\omega + d\alpha + \beta \wedge d\varphi(z, s_1, \tilde{s}')) = d\omega + d\beta \wedge d\varphi(z, s_1, \tilde{s}') = 0$, $\alpha, \beta \in \Omega^{n-1}$ in $\mathcal{B}_\varphi(\tilde{s}')$, entails the existence of $\alpha' \in \Omega^{n-1}$ such that $d\omega = d\alpha' \wedge d\varphi(z, s_1, \tilde{s}')$, this in turn together with the de Rham lemma entails $\omega = \alpha' \wedge d\varphi(z, s_1, \tilde{s}') + d\beta'$ for some $\beta' \in \Omega^{n-1}$

To see the surjectivity, it is enough to check that for every $\gamma \in \Omega^n$ the equation $d\omega = \gamma$ is solvable. **Q.E.D.**

Let us introduce a module for $\tilde{s}' = (\tilde{s}_2, \cdots, \tilde{s}_m) \in U$,

$$Q_\varphi(\tilde{s}') := \frac{\Omega^n_{\mathbb{C}^n}}{d_z\varphi(z, s_1, \tilde{s}') \wedge \Omega^{n-1}_{\mathbb{C}^n}} \cong \frac{\mathbb{C}[z]}{(d_z\varphi(z, s_1, \tilde{s}'))\mathbb{C}[z]}, \tag{2.6}$$

that is a free $\mathbb{C}[s_1]$ module of rank μ because it is isomorphic to

$$\oplus_{\{s_1 : Z_{(s_1,\tilde{s}')} \text{ singular}\}} \oplus_{z: singular\ points\ on\ Z_{(s_1,\tilde{s}')}} \mathbb{C}^{\mu(z)},$$

with $\mu(z)$: the Milnor number of the singular point $z \in Z_{(s_1,\tilde{s}')}$. Let us denote its basis by

$$\{g_1 dz, \cdots, g_\mu dz\}, \tag{2.7}$$

such that the polynomials $\{g_1(z), \cdots, g_\mu(z)\}$ consist a basis of the RHS of (2.6) as a free $\mathbb{C}[s_1]$ module.

According to [3],p.218, lines 5-6, the following is a locally trivial fibration,

$$Z_{(s_1,s')} \longrightarrow (s_1, s') \in \mathbb{C} \times U \setminus D_\varphi.$$

This yields the next statement.

Corollary 2.3. *We can choose a basis* $\{\omega_1, \cdots, \omega_\mu\}$ *of* $\mathcal{P}_\varphi(\tilde{s}')$ *independent of* $\tilde{s}' \in U$.

Due to the construction of U, we can consider the ring $\mathcal{O}_U$ of holomorphic functions on U. By the analytic continuation with respect to the parameter $s' \in U$, we see the following.

Lemma 2.4. *The modules $\mathcal{B}_\varphi(s')$, $\mathcal{P}_\varphi(s')$, $Q_\varphi(s')$ are free $\mathbb{C}[s_1]\otimes\mathcal{O}_U$ modules of rank μ.*

As the deformation polynomials $e_1,\cdots,e_\mu$ arise from the special form of $\Psi(x,t,z)$ we are obliged to impose the following assumption.

Assumption II We assume that we can adopt $e_i(z)$ of (1.5), (1.6) as $g_i(z)$ in (2.7) $i=1,\cdots,\mu$ and they serve as a basis of $Q_\varphi(s')$ as a free $\mathbb{C}[s_1]\otimes\mathcal{O}_U$ module.

For the sake of simplicity, let us denote by $mod(d_z(\varphi(z,0)+\sum_{j=2}^m s_je_j(z)))$ the residue class modulo the ideal $(d_z(\varphi(z,0)+\sum_{j=2}^m s_je_j(z)))\mathbb{C}[z,s_1]\otimes\mathcal{O}_U$ in $\mathbb{C}[z,s_1]\otimes\mathcal{O}_U$. By virtue of the freeness of $Q_\varphi(s')$, this residue class is uniquely determined. Our assumption (1.5), (1.6) together with the Weierstrass preparation theorem gives us a decomposition as follows,

$$(\varphi(z,0)+\sum_{j=2}^m s_je_j(z))\cdot\frac{\partial\varphi(z,s)}{\partial s_i}$$

$$\equiv\sum_{\ell=1}^{\mu}\sigma_i^\ell(s')\frac{\partial\varphi(z,s)}{\partial s_\ell}\ mod(d_z(\varphi(z,0)+\sum_{j=2}^m s_je_j(z))),\ 1\le i\le\mu \tag{2.8}$$

$$\frac{\partial\varphi(z,s)}{\partial s_i}\equiv\sum_{\ell=1}^{\mu}\sigma_i^\ell(s')\frac{\partial\varphi(z,s)}{\partial s_\ell}\ mod(d_z(\varphi(z,0)+\sum_{j=2}^m s_je_j(z))),\ \mu+1\le i\le m, \tag{2.9}$$

with $\sigma_i^\ell(s')\in\mathcal{O}_U$. In fact, according to an argument used in [4],Theorem A4, [10], Proposition 2 (both treat liftable vector fields in local case but they are valid for our situation), the following vector fields are tangent to the discriminant D_φ,

$$\vec{v}_i:=(s_1+\sigma_i^i(s'))\frac{\partial}{\partial s_i}+\sum_{\ell=1,\ell\neq i}^{\mu}\sigma_i^\ell(s')\frac{\partial\varphi(z,s)}{\partial s_\ell},\ 1\le i\le\mu \tag{2.10}$$

Here we recall the Assumption I, (i) that allows us to adopt $(s_1,s_2,\cdots,s_\nu)$, $\nu\ge\mu$ as the local coordinates of $\mathbb{C}\times U$.

$$\vec{v}_i:=-\frac{\partial}{\partial s_i}+\sum_{\ell=1}^{\mu}\sigma_i^\ell(s')\frac{\partial}{\partial s_\ell},\ \mu+1\le i\le\nu, \tag{2.11}$$

Evidently they are linearly independent over $\mathbb{C}[s_1]\otimes\mathcal{O}_U$ because of the presence of the term $s_1\frac{\partial}{\partial s_i}$ for every $1\le i\le\mu$ and $-\frac{\partial}{\partial s_i}$ for $\mu+1\le i\le\nu$. Therefore they form a $\mathbb{C}[s_1]\otimes\mathcal{O}_U$free module of rank ν. Let us introduce the following matrix of which the i−th row corresponds to the vector $\vec{v}_i$.

$$\Sigma(s) := \begin{pmatrix} s_1+\sigma_1^1(s') & \sigma_1^2(s') & \cdots & \sigma_1^{\mu}(s') & 0 & \cdots & 0 & 0 \\ \sigma_2^1(s') & s_1+\sigma_2^2(s') & \cdots & \sigma_2^{\mu}(s') & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{\mu}^1(s') & \sigma_{\mu}^2(s') & \cdots & s_1+\sigma_{\mu}^{\mu}(s') & 0 & \cdots & 0 & 0 \\ \sigma_{\mu+1}^1(s') & \sigma_{\mu+1}^2(s') & \cdots & \sigma_{\mu+1}^{\mu}(s') & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{\nu-1}^1(s') & \sigma_{\nu-1}^2(s') & \cdots & \sigma_{\nu-1}^{\mu}(s') & 0 & \cdots & -1 & 0 \\ \sigma_{\nu}^1(s') & \sigma_{\nu}^2(s') & \cdots & \sigma_{\nu}^{\mu}(s') & 0 & \cdots & 0 & -1 \end{pmatrix}. \quad (2.12)$$

In fact the following $\mu \times \mu$ submatrix of $\Sigma(s)$ contains the essential geometrical informations on D_{φ}.

$$\tilde{\Sigma}(s) := \begin{pmatrix} s_1+\sigma_1^1(s') & \sigma_1^2(s') & \cdots & \sigma_1^{\mu}(s') \\ \sigma_2^1(s') & s_1+\sigma_2^2(s') & \cdots & \sigma_2^{\mu}(s') \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\mu}^1(s') & \sigma_{\mu}^2(s') & \cdots & s_1+\sigma_{\mu}^{\mu}(s') \end{pmatrix}. \quad (2.13)$$

Theorem 2.5. *1) The algebra $Der_{\mathbb{C}\times U}(log\ D_{\varphi})$ of tangent fields to D_{φ} as a free $\mathbb{C}[s_1]\otimes\mathcal{O}_U$ is generated by the vectors v_i, $1 \leq i \leq \nu$ of* (2.10), (2.11).

2) The discriminantal loci D_{φ} is given by the equation

$$D_{\varphi} = \{s \in \mathbb{C}\times U : det\ \tilde{\Sigma}(s) = 0\}.$$

3) The preimage of D_{φ} by the mapping ι contains the wave front $LW = \cup_{t\in\mathbb{C}} W_t \subset \mathbb{C}^{n+1}$ i.e. $LW \subset \iota^{-1}(D_{\varphi})$.

Proof The tangency of vector fields $\vec{v}_i$'s to D_{φ} and their independence over $\mathbb{C}[s_1]\otimes\mathcal{O}_U$ have already been shown.

First we shall prove 2). By virtue of the tangency of $\vec{v}_i$'s to D_{φ} and the equality,

$$\vec{v}_1 \wedge \cdots \wedge \vec{v}_{\nu} = det\ \Sigma(s)\partial_{s_1} \wedge \cdots \wedge \partial_{s_{\nu}},$$

the function $det\ \Sigma(s)$ shall vanish on D_{φ}. The statement on $Q_{\varphi}(s')$ of the Lemma 2.4 tells us that

$$\sharp\{s \in \mathbb{C}\times U : s_1 = const \cap D_{\varphi}\} = \mu, \quad (2.14)$$

in taking the multiplicity into account.

From (2.12), (2.13) we see that

$$\pm det\ \Sigma(s) = det\ \tilde{\Sigma}(s) = s_1^{\mu} + d_1(s')s_1^{\mu-1} + \cdots + d_{\mu}(s'),$$

with $d_i(s') \in \mathcal{O}_U$, $1 \leq i \leq \mu$. Thus the Weierstrass polynomial in s_1 ,$det\ \tilde{\Sigma}(s)$ shall be divided by the defining equation of D_{φ} which turns out to be also a Weierstrass polynomial in s_1 of degree μ. This proves 2).

Now we shall show that every vector $\vec{v}$ tangent to D_φ admits a decomposition like

$$\vec{v} = \sum_{i=1}^{\nu} a_i(s)\vec{v}_i, \tag{2.15}$$

for some $a_i(s) \in \mathbb{C}[s_1] \otimes \mathcal{O}_U$. For every i the following expression shall vanish on D_φ, because of the tangency of all vectors taking part in it,

$$\vec{v}_1 \wedge \cdots \wedge \vec{v}_{i-1} \wedge \vec{v} \wedge \vec{v}_{i+1} \wedge \cdots \wedge \vec{v}_\nu.$$

Therefore there exists $a_i(s) \in \mathbb{C}[s_1] \otimes \mathcal{O}_U$ such that the above expression equals to $a_i(s) det\Sigma(s)\partial_{s_1} \wedge \cdots \wedge \partial_{s_m}$. This means that the vector $\vec{v} - \sum_{i=1}^{\nu} a_i(s)\vec{v}_i$ defines a zero vector at every $s \notin D_\varphi$, as the vectors $\vec{v}_1, \cdots, \vec{v}_\nu$ form a frame outside D_φ. By the continuity argument on holomorphic functions, we see that the decomposition holds everywhere on $\mathbb{C} \times U$.

The statement 3) follows from Lemma 1.1, (1.4) and the definition (1.7) of the mapping ι. **Q.E.D.**

3 Gauss-Manin system for a tame polynomial

In this section, we willl show that the above matrix $\tilde{\Sigma}(s)$, (2.13) can be obtained as the coefficient of the Gauss-Manin system defined for a tame polynomial $\varphi(z,s)$.

According to Lemma 2.4, every $\omega \in \mathcal{P}_\varphi(s')$ admits a unique decomposition as follows,

$$\omega = \sum_{i=1}^{\mu} a_i(s)\omega_i, \quad s \in \mathbb{C} \times U. \tag{3.1}$$

A generalisation of theorem 0.2 of [6] tells us that the following equivalence holds for every holomorphic $n-1$ form ω,

$$\forall s \in \mathbb{C} \times U, \omega|_{Z_s} = 0 \ \text{ in } \ H^{n-1}(Z_s) \Leftrightarrow \omega = 0 \ \text{ in } \ \mathcal{P}_\varphi(s'). \tag{3.2}$$

We can prove the above statement (3.2) for every $n \geq 2$ in following a slightly modified argument explained in §2 of [6].

This theorem yields a corollary that ensures us the following equality for every vanishing cycle $\delta(s) \in H_{n-1}(Z_s)$,

$$\int_{\delta(s)} \omega = \sum_{i=1}^{\mu} a_i(s) \int_{\delta(s)} \omega_i, s \in \mathbb{C} \times U, \tag{3.3}$$

for some $a_i(s) \in \mathbb{C}[s_1] \otimes \mathcal{O}_U$, $1 \leq i \leq \mu$. To show this along with the argument by L.Gavrilov [6], we simply need to replace his Lemma 2.2 by [5], Corollary 0.7.

Here we remark that for the basis of $\{e_1(z)dz, \cdots, e_\mu(z)dz\}$ of $Q_\varphi(\tilde{s}')$ we can choose the basis $\{\omega_1, \cdots, \omega_\mu\}$ of $\mathcal{P}_\varphi(\tilde{s}')$ such that

$$d\omega_i = e_i(z)dz + d_z\varphi(z,s) \wedge \epsilon_i,$$

for some $\epsilon_i \in \Omega^{n-1}$. That is to say, for every $\omega \in \Omega^{n-1}$ we can find the following two types of decomposition

$$\omega = \sum_{i=1}^{\mu} c_i(s')d\omega_i + d_z\varphi(z,s) \wedge d\xi,$$

$$= \sum_{i=1}^{\mu} c_i(s')(e_i(z)dz + d_z\varphi(z,s) \wedge \epsilon_i) + d_z\varphi(z,s) \wedge \eta,$$

for some $c_i(s') \in \mathcal{O}_U$, $\xi \in \Omega^{n-2} \otimes \mathcal{O}_U$, $\eta \in \Omega^{n-1} \otimes \mathcal{O}_U$. In other words, for every $\eta \in \Omega^{n-1} \otimes \mathcal{O}_U$ one can find $\tilde{\xi} \in \Omega^{n-2} \otimes \mathcal{O}_U$ and $c_i(s')$, ξ as above that satisfy

$$\eta = -\sum_{i=1}^{\mu} c_i(s')\epsilon_i + d\xi + d_z\varphi(z,s) \wedge d\tilde{\xi}.$$

If we take ϵ_i as some representatives of $\mathcal{P}_\varphi(\tilde{s}')$, the above statement is reduced to that on $\mathcal{P}_\varphi(\tilde{s}')$ of Lemma 2.4.

As E.Brieskorn [2] showed, the following equality holds if we understand it as the property of the holomorphic sections in the cohomology bundle $H^{n-1}(Z_s)$ defined as the Leray's residue $\omega/d_z\varphi(z,s)$ for $\omega \in \Omega^n$,

$$(\frac{\partial}{\partial s_1})^{-1} d\eta = d_z\varphi(z,s) \wedge \eta.$$

This yields that

$$(\frac{\partial}{\partial s_1})^{-1}\mathcal{B}_\varphi(\tilde{s}') = d_z\varphi(z,s) \wedge \Omega^{n-1}/d_z\varphi(z,s) \wedge d\Omega^{n-2},$$

$$Q_\varphi(\tilde{s}') = \mathcal{B}_\varphi(\tilde{s}')/(\frac{\partial}{\partial s_1})^{-1}\mathcal{B}_\varphi(\tilde{s}'),$$

we see that $\{e_1(z)dz, \cdots, e_\mu(z)dz\}$ is a basis of $\mathcal{B}_\varphi(\tilde{s}')$ as an $\mathcal{O}_U[(\frac{\partial}{\partial s_1})^{-1}]$ module.

For such ω_i's we have a decomposition in $Q_\varphi(\tilde{s}')$ as follows,

$$(\varphi(z,s) - s_1)d\omega_i = \sum_{\ell=1}^{\mu} \sigma_i^\ell(s')d\omega_\ell + d_z\varphi(z,s) \wedge \eta_i, \quad 1 \leq i \leq \mu \tag{3.4}$$

$\eta_i \in \Omega^{n-1}$. We see that (3.4) is equivalent to (2.8). This relation immediately entails the following equality for every $\delta(s) \in H_{n-1}(Z_s)$,

$$s_1\frac{\partial}{\partial s_1}\int_{\delta(s)} \omega_i + \sum_{\ell=1}^{\mu} \sigma_i^\ell(s')\frac{\partial}{\partial s_1}\int_{\delta(s)} \omega_\ell + \int_{\delta(s)} \eta_i = 0, \tag{3.5}$$

in view of the fact $\int_{\delta(s)} \varphi(z,s)\frac{d\omega_i}{d_z\varphi(z,s)} = 0$ and the Leray's residue theorem

$$\frac{\partial}{\partial s_1}\int_{\delta(s)} \omega_i = \int_{\delta(s)} \frac{d\omega_i}{d_z\varphi(z,s)}.$$

After (3.3), every $\int_{\delta(s)} \eta_i$ admits an unique decomposition

$$\int_{\delta(s)} \eta_i = \sum_{j=1}^{\mu} b_i^j(s) \int_{\delta(s)} \omega_j, s \in \mathbb{C} \times U, \tag{3.6}$$

for some $b_i^j(s) \in \mathbb{C}[s_1] \times \mathcal{O}_U$, $1 \leq i, j \leq \mu$.

Let us consider a vector of fibre integrals

$$\mathbb{I}_Q :=^t (\int_{\delta(s)} \omega_1, \cdots, \int_{\delta(s)} \omega_\mu). \tag{3.7}$$

In summary we get

Proposition 3.1. *1) For a vector $\mathbb{I}_Q$,* (3.5) *we have the following Gauss-Manin system*

$$\tilde{\Sigma} \cdot \frac{\partial}{\partial s_1} \mathbb{I}_Q + B(s) \mathbb{I}_Q = 0, \tag{3.8}$$

where $B(s) = \left(b_i^j(s)\right)_{1 \leq i,j \leq \mu}$ for functions determined in (3.6).

2) The discriminantal loci D_φ of the tame polynomial $\varphi(z,s)$, $s \in \mathbb{C} \times U$ has an expression,

$$D_\varphi = \{s \in \mathbb{C} \times U : det\, \tilde{\Sigma}(s) = 0\},$$

that corresponds to the singular loci of the system (3.8).

Remark 3.1. *To see that the two statements on D_φ do not mean a simple coincidence, one may consult [10] Theorem 2.3 where he finds a description of the Gauss-Manin system for Leray's residues by means of the tangent vector fields to the discriminant loci.*

4 Free and almost free wave fronts

Now we recall that the freeness of $Der_{\mathbb{C}\times U}(log\ D_\varphi)$ as a $\mathbb{C}[s_1] \otimes \mathcal{O}_U$ module, proven in the Theorem 2.5, means that D_φ defines a free divisor (in the sense of K.Saito) in the neighbourhood of every point $s \in D_\varphi$. We define the logarithmic tangent space $T_s^{log} D_\varphi$ to D_φ at s:

$$T_s^{log} D_\varphi = \{\vec{v}(s) : \vec{v}(s) \in Der_{\mathbb{C}\times U}(log\ D_\varphi)_s\} \tag{4.1}$$

We follow the presentation by David Mond [8] on the free and almost free divisors though the latter has been first introduced by J.N.Damon. To discuss when the large wave front LW becomes a free divisor, we need to make use of the notion of algebraic transversaliy. We recall here the Assumption I, (ii) on the image of the mapping ι that entails the following inclusion relation,

$$d_{x,t}\iota(T_{(x,t)}\mathbb{C}^{n+2}) \subset T_{\iota(x,t)}(\mathbb{C} \times U),$$

for (x,t) in the neighbourhood of (x_0,t_0).

Definition 4.1. *The mapping ι is algebraically transverse to D_φ at $(x_0,t_0) \in \mathbb{C}^{n+2}$ if and only if*

$$d_{x,t}\iota(T_{(x_0,t_0)}\mathbb{C}^{n+2}) + T^{log}_{\iota(x_0,t_0)} D_\varphi = T_{\iota(x_0,t_0)}(\mathbb{C} \times U). \tag{4.2}$$

Lemma 4.1. *([8] Jacobian criterion for freeness) The divisor $\iota^{-1}(D_\varphi)$ is free if and only if ι is algebraically transverse to D_φ.*

To state a criterion of the freeness of $\iota^{-1}(D_\varphi)$, we need the following $m \times (\nu + n + 2)$ matrix $T(x,t)$.

$$T(x,t) = \begin{pmatrix} s_1 + \sigma_1^1(s'(x,t)) & \cdots & \sigma_1^\mu(s'(x,t)) & 0 & \cdots & 0 & \cdots & 0 \\ \sigma_2^1(s'(x,t)) & \cdots & \sigma_2^\mu(s'(x,t)) & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \sigma_\mu^1(s'(x,t)) & \cdots & s_1 + \sigma_\mu^\mu(s'(x,t)) & 0 & \cdots & 0 & \cdots & 0 \\ \sigma_{\mu+1}^1(s'(x,t)) & \cdots & \sigma_{\mu+1}^\mu(s'(x,t)) & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ \sigma_\nu^1(s'(x,t)) & \cdots & \sigma_\nu^\mu(s'(x,t)) & 0 & \cdots & -1 & \cdots & 0 \\ s_1(x,t)_{x_1} & \cdots & s_\mu(x,t)_{x_1} & s_{\mu+1}(x,t)_{x_1} & \cdots & s_\nu(x,t)_{x_1} & \cdots & s_m(x,t)_{x_1} \\ s_1(x,t)_{x_2} & \cdots & s_\mu(x,t)_{x_2} & s_{\mu+1}(x,t)_{x_2} & \cdots & s_\nu(x,t)_{x_2} & \cdots & s_m(x,t)_{x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_1(x,t)_{x_{n+1}} & \cdots & s_\mu(x,t)_{x_{n+1}} & s_{\mu+1}(x,t)_{x_{n+1}} & \cdots & s_\nu(x,t)_{x_{n+1}} & \cdots & s_m(x,t)_{x_{n+1}} \\ s_1(x,t)_t & \cdots & s_\mu(x,t)_t & s_{\mu+1}(x,t)_t & \cdots & s_\nu(x,t)_t & \cdots & s_m(x,t)_t \end{pmatrix}. \tag{4.3}$$

The first ν rows of the $T(x,t)$ correspond to those of $\Sigma(\iota(x,t))$ while the $(\nu+i)-$th row corresponds to $\frac{\partial}{\partial x_i}\iota(x,t)$, $1 \leq i \leq n+1$ and the last row to $\frac{\partial}{\partial t}\iota(x,t)$ for $\iota(x,t)$ of (1.7).

The Lemma 4.1 yields immediately the following statement in view of the Theorem 2.5.

Proposition 4.2. *The divisor $\iota^{-1}(D_\varphi)$ is free in the neighbourhood of (x,t) if and only if rank $T(x,t) \geq \nu$.*

After Theorem 2.5, in the neighbourhood of each of its point s, the hypersurface D_φ defines a germ of free divisor.

Definition 4.2. *The germ of hypersurface $\iota^{-1}(D_\varphi)$ at $(x_0,t_0) \in \mathbb{C}^{n+2}$ is an almost free divisor based on the germ of free divisor D_φ at $\iota(x_0,t_0) \in \mathbb{C} \times U$ if there is a map $i_0 : \iota^{-1}(D_\varphi) \to D_\varphi$ which is algebraically transverse to D_φ except at (x_0,t_0) such that $\iota^{-1}(D_\varphi) = i_0^{-1}(D_\varphi)$.*

In view of this definition, we get a criterion so that $\iota^{-1}(D_\varphi)$ be an almost free divisor.

Proposition 4.3. *The germ of hypersurface $\iota^{-1}(D_\varphi)$ at $(x_0,t_0) \in \mathbb{C}^{n+2}$ is an almost free divisor based on the germ of free divisor D_φ at $\iota(x_0,t_0) \in \mathbb{C} \times U$ if the following inequality holds at an isolated point $(x_0,t_0) \in \iota^{-1}(D_\varphi)$,*

$$rank\ \Sigma(\iota(x_0,t_0)) + rank\ d_{x,t}\iota(x_0,t_0) < \nu, \tag{4.4}$$

while at other points $(x,t) \neq (x_0,t_0)$ in the neighbourhood of (x_0,t_0) , rank $T(x,t) \geq \nu$.

5 Examples

1. Wave propagation on the plane

Let us consider the following initial wave front on the plane $Y := \{(z,u) \in \mathbb{C}^2; az^2 + z^4 + u = 0\}, z =$ i.e. $F(z) = az^2 + z^4$ for some real non-zero constant a. In this case our phase function has the following expression

$$\begin{aligned}\Psi(x,t,z) &= (x_1 + az^2 + z^4 + (x_2 - z)(2az + 4z^3))^2 - t^2(1 + (2az + 4z^3)^2),\\ &= -t^2 + x_2^2 + 4ax_1x_2z + (-4a^2t^2 + 4a^2x_1^2 - 2ax_2)z^2\\ &\quad(-4a^2x_1 + 8x_1x_2)z^3 + (a^2 - 16at^2 + 16ax_1^2 - 6x_2)z^4\\ &\quad -20ax_1z^5 + (6a - 16t^2 + 16x_1^2)z^6 - 24x_1z^7 + 9z^8. \qquad (5.1)\end{aligned}$$

It is easy to see that $(x_1, x_2, t) = (0, -1/2a, 1/2a)$ is a focal point with a singular point $(z,u) = (0,0)$ and the Milnor number $\mu(0) = 3$ (A_3 singularity i.e. the swallow tail) if $a \neq 1$ and $\mu(0) = 5$ (A_5 singularity) if $a = 1$,

$$\Psi(0, -a/2, a/2, z) = (-(1/a) + a^2)z^4 + (-(4/a^2) + 6a)z^6 + 9z^8. \qquad (5.2)$$

The quotient ring (1.5) for this $\Psi(0, -1/2a, 1/2a, z)$ has dimension $\mu = 7$.

Especially we can choose $e_i = z^{i-1}$, $i = 1, \cdots, 7$ as the basis (2.7). Now, in view of (5.1) we introduce additional deformation polynomials $e_8 = z^7$, together with entries of the mapping ι (1.7),

$$s_1 = -t^2 + x_2^2, s_2 = 4ax_1x_2, s_3 = -4a^2t^2 + 4a^2x_1^2 - 2ax_2, s_4 = -4a^2x_1 + 8x_1x_2,$$

$$s_5 = a^2 - 16at^2 + 16ax_1^2 - 6x_2, s_6 = -20ax_1, s_7 = 6a - 16t^2 + 16x_1^2, s_8 = -24x_1. \quad (5.3)$$

$$\varphi(z,s) = 9z^8 + \sum_{i=1}^{8} s_i z^{i-1}.$$

In this case, the constructible set U of the Assumption I,(i) coincides with $\mathbb{C}^7$.

By the aid of the computer algebra system SINGULAR, we calculate the residue class $mod(d_z(\varphi(z,0) + \sum_{j=2}^{m} s_j e_j(z)))$ of the following polynomials that illustrate (2.8).

$\varphi(z,s) \equiv (1/4*s_7 - 7/576*s_8^2)*z^6 + (3/8*s_6 - 1/96*s_7*s_8)*z^5 + (1/2*s_5 - 5/576*s_6*s_8)*z^4 + (5/8*s_4 - 1/144*s_5*s_8)*z^3 + (3/4*s_3 - 1/192*s_4*s_8)*z^2 + (7/8*s_2 - 1/288*s_3*s_8)*z + (s_1 - 1/576*s_2*s_8)$

$z*\varphi(z,s) \equiv (3/8*s_6 - 5/144*s_7*s_8 + 49/41472*s_8^3)*z^6 + (1/2*s_5 - 5/576*s_6*s_8 - 1/48*s_7^2 + 7/6912*s_7*s_8^2)*z^5 + (5/8*s_4 - 1/144*s_5*s_8 - 5/288*s_6*s_7 + 35/41472*s_6*s_8^2)*z^4 + (3/4*s_3 - 1/192*s_4*s_8 - 1/72*s_5*s_7 + 7/10368*s_5*s_8^2)*z^3 + (7/8*s_2 - 1/288*s_3*s_8 - 1/96*s_4*s_7 + 7/13824*s_4*s_8^2)*z^2 + (s_1 - 1/576*s_2*s_8 - 1/144*s_3*s_7 + 7/20736*s_3*s_8^2)*z + (-1/288*s_2*s_7 + 7/41472*s_2*s_8^2)$

$z^2*\varphi(z,s) \equiv (1/2*s_5 - 13/288*s_6*s_8 - 1/48*s_7^2 + 91/20736*s_7*s_8^2 - 343/2985984*s_8^4)*z^6 + (5/8*s_4 - 1/144*s_5*s_8 - 7/144*s_6*s_7 + 35/41472*s_6*s_8^2 + 5/1728*s_7^2*s_8 - 49/497664*s_7*s_8^3)*z^5 + (3/4*s_3 - 1/192*s_4*s_8 - 1/72*s_5*s_7 + 7/10368*s_5*s_8^2 - 5/192*s_6^2 + 25/10368*s_6*s_7*s_8 - 245/2985984*s_6*s_8^3)*z^4 + (7/8*s_2 - 1/288*s_3*s_8 - 1/96*s_4*s_7 + 7/13824*s_4*s_8^2 - 1/48*s_5*s_6 + 5/2592*s_5*s_7*s_8 - 49/746496*s_5*s_8^3)*z^3 +$

$(s_1-1/576*s_2*s_8-1/144*s_3*s_7+7/20736*s_3*s_8^2-1/64*s_4*s_6+5/3456*s_4*s_7*s_8-49/995328*s_4*s_8^3)*z^2+(-1/288*s_2*s_7+7/41472*s_2*s_8^2-1/96*s_3*s_6+5/5184*s_3*s_7*s_8-49/1492992*s_3*s_8^3)*z+(-1/192*s_2*s_6+5/10368*s_2*s_7*s_8-49/2985984*s_2*s_8^3)$

$z^3*\varphi(z,s)\equiv(5/8*s_4-1/18*s_5*s_8-7/144*s_6*s_7+217/41472*s_6*s_8^2+17/3456*s_7^2*s_8-49/93312*s_7*s_8^3+2401/214990848*s_8^5)*z^6+(3/4*s_3-1/192*s_4*s_8-1/18*s_5*s_7+7/10368*s_5*s_8^2-5/192*s_6^2+1/162*s_6*s_7*s_8-245/2985984*s_6*s_8^3+1/576*s_7^3-91/248832*s_7^2*s_8^2+343/35831808*s_7*s_8^4)*z^5+(7/8*s_2-1/288*s_3*s_8-1/96*s_4*s_7+7/13824*s_4*s_8^2-1/18*s_5*s_6+5/2592*s_5*s_7*s_8-49/746496*s_5*s_8^3+65/20736*s_6^2*s_8+5/3456*s_6*s_7^2-455/1492992*s_6*s_7*s_8^2+1715/214990848*s_6*s_8^4)*z^4+(s_1-1/576*s_2*s_8-1/144*s_3*s_7+7/20736*s_3*s_8^2-1/64*s_4*s_6+5/3456*s_4*s_7*s_8-49/995328*s_4*s_8^3-1/36*s_5^2+13/5184*s_5*s_6*s_8+1/864*s_5*s_7^2-91/373248*s_5*s_7*s_8^2+343/53747712*s_5*s_8^4)*z^3+(-1/288*s_2*s_7+7/41472*s_2*s_8^2-1/96*s_3*s_6+5/5184*s_3*s_7*s_8-49/1492992*s_3*s_8^3-1/48*s_4*s_5+13/6912*s_4*s_6*s_8+1/1152*s_4*s_7^2-91/497664*s_4*s_7*s_8^2+343/71663616*s_4*s_8^4)*z^2+(-1/192*s_2*s_6+5/10368*s_2*s_7*s_8-49/2985984*s_2*s_8^3-1/72*s_3*s_5+13/10368*s_3*s_6*s_8+1/1728*s_3*s_7^2-91/746496*s_3*s_7*s_8^2+343/107495424*s_3*s_8^4)*z+(-1/144*s_2*s_5+13/20736*s_2*s_6*s_8+1/3456*s_2*s_7^2-91/1492992*s_2*s_7*s_8^2+343/214990848*s_2*s_8^4)$

$z^4*\varphi(z,s)\equiv(3/4*s_3-19/288*s_4*s_8-1/18*s_5*s_7+7/1152*s_5*s_8^2-5/192*s_6^2+113/10368*s_6*s_7*s_8-49/82944*s_6*s_8^3+1/576*s_7^3-35/41472*s_7^2*s_8^2+6517/107495424*s_7*s_8^4-16807/15479341056*s_8^6)*z^6+(7/8*s_2-1/288*s_3*s_8-1/16*s_4*s_7+7/13824*s_4*s_8^2-1/18*s_5*s_6+17/2592*s_5*s_7*s_8-49/746496*s_5*s_8^3+65/20736*s_6^2*s_8+19/3456*s_6*s_7^2-553/746496*s_6*s_7*s_8^2+1715/214990848*s_6*s_8^4-17/41472*s_7^3*s_8+49/1119744*s_7^2*s_8^3-2401/2579890176*s_7*s_8^5)*z^5+(s_1-1/576*s_2*s_8-1/144*s_3*s_7+7/20736*s_3*s_8^2-17/288*s_4*s_6+5/3456*s_4*s_7*s_8-49/995328*s_4*s_8^3-1/36*s_5^2+11/1728*s_5*s_6*s_8+1/864*s_5*s_7^2-91/373248*s_5*s_7*s_8^2+343/53747712*s_5*s_8^4+35/10368*s_6^2*s_7-1085/2985984*s_6^2*s_8^2-85/248832*s_6*s_7^2*s_8+245/6718464*s_6*s_7*s_8^3-12005/15479341056*s_6*s_8^5)*z^4+(-1/288*s_2*s_7+7/41472*s_2*s_8^2-1/96*s_3*s_6+5/5184*s_3*s_7*s_8-49/1492992*s_3*s_8^3-1/18*s_4*s_5+13/6912*s_4*s_6*s_8+1/1152*s_4*s_7^2-91/497664*s_4*s_7*s_8^2+343/71663616*s_4*s_8^4+1/324*s_5^2*s_8+7/2592*s_5*s_6*s_7-217/746496*s_5*s_6*s_8^2-17/62208*s_5*s_7^2*s_8+49/1679616*s_5*s_7*s_8^3-2401/3869835264*s_5*s_8^5)*z^3+(-1/192*s_2*s_6+5/10368*s_2*s_7*s_8-49/2985984*s_2*s_8^3-1/72*s_3*s_5+13/10368*s_3*s_6*s_8+1/1728*s_3*s_7^2-91/746496*s_3*s_7*s_8^2+343/107495424*s_3*s_8^4-5/192*s_4^2+1/432*s_4*s_5*s_8+7/3456*s_4*s_6*s_7-217/995328*s_4*s_6*s_8^2-17/82944*s_4*s_7^2*s_8+49/2239488*s_4*s_7*s_8^3-2401/5159780352*s_4*s_8^5)*z^2+(-1/144*s_2*s_5+13/20736*s_2*s_6*s_8+1/3456*s_2*s_7^2-91/1492992*s_2*s_7*s_8^2+343/214990848*s_2*s_8^4-5/288*s_3*s_4+1/648*s_3*s_5*s_8+7/5184*s_3*s_6*s_7-217/1492992*s_3*s_6*s_8^2-17/124416*s_3*s_7^2*s_8+49/3359232*s_3*s_7*s_8^3-2401/7739670528*s_3*s_8^5)*z+(-5/576*s_2*s_4+1/1296*s_2*s_5*s_8+7/10368*s_2*s_6*s_7-217/2985984*s_2*s_6*s_8^2-17/248832*s_2*s_7^2*s_8+49/6718464*s_2*s_7*s_8^3-2401/15479341056*s_2*s_8^5)$

We omit $z^5*\varphi(z,s)$, $z^6*\varphi(z,s)$. The vector (2.9) is given as follows

$$-72z^7\equiv(s_1,2s_2,3s_3,4s_4,5s_5,6s_6,7s_7).$$

We list the rows of the matrix $\iota^*(\Sigma)(x,t)$ below. In this way we introduce 8 vector fields $w_i(x,t)\in\mathbb{C}^8$, $1\le i\le 7$.

$w_1(x,t)=(-t^2+1/6ax_1^2x_2+x_2^2, 1/3ax_1(a(-t^2+x_1^2)+10x_2), a^2(-3t^2+(5x_1^2)/2)-(3ax_2)/2+x_1^2x_2, 1/3x_1(-7a^2-8at^2+8ax_1^2+12x_2), a^2/2-8at^2+(23ax_1^2)/6-3x_2, -6ax_1-$

$4t^2x_1 + 4x_1^3, (3a)/2 - 4t^2 - 3x_1^2, 0)$

$w_2(x,t) = (0, 0, 0, 1/36(-3a^3 + a^2(-52t^2 + 48x_1^2) - 48t^2x_2 - 4a(32t^4 - 8t^2x_1^2 - 24x_1^4 + 9x_2)), -(1/36)x_1(9a^2 + 296at^2 + 54ax_1^2 - 144x_2), -(a^2/4) - 4at^2 - (16t^4)/3 + (10ax_1^2)/3 + (4t^2x_1^2)/3 + 4x_1^4 - 3x_2, -(1/6)x_1(15a + 80t^2 + 18x_1^2), 0)$

$w_3(x,t) = (1/108ax_1^2(15a+80t^2+18x_1^2)x_2, 1/54ax_1(-15a^2(t^2-x_1^2) - 28t^2x_2 - 2a(40t^4 - 31t^2x_1^2 - 9x_1^4 + 6x_2)), 1/36(-16a^2t^4 - 21a^3x_1^2 + 30ax_1^2x_2 + 3a^2(-2x_1^4 + x_2) + 36x_2(x_1^4 + x_2) + 2t^2(-18 + 3a^3 - 38a^2x_1^2 - 4ax_2 + 80x_1^2x_2)), 1/54x_1(21a^3 - 2a^2(67t^2 - 60x_1^2) - 168t^2x_2 + a(-640t^4 + 496t^2x_1^2 + 36(4x_1^4 + 3x_2))), 1/108(-9a^3 - 3a^2(52t^2 + 77x_1^2) - 144t^2x_2 - 2a(192t^4 + 952t^2x_1^2 + 81x_1^4 + 54x_2)), 1/9x_1(9a^2 + a(-44t^2 + 30x_1^2) + 4(-40t^4 + 31t^2x_1^2 + 9(x_1^4 + x_2))), 1/36(-9a^2 - 18a(8t^2 + 5x_1^2) - 4(48t^4 + 268t^2x_1^2 + 27(x_1^4 + x_2))), 0)$

$w_4(x,t) = (1/648ax_1x_2(9a^2 + 18a(8t^2 + 5x_1^2) + 4(48t^4 + 268t^2x_1^2 + 27(x_1^4 + x_2))), -(1/648)a(18a^3(t^2 - x_1^2) + 9a^2(32t^4 - 12t^2x_1^2 - 20x_1^4 + x_2) + 4x_2(48t^4 + 148t^2x_1^2 + 27x_2) + 8a(48t^6 + 220t^4x_1^2 - 27x_1^2(x_1^4 + x_2) + t^2(-241x_1^4 + 45x_2))), 1/216x_1(-9a^4 - 6a^3(34t^2 + 5x_1^2) + 4a(44t^2 + 45x_1^2)x_2 - 2a^2(256t^4 + 412t^2x_1^2 + 18x_1^4 + 69x_2) + 8x_2(48t^4 + 268t^2x_1^2 + 27(x_1^4 + x_2))), 1/648(9a^4 + 36a^3(3t^2 - 4x_1^2) + 4a^2(-600t^4 + 142t^2x_1^2 + 360x_1^4 + 27x_2) - 24t^2(27 + 48t^2x_2 + 148x_1^2x_2) - 16a(192t^6 + 880t^4x_1^2 - 108x_1^2(x_1^4 + x_2) + t^2(-964x_1^4 + 171x_2))), -(1/648)x_1(-27a^3 + 6a^2(868t^2 + 135x_1^2) + 2016t^2x_2 + 4a(3120t^4 + 5212t^2x_1^2 + 243x_1^4 + 351x_2)), 1/216(9a^3 + 24a^2(2t^2 - 5x_1^2) - 4a(336t^4 + 40t^2x_1^2 - 9(20x_1^4 + 3x_2)) - 32(48t^6 + 220t^4x_1^2 - 27x_1^2(x_1^4 + x_2) + t^2(-241x_1^4 + 36x_2))), 1/108x_1(45a^2 - 6a(256t^2 + 45x_1^2) - 4(816t^4 + 1504t^2x_1^2 + 81(x_1^4 + x_2))), 0)$

$w_5(x,t) = ((ax_1^2x_2(-45a^2 + 6a(256t^2 + 45x_1^2) + 4(816t^4 + 1504t^2x_1^2 + 81(x_1^4 + x_2))))/1944, -(1/972)ax_1(-45a^3(t^2 - x_1^2) + 6a^2(256t^4 - 211t^2x_1^2 - 45x_1^4 - 6x_2) + 56t^2(24t^2 + 25x_1^2)x_2 + 4a(816t^6 + 688t^4x_1^2 - 81x_1^2(x_1^4 + x_2) + t^2(-1423x_1^4 + 219x_2))), 1/648(-9a^4(2t^2 - 7x_1^2) - 3a^3(96t^4 + 476t^2x_1^2 + 30x_1^4 + 3x_2) - 4ax_2(48t^4 - 620t^2x_1^2 + 27(-5x_1^4 + x_2)) + 8x_1^2x_2(816t^4 + 1504t^2x_1^2 + 81(x_1^4 + x_2)) - 2a^2(192t^6 + 2512t^4x_1^2 + 54x_1^6 + 99x_1^2x_2 + 4t^2(511x_1^4 + 45x_2))), 1/972x_1(-63a^4 + 30a^3(7t^2 - 12x_1^2) - 336t^2(24t^2 + 25x_1^2)x_2 - 4a^2(3240t^4 - 2357t^2x_1^2 - 540x_1^4 + 81x_2) - 8a(3264t^6 + 2752t^4x_1^2 - 324x_1^2(x_1^4 + x_2) + t^2(-5692x_1^4 + 801x_2))), (1/1944)(27a^4 + 9a^3(36t^2 + 77x_1^2) - 6a^2(1200t^4 + 6116t^2x_1^2 + 405x_1^4 - 54x_2) - 72t^2(27 + 48t^2x_2 + 148x_1^2x_2) - 4a(2304t^6 + 30960t^4x_1^2 + 729x_1^2(x_1^4 + x_2) + 4t^2(6508x_1^4 + 513x_2))), 1/162x_1(-27a^3 - 30a^2(2t^2 + 3x_1^2) - 4a(936t^4 - 458t^2x_1^2 + 27(-5x_1^4 + x_2)) - 8(816t^6 + 688t^4x_1^2 - 81x_1^2(x_1^4 + x_2) + t^2(-1423x_1^4 + 144x_2))), 1/648(27a^3 + 18a^2(8t^2 + 15x_1^2) - 12a(336t^4 + 1832t^2x_1^2 - 27(-5x_1^4 + x_2)) - 8(576t^6 + 8352t^4x_1^2 + 243x_1^2(x_1^4 + x_2) + t^2(7636x_1^4 + 432x_2))), 0)$

$w_6(x,t) = ((ax_1x_2(-27a^3 - 18a^2(8t^2 + 15x_1^2) + 12a(336t^4 + 1832t^2x_1^2 - 27(-5x_1^4 + x_2)) + 8(576t^6 + 8352t^4x_1^2 + 243x_1^2(x_1^4 + x_2) + t^2(7636x_1^4 + 432x_2))))/11664, (1/11664)a(54a^4(t^2 - x_1^2) + 9a^3(32t^4 + 28t^2x_1^2 - 60x_1^4 + 3x_2) - 32t^2x_2(144t^4 + 1476t^2x_1^2 + 781x_1^4 + 108x_2) - 24a^2(336t^6 + 1496t^4x_1^2 + 27x_1^2(-5x_1^4 + x_2) - t^2(1697x_1^4 + 33x_2)) - 4a(2304t^8 + 31104t^6x_1^2 + 16t^4(-179x_1^4 + 171x_2) + 4t^2(-7393x_1^6 + 609x_1^2x_2) - 81(12x_1^8 + 12x_1^4x_2 + x_2^2))), (1/3888)x_1(27a^5 + 18a^4(18t^2 + 5x_1^2) - 6a^3(1696t^4 + 2820t^2x_1^2 + 90x_1^4 - 69x_2) + 8ax_2(336t^4 + 4796t^2x_1^2 - 81(-5x_1^4 + x_2)) + 16x_2(576t^6 + 8352t^4x_1^2 + 243x_1^2(x_1^4 + x_2) + t^2(7636x_1^4 + 432x_2)) - 4a^2(4416t^6 + 19456t^4x_1^2 + 27x_1^2(6x_1^4 + 11x_2) + 4t^2(2395x_1^4 + 453x_2))), (1/11664)(-27a^5 - 36a^4(t^2 - 12x_1^2) - 192t^2x_2(144t^4 + 1476t^2x_1^2 + 781x_1^4 + 108x_2) + 12a^3(96t^4 - 142t^2x_1^2 - 9(40x_1^4 + 3x_2)) - 16a^2(4176t^6 + 19428t^4x_1^2 + 324x_1^2(-5x_1^4 + x_2) - t^2(19583x_1^4 + 189x_2)) - 32a(2304t^8 + 31104t^6x_1^2 - 972x_1^4(x_1^4 + x_2) + 16t^4(-179x_1^4 + 162x_2) + t^2(-29572x_1^6 + 1971x_1^2x_2))), -(1/11664)x_1(81a^4 - 90a^3(68t^2 + 27x_1^2) + 36a^2(7120t^4 + 12124t^2x_1^2 + 405x_1^4 - 117x_2) + 4032t^2(24t^2 + 25x_1^2)x_2 + 8a(53568t^6 +$

$241824t^4x_1^2+2187x_1^2(x_1^4+x_2)+4t^2(30649x_1^4+5103x_2))), -(a^4/144)-(256t^8)/27-128t^6x_1^2+1/54a^3(6t^2+5x_1^2)+t^4((2864x_1^4)/243-(80x_2)/9)+4x_1^4(x_1^4+x_2)-1/324a^2(96t^4+536t^2x_1^2+180x_1^4+27x_2)+t^2(-1+(29572x_1^6)/243-(64x_1^2x_2)/27)-2/243a(1152t^6+5964t^4x_1^2+81x_1^2(-5x_1^4+x_2)+2t^2(-2155x_1^4+54x_2)), -((x_1(135a^3-18a^2(16t^2+45x_1^2)+36a(2032t^4+3664t^2x_1^2-27(-5x_1^4+x_2))+8(13824t^6+66720t^4x_1^2+729x_1^2(x_1^4+x_2)+8t^2(4547x_1^4+594x_2))))/1944), 0)$

$w_7(x,t) = ((ax_1^2x_2(135a^3-18a^2(16t^2+45x_1^2)+36a(2032t^4+3664t^2x_1^2-27(-5x_1^4+x_2))+8(13824t^6+66720t^4x_1^2+729x_1^2(x_1^4+x_2)+8t^2(4547x_1^4+594x_2))))/34992, (1/17496)ax_1(-135a^4(t^2-x_1^2)+18a^3(16t^4+29t^2x_1^2-45x_1^4-6x_2)-16t^2x_2(3024t^4+10416t^2x_1^2+3367x_1^4+864x_2)-36a^2(2032t^6+1632t^4x_1^2+27x_1^2(-5x_1^4+x_2)-t^2(3529x_1^4+25x_2))-8a(13824t^8+52896t^6x_1^2-729x_1^4(x_1^4+x_2)+8t^4(-3793x_1^4+1071x_2)+t^2(-35647x_1^6+99x_1^2x_2))), (1/11664)(27a^5(2t^2-7x_1^2)+9a^4(32t^4+60t^2x_1^2+30x_1^4+3x_2)-6a^3(1344t^6+18176t^4x_1^2+270x_1^6-99x_1^2x_2-4t^2(-3799x_1^4+33x_2))-8ax_2(576t^6-12384t^4x_1^2+243x_1^2(-5x_1^4+x_2)+4t^2(-7463x_1^4+108x_2))+16x_1^2x_2(13824t^6+66720t^4x_1^2+729x_1^2(x_1^4+x_2)+8t^2(4547x_1^4+594x_2))-4a^2(2304t^8+58752t^6x_1^2+16t^4(8161x_1^4+171x_2)+4t^2(10795x_1^6+3021x_1^2x_2)-81(-6x_1^8-11x_1^4x_2+x_2^2))),$
$(1/17496)x_1(189a^5+18a^4(13t^2+60x_1^2)-36a^3(192t^4+167t^2x_1^2+180x_1^4-27x_2)-96t^2x_2(3024t^4+10416t^2x_1^2+3367x_1^4+864x_2)-8a^2(76176t^6+69168t^4x_1^2+972x_1^2(-5x_1^4+x_2)+t^2(-123677x_1^4+621x_2))-16a(55296t^8+211584t^6x_1^2-2916x_1^4(x_1^4+x_2)+32t^4(-3793x_1^4+999x_2)-t^2(142588x_1^6+2151x_1^2x_2))), (1/34992)(-81a^5-27a^4(4t^2+77x_1^2)+18a^3(192t^4+116t^2x_1^2+405x_1^4-54x_2)-576t^2x_2(144t^4+1476t^2x_1^2+781x_1^4+108x_2)-12a^2(16704t^6+230112t^4x_1^2+t^2(196468x_1^4-756x_2)-729x_1^2(-5x_1^4+x_2))-8a(27648t^8+718848t^6x_1^2+6561x_1^4(x_1^4+x_2)+96t^4(17017x_1^4+324x_2)+4t^2(138634x_1^6+35613x_1^2x_2))), (1/2916)x_1(81a^4+18a^3(58t^2+15x_1^2)-36a^2(240t^4+254t^2x_1^2+45x_1^4-9x_2)-8a(21312t^6+25104t^4x_1^2+243x_1^2(-5x_1^4+x_2)+2t^2(-14197x_1^4+648x_2))-16(13824t^8+52896t^6x_1^2-729x_1^4(x_1^4+x_2)+8t^4(-3793x_1^4+783x_2)-t^2(35647x_1^6+2448x_1^2x_2))), 1/11664(-81a^4+162a^3(8t^2-5x_1^2)-36a^2(96t^4+424t^2x_1^2+27(-5x_1^4+x_2))-24a(4608t^6+66528t^4x_1^2-243x_1^2(-5x_1^4+x_2)+8t^2(7463x_1^4+54x_2))-16(6912t^8+190080t^6x_1^2+2187x_1^4(x_1^4+x_2)+48t^4(9551x_1^4+135x_2)+t^2(729+165916x_1^6+34992x_1^2x_2))), 0)$

$w_8(x,t) = (4ax_1x_2, 2(-4a^2t^2+4a^2x_1^2-2ax_2), 3(-4a^2x_1+8x_1x_2), 4(a^2-16at^2+16ax_1^2-6x_2), -100ax_1, 6(6a-16t^2+16x_1^2), -168x_1, 72)$

At the focal point $(x,t) = (0,-1/2a,1/2a)$ the matrix $\iota^*(\Sigma)(0,-1/2a,1/2a)$ has the following form with rank 5 if $a \neq 1$ and rank 3 if $a = 1$.

$$\begin{pmatrix}
0 & 0 & 0 & 0 & (-1+a^3)/(2a) & 0 & -(1/a^2)+(3a)/2 & 0 \\
0 & 0 & 0 & A_1 & 0 & A_2 & 0 & 0 \\
0 & 0 & 0 & 0 & A_1 & 0 & A_2 & 0 \\
0 & 0 & 0 & A_3 & 0 & A_4 & 0 & 0 \\
0 & 0 & 0 & 0 & A_3 & 0 & A_4 & 0 \\
0 & 0 & 0 & A_5 & 0 & A_6 & 0 & 0 \\
0 & 0 & 0 & 0 & A_5 & 0 & A_6 & 0 \\
0 & 0 & 0 & 4(-1+a^3)/a & 0 & 6(-(4/a^2)+6a) & 0 & 72
\end{pmatrix} \tag{5.4}$$

where $A1 = \frac{-(2-5a^3+3a^6)}{36a^3}$, $A_2 = \frac{-(4-6a^3+3a^6)}{12a^4}$, $A_3 = \frac{-4+10a^3-9a^6+3a^9}{216a^5}$, $A_4 = \frac{(-2+a^3)^2(-2+3a^3)}{72a^6}$, $A_5 = -\frac{(-2+a^3)^2(2-5a^3+3a^6)}{1296a^7}$, $A_6 = -\frac{16-56a^3+68a^6-30a^9+3a^12}{432a^8}$.

Thus together with the data

$$d_{x,t}\iota(0,-1/2a,1/2a) \tag{5.5}$$

$$= \begin{pmatrix} 0 & -2 & 0 & -(4/a)-4a^2 & 0 & -20a & 0 & -24 \\ 0 & -(1/a) & 0 & -2a & 0 & -6 & 0 & 0 \\ -(1/a) & 0 & -4a & 0 & -16 & 0 & -(16/a) & 0 \end{pmatrix}$$

we conclude that $rank\ T(0,-1/2a,1/2a) = 8 = \nu$ if $a \neq 1$. Therefore after Proposition 4.2, the germ of the large wave front LW defines a free divisor in the neighbourhood of the focal point $(0,-1/2a,1/2a)$ for $a \neq 1$.

In the case $a = 1$, $rank\ \iota^*(\Sigma)(0,-1/2,1/2) = rank\ \iota^*(\tilde{\Sigma})(0,-1/2,1/2)+1 = 3$ and

$$rank\ T(0,-1/2,1/2) = 6 < 8. \tag{5.6}$$

We see that the focal point $(0,-1/2,1/2)$ is an isolated point after the following reasoning. The matrices above (5.4), (5.5) entail the following relationship

$$span_{\mathbb{C}}\{v_1(\iota(0,-1/2,1/2)),\cdots,v_8(\iota(0,-1/2,1/2))\}$$
$$\cap span_{\mathbb{C}}\{\frac{\partial\iota}{\partial t},\frac{\partial\iota}{\partial x_1},\frac{\partial\iota}{\partial x_2}\}_{(0,-1/2,1/2)} = \{0\}.$$

This means that the germ of the integral varieties of the vector fields $\{v_1(s),\cdots,v_8(s)\}$ (i.e. the stratum of A_5 singularities of the discriminantal loci $D_{\varphi,\iota(0,-1/2,1/2)}$) and the image $\iota(\mathbb{C}^3)$ intersect transversally at $\iota(0,-1/2,1/2)$. In addition to that we can verify that the limit of tangent vectors to the stratum of A_4 singularities adjacent to A_5 stratum near $\iota(0,-1/2,1/2)$ generated by the rows of the following matrix

$$lim_{s_5\to 0}\frac{\Sigma(\iota(0,-1/2,1/2)+(0,0,0,0,s_5,0,0))-\iota^*(\Sigma)(0,-1/2,1/2)}{s_5}$$

$$= \frac{\partial\Sigma(\iota(0,-1/2,1/2)+(0,0,0,0,s_5,0,0))}{\partial s_5}|_{s_5=0}$$

$$= \begin{pmatrix} 0&0&0&0&0&3/8&0&0 \\ 0&0&0&0&-(5/144)&0&3/8&0 \\ 0&0&0&0&0&-(7/72)&0&0 \\ 0&0&0&0&5/864&0&-(7/72)&0 \\ 0&0&0&0&0&19/864&0&0 \\ 0&0&0&0&-(5/5184)&0&19/864&0 \\ 0&0&0&0&0&-(1/216)&0&0 \\ 0&0&0&0&5&0&0&72 \end{pmatrix}$$

are linearly independent of a vector from $span_{\mathbb{C}}\{\frac{\partial\iota}{\partial t},\frac{\partial\iota}{\partial x_1},\frac{\partial\iota}{\partial x_2}\}_{(0,-1/2,1/2)}$.

This means that $(0,-1/2,1/2)$ is an isolated point on $LW \subset \iota^{-1}(D_\varphi)$ with the property (5.6). Upshot is the almost freenes of the large wave front germ at the focal point after Proposition 4.3.

In summary we established

Proposition 5.1. *The germ of the large wave front LW at the focal point* $(x_1, x_2, t) = (0, -1/2a, 1/2a)$ *defines a free divisor if* $a \neq 1$. *If* $a = 1$ *it defines an almost free divisor germ at the focal point* $(x_1, x_2, t) = (0, -1/2, 1/2)$.

2. Wave propagation in the 3 dimensional space

Now we consider the following initial wave front in the 3−dimensional space, $Y := \{(z,u) \in \mathbb{C}^2 : -\frac{1}{2}(k_1 z_1^2 + k_2 z_2^2) + u = 0\}$, i.e. $F(z) = -\frac{1}{2}(k_1 z_1^2 + k_2 z_2^2)$ for $0 < k_1 < k_2$. In this case our phase function has the following expression

$$\begin{aligned}\Psi(x,t,z) = {} & (-x_3 + k_1(x_1 - z_1)z_1 + k_2(x_2 - z_2)z_2 + 1/2(k_1 z_1^2 + k_2 z_2^2))^2 - t^2(1 + k_1^2 z_1^2 + k_2^2 z_2^2),\\ = {} & -t^2 + x_3^2 - k_1^2 x_1 z_1^3 + (k_1^2 z_1^4)/4 - 2k_2 x_3 (x_2 - z_2) z_2\\ & -k_2^2 t^2 z_2^2 - k_2 x_3 z_2^2 + k_2^2 (x_2 - z_2)^2 z_2^2 + k_2^2 (x_2 - z_2) z_2^3 + (k_2^2 z_2^4)/4\\ & + z_1^2(-k_1^2 t^2 + k_1^2 x_1^2 + k_1 x_3 - k_1 k_2 (x_2 - z_2) z_2 - 1/2 k_1 k_2 z_2^2)\\ & + z_1(-2k_1 x_1 x_3 + 2k_1 k_2 x_1 (x_2 - z_2) z_2 + k_1 k_2 x_1 z_2^2) \qquad (5.7)\end{aligned}$$

It is easy to see that the point $(x_1, x_2, x_3, t) = (0, 0, 1/k_1, 1/k_1)$ is a focal point with a singular point $(z, u) = (0, 0)$ and the Milnor number $\mu(0) = 3$. We have the following tame polynomial,

$$\Psi(0, 0, 1/k_1, 1/k_1, z) = (k_1^4 z_1^4 + 4k_1 k_2 z_2^2 - 4k_2^2 z_2^2 + 2k_1^3 k_2 z_1^2 z_2^2 + k_1^2 k_2^2 z_2^4)/4k_1^2.$$

As a matter of fact, the polynomial $\Psi(0, 0, 1/k_1, 1/k_1, z)$ satisfies the criterion on the presence of A_3 singularity at the origin mentioned in [7], Theorem 2.2, (2). The situation is the same at another focal point $(x_1, x_2, x_3, t) = (0, 0, 1/k_2, 1/k_2)$. The quotient ring (1.5) for this $\Psi(0, 0, 1/k_1, 1/k_1, z)$ has dimension $\mu = 5$.

We can choose

$$\{e_1, e_2, e_3, e_4, e_5\} = \{1, z_1, z_1^2, z_2, z_2^2\}$$

as the basis (2.7). In view of (5.7), we introduce addtional deformation monomials $e_6 = z_1 * z_2, e_7 = z_2^3$, $e_8 = z_1^3$, $e_9 = z_1^2 * z_2$, $e_{10} = z_1 * z_2^2$ together with the entries of the mapping ι,

$$\begin{aligned} s_1 = -t^2 + x_3^2, s_2 = -2k_1 x_1 x_3, s_3 = -k_1^2 t^2 + k_1^2 x_1^2 + k_1 x_3, s_4 = -2k_2 x_2 x_3\\ s_5 = -(k_2/k_1) + k_2^2/k_1^2 - k_2^2 t^2 + k_2^2 x_2^2 + k_2 x_3, s_6 = 2k_1 k_2 x_1 x_2\\ s_7 = -k_2^2 x_2, s_8 = -k_1^2 x_1, s_9 = -k_1 k_2 x_2, s_{10} = -k_1 k_2 x_1.\end{aligned}$$

It turns out that the image of the mapping $\iota(\mathbb{C}^4) \subset \mathbb{C}^{10}$ is contained in a constructible set $\mathbb{C} \times U$ where the value of the matrix $\Sigma(s)$ is well-defined at each point $s \in \mathbb{C} \times U$. Therefore

$$dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{d_z(\Psi(x,t,z))\mathbb{C}[z]} = 5,$$

for every $(x,t) \in \mathbb{C}^4$. This means that the Assumption I,(ii) is satisfied. By direct calculation with the aid of SINGULAR, we can verify that $dim\ U = 5$. This can be seen from the fact that

$$dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{d_z(\Psi(0,0,1/k_1,1/k_1,z) + \sum_{i=1}^{6} s_i e_i)\mathbb{C}[z]} = 5,$$

while

$$dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{d_z(\Psi(0,0,1/k_1,1/k_1,z) + \sum_{i=1}^{6} s_i e_i + s_j e_j)\mathbb{C}[z]} = 7,$$

for $j = 7, 8, 9, 10$. This implies that the Assumption I,(i) is satisfied with $\nu = 6$.

At the focal point $(x_1, x_2, x_3, t) = (0, 0, 1/k_1, 1/k_1)$ the matrix $\iota^*(\Sigma)$ has the following form with rank 3

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -k_2(k_1-k_2)/2k_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (k_1-k_2)^2/k_1^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & (k_1-k_2)^2/k_1^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Together with the data

$$d_{x,t}\iota(0,0,1/k_1,1/k_1) =$$

$$\begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 & -k_1^2 & 0 & -k_1k_2 \\ 0 & 0 & 0 & -2k_2/k_1 & 0 & 0 & -k_2^2 & 0 & -k_1k_2 & 0 \\ 2/k_1 & 0 & k_1 & 0 & k_2 & 0 & 0 & 0 & 0 & 0 \\ -2/k_1 & 0 & -2k_1 & 0 & -2k_2^2/k_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we see that the $rank\ T(0,0,1/k_1,1/k_1) = 7 \geq \nu$. By virtue of the Proposition 4.3, we see that the wave front defines a free divisor germ in the neighbourhood of the focal point $(0,0,1/k_1,1/k_1)$.

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