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Bifurcation of indices in one-parameter deformations

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This short note is a résumé of our preprint [5].

Let $S_{\epsilon}^{n-1}(a)$ be the $(n-1)$-dimensional sphere with center a and radius ϵ. For a map-germ $f:(\mathbb{R}^{n},a)\to(\mathbb{R}^{n},0)$ with an isolated zero $f^{-1}(0)=\{a\}$ near a, we can consider the mapping $f/\|f\|:S_{\epsilon}^{n-1}(a)\to S_{1}^{n-1}$ (ϵ : sufficiently small) and its topological mapping degree. We call this degree the *index of f at a* and denote it by $\mathrm{ind}_{a}[f]$ ([1, Chap. 5]).

Let $\mathbb{R}\{x\}$ be the convergent power series ring in variables $x_{1},\cdots,x_{n}$ $(x=(x_{1},\cdots,x_{n})\in \mathbb{R}^{n})$ and (f_{0}) be the ideal in $\mathbb{R}\{x\}$ generated by components of f_{0}. We consider a polynomial map-germ $f_{0}:(\mathbb{R}^{n},0)\to(\mathbb{R}^{n},0)$ with $\dim_{\mathbb{R}}\mathbb{R}\{x\}/(f_{0})<\infty$. Then f_{0} has an isolated zero $f_{0}^{-1}(0)=\{0\}$ near 0 and the index of f_{0} at the origin is defined.

Let $f:(\mathbb{R}^{n}\times \mathbb{R},0)\to(\mathbb{R}^{n},0)$ be any polynomial one-parameter deformation of f_{0} and $g:(\mathbb{R}^{n}\times \mathbb{R},0)\to(\mathbb{R},0)$ be any polynomial function-germ. We denote $f_{t}(x)=f(x,t)$ and $g_{t}(x)=g(x,t)$.

When we deform f_{0} to f_{t} $(t>0)$, the zero of f_{0} bifurcates to the zeros of f_{t}. The sum of the indices over these zeros of f_{t} is equal to $\mathrm{ind}_{0}[f_{0}]$. Then we are interested in the subsets of zeros which bifurcate into the region $g_{t}>0$ $(<0,=0$ respectively). Our aim is to construct certain map-germs, the indices of which at the origin give the sums of indices of f_{t} over these subsets of zeros.

For any polynomial function-germ $\varphi:(\mathbb{R}^{n}\times \mathbb{R},0)\to(\mathbb{R},0)$ and any positive integer l, we define

$$L(l)=L(l;f,\varphi)=\dim_{\mathbb{R}}\mathbb{R}\{x,t\}/(f,\varphi+t^{l}).$$

It turns out $\{L(l)\}$ is an arithmetic sequence for sufficiently large integers l.

Definition. We define "$l_{0}=l_{0}(f,\varphi)$" by the minimal exponent in the set of integers l, for which $L(l)$ is finite and satisfies the general term of the arithmetic sequence determined by large integers l.

We define $I_{+}=I(f,t>0;g>0)$ by the sum of the indices over the zeros of f_{t} $(t>0)$, which bifurcate to the region $g_{t}>0$ from the zero of f_{0}, and we define $I_{-}=I(f,t>0;g<0)$, $I_{0}=I(f,t>0;g=0)$ similarly.

We call a function $\varphi(x,t)$ has the symmetry with respect to the hyperplane $t = 0$, if $\varphi(x,t) = \varphi(x,-t)$. For any function $\varphi(x,t)$, by setting $\varphi'(x,t) = \varphi(x,t^2)$, $\varphi'(x,t)$ has the symmetry with respect to $t = 0$.

Theorem. *We assume that f and g have the symmetry with respect to $t = 0$.*

(i) for any even integer l with $l \geq l_0(f,tg)+1$, if we consider $F' = (f, tg + t^l) : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$, then we obtain

$$\mathrm{ind}_0[F'] = I_+ - I_-.$$

(ii) for any odd integer l with $l \geq l_0(f,g)+1$, if we consider $F'' = (f, g + t^l) : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$, then we obtain

$$\mathrm{ind}_0[F''] = I_0.$$

The sum of indices of f_t over all the zeros, which bifucate from the zero of f_0, is $I_+ + I_- + I_0 = \mathrm{ind}_0[f_0]$. Therefore we obtain the formula about each of I_+, I_- and I_0.

Remark. For $F = (f, tg) : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$, $\dim_{\mathbb{R}} \mathbb{R}\{x,t\}/(f,tg)$ may not be finite. But if $\dim_{\mathbb{R}} \mathbb{R}\{x,t\}/(f,tg) < \infty$, then we can prove the formula

$$\mathrm{ind}_0[F] = I_+ - I_-,$$

by a similar method as in [3], [4].

There are related works: (i) for a generic f and the function $g(x,t) = t$, f_t has the only simple zeros outside $t = 0$. The number of the zeros in $t > 0$ (< 0 respectively) is studied in [3]. (ii) for generic f and g, the deformation f_t has the only simple zeros outside $g_t^{-1}(0)$. The number of the zeros in $g_t > 0$ (< 0 respectively) can be obtained by modifying the method developed in [4].

In our case: Since f is arbitrary, f_t may have multiple zeros, and since g is arbitrary, some zeros of f_t may belong to $g_t^{-1}(0)$. Because we want to treat arbitrary bifurcations of the zero of f_0, these non-generic situations can not be avoided. Therefore our results are different from theirs.

In the example below, we need the next proposition and lemma. Both of these are considered for the complexifications of f_0, f and φ. We consider a polynomial map-germ $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with $\dim_{\mathbb{C}} \mathbb{C}\{x\}/(f_0) < \infty$. Let $f : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ be any polynomial one-parameter deformation of f_0 and $\varphi : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be any polynomial function-germ. We have $L(l) = L(l; f, \varphi) = \dim_{\mathbb{C}} \mathbb{C}\{x,t\}/(f, \varphi + t^l)$.

Proposition. *For sufficiently large integers l,*

$$L(l+1) - L(l) = \left[\begin{array}{l} \textit{the sum of the multiplicities over the zeros of } f_t \ (t \neq 0), \\ \textit{which bifurcate into the hypersurface } \varphi_t = 0 \textit{ from the zero of } f_0 \end{array} \right].$$

Lemma. *If we set $l_1 = \deg(f_1) \times \cdots \times \deg(f_n) \times \deg(\varphi) + 1$, then $L(l_1)$ is finite and satisfies the general term of the arithmetic sequence determined by large integers l.*

Example. (i) for $f, g : (\mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$, we consider $f = x^2(x - t)$ and $g = x$. Symmetrizing these with respect to $t = 0$, we obtain $f' = x^2(x - t^2)$ and $g' = x$.

(i-a) for $\varphi = tg'$, we consider $L(l) = L(l; f', tg')$. By Lemma, we have the exponent $l_1 = 4 \times 2 + 1 = 9$. Starting from this l_1, we obtain Table 1. (We use "Singular" to calculate $L(l)$ [2, Chap. A].) The minimal exponent l_0 which we want is 3. Since

Table 1: (i-a)

l	9	8	7	6	5	4	3	2	1
$L(l)$	21	19	17	15	13	11	9	6	3

4 is the minimal even integer which is greater than 3, by (i) of Theorem, we obtain $\text{ind}_0[(f', tg' + t^4)] = I_+ - I_-$.

(i-b) for $\varphi = g'$, we consider $L(l) = L(l; f', g')$. By Lemma, we have the exponent $l_1 = 4 \times 1 + 1 = 5$. Starting from this l_1, we obtain Table 2. The minimal exponent l_0

Table 2: (i-b)

l	5	4	3	2	1
$L(l)$	12	10	8	6	3

which we want is 2. Since 3 is the minimal odd integer which is greater than 2, by (ii) of Theorem, we obtain $\text{ind}_0[(f', g' + t^3)] = I_0$.

(ii) sometimes, we want to deform f_0 such that two simple zeros, which are conjugate each other, run away to the imaginary region. By using our results, we can algebraically determine whether or not a given deformation f is in this situation: A polynomial map-germ $f_0 : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ with $\dim_{\mathbb{R}} \mathbb{R}\{x\}/(f_0) < \infty$ and a polynomial one-parameter deformation $f : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ of f_0 are given. We set $\mu = \dim_{\mathbb{R}} \mathbb{R}\{x\}/(f_0)$ and consider f_t for $t > 0$. If $\dim_{\mathbb{R}} \mathbb{R}\{x\}/(f_t) = \mu - 2$, then $\mu - 2$ multiple zero remains at the origin. We take $\varphi = J(f_t)$ ($J(f_t)$: the Jacobian of f_t) and consider $L(l) = L(l; f, J(f_t))$. If $L(l+1) - L(l) = \mu - 2$ (l : sufficiently large integer), by Proposition, two simple zeros bifurcate outside $J(f_t) = 0$. We symmetrize f to f' with respect to $t = 0$, then $J(f'_t)$ has the symmetry too. If $\text{ind}_0[(f', tJ(f'_t) + t^l)] = 0$ (l : sufficiently large even integer), by (i) of Theorem, these two simple zeros of f_t ($t > 0$) are not real.

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