数理解析研究所講究録
第 1665 巻 2009 年 5-8

The reduced length of a polynomial with complex or real coefficients

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Let for a polynomial $P\in\mathbb{C}[x]$, $P(x)=\sum_{i=0}^{d}a_{i}x^{d-i}=a_{0}\prod_{i=1}^{d}(x-\alpha_{i})$, $P^{*}(x)=\sum_{i=0}^{d}a_{i}x^{i}$, $L(P)=\sum_{i=0}^{d}|a_{i}|$, $M(P)=|a_{0}|\prod_{i=1}^{d}\max\{1, |\alpha_{i}|\}$ and let $\mathbb{C}[x]^{1}$, $\mathbb{R}[x]^{1}$ denote the set of monic polynomials over $\mathbb{C}$ or $\mathbb{R}$, respectively.

$L(P)$ is called the length of P. Following A. Dubickas [1] we consider $l(P)$, the reduced length of P defined by the formula

$$l(P)=\inf_{G\in \mathbb{C}[x]^{1}}L(PG),$$

which for $P\in\mathbb{R}[x]$ reduces to

$$l(P)=\inf_{G\in \mathbb{R}[x]^{1}}L(PG). \tag{1}$$

Actually Dubickas considered only the case $P\in\mathbb{R}[x]$ and called the reduced length of P the quantity $\min\{l(P), l(P^{*})\}$. For $P\in\mathbb{R}[x]$ some of the following results of [6] are due to him.

Proposition 1. *Suppose that $\omega, \eta, \psi\in\mathbb{C}$, $|\omega|\geq 1$, $|\eta|<1$, then for every $Q\in\mathbb{C}[x]$*

(i) $l(\psi Q)=|\psi|l(Q)$,

(ii) $l(x+\omega)=1+|\omega|$,

(iii) *if* $T(x)=Q(x)(x-\eta)$, *then* $l(T)=l(Q)$,

(iv) $l(\overline{Q})=l(Q)$, *where* $\overline{Q}$ *denotes the complex conjugate of* Q.

Proposition 2. *For all P, Q in $\mathbb{C}[x]^1$, all $\eta \in \mathbb{C}$ with $|\eta| = 1$ and all positive integers k*

(i) $\max\{l(P), l(Q)\} \leq l(PQ) \leq l(P)l(Q)$,

(ii) $M(P) \leq l(P)$,

(iii) $l(P(\eta x)) = l(P(x))$,

(iv) $l(P(x^k)) = l(P(x))$.

The main problem consists in finding an algorithm of computing $l(P)$ for a given P. An apparently similar problem in which P and G in formula (1) are restricted to polynomials with integer coefficients has been considered in [2] and [3], however the restriction makes a great difference. Coming back to our problem Proposition 1 (iii) shows that it is enough to consider P with no zeros inside the unit circle. The case of zeros on the unit circle is treated in the following two theorems.

Theorem 1. *Let $P \in \mathbb{C}[x]$,$Q \in \mathbb{C}[x]^1$ and Q have all zeros on the unit circle. Then for all $m \in \mathbb{N}$*

$$l\left(PQ^m\right) = l(PQ).$$

Theorem 2. *If $P \in \mathbb{C}[x]^1 \setminus \mathbb{C}$ has all zeros on the unit circle, then $l(P) = 2$ with $l(P)$ attained, if all zeros are roots of unity and simple ($l(P)$ is attained means that $l(P) = L(Q)$, where $Q/P \in \mathbb{C}[x]^1$).*

Proofs for $P \in \mathbb{R}[x]$ are given in [4], proofs for $P \in \mathbb{C}[x]$ are essentially the same. We have further (see [6]).

Theorem 3. *Let $P = P_0P_1$, where $P_0 \in \mathbb{C}[x]$, $P_1 \in \mathbb{C}[x]^1$, $L(P_0) \leq 2|P_0(0)|$. Then*

$$l(P) \geq L\left(P_0\right) + \left(2\left|P_0(0)\right| - L\left(P_0\right)\right)\left(l\left(P_1\right) - 1\right).$$

Corollary 1. *If $P \in \mathbb{C}[x]$ and $L(P) \leq 2|P(0)|$, then*

$$l(P) = L(P).$$

Conversely, if $l(P) = L(P)$ and all coefficients of P are real and positive, then $L(P) \leq 2P(0)$.

Corollary 2. *If $P(x) = (x-\alpha)(x-\beta)$, where $|\alpha| \geq |\beta| \geq 1$, then*

$$l(P) \geq 1 + |\alpha| - |\beta| + |\alpha\beta|$$

with equality if $\alpha/\beta \in \mathbb{R}$ and either $\alpha/\beta < 0$ or $|\beta| = 1$.

Corollary 3. *Let $P = P_0P_1$, where $P_\nu \in \mathbb{C}[x]$ $(\nu = 0,1)$, $\deg P_1 \geq 1$ and all zeros z of P_ν satisfy $|z| > 1$ for $\nu = 0$, $|z| = 1$ for $\nu = 1$. If*

$$l\,(P_0) = L\,(P_0)\,, \tag{2}$$

then

$$l\,(P) \geq 2M\,(P)\,. \tag{3}$$

It remains a problem, whether (3) holds without the assumption (2). The following results of [6] point towards an affirmative answer.

Theorem 4. *If $P \in \mathbb{C}[x] \setminus \{0\}$ has a zero z with $|z| = 1$, then*

$$L(P) > \sqrt{2}M(P), \quad l(P) \geq \sqrt{2}M(P).$$

Theorem 5. *If $P(x) = (x-\alpha)(x-\beta)(x-1)$, where α,β are real and at least one of them is positive, then* (3) *holds.*

The question of validity of (3) for all polynomials P on $\mathbb{C}$ is equivalent to the following

Problem 1. Is it true that for all polynomials P in $\mathbb{C}[x]$ with a zero on the unit circle $L(P) \geq 2M(P)$?

The following theorems like Theorem 5 concern P in $\mathbb{R}[x]$.

Theorem 6 ([4], Theorem 1). *If $P \in \mathbb{R}[x]^1$ is of degree d with $P(0) \neq 0$, then $l(P) = \inf\limits_{Q \in S_d(P)} L(Q)$, where $S_d(P)$ is the set of all polynomials in $\mathbb{R}[x]^1$ divisible by P with $Q(0) \neq 0$ and with at most $d+1$ non-zero coefficients, all belonging to the field $K(P)$, generated by the coefficients of P.*

Theorem 7 ([4], Theorem 2). *If $P \in \mathbb{R}[x]$ has all zeros outside the unit circle, then $l(P)$ is attained and effectively computable, moreover $l(P) \in K(P)$.*

Theorem 8 ([5], Theorem 1). *Let* $P(x) = \prod_{i=1}^{3}(x - \alpha_i)$, *where* α_i *distinct,* $|\alpha_1| \geq |\alpha_2| > |\alpha_3| = 1$. *Then* $l(P)$ *is effectively computable.*

Theorem 9 ([5], Theorem 2). *Let* $P(x) = (x - \alpha)(x^2 - \varepsilon)$, *where* $|\alpha| > 1$, $\varepsilon = \pm 1$. *Then*

$$l(P) = 2(|\alpha| + 1 - |\alpha|^{-1}).$$

The following problem remains open

Problem 2. How to compute $l(2x^3 + 3x^2 + 4)$?

References

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