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Diophantine approximations with leaping convergents

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1 Introduction

Every real number α can be expressed as its simple continued fraction expansion as

$$\alpha=[a_0;a_1,a_2,\dots]=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots}}},$$

where a_0 is an integer and a_n $(n=1,2,\dots)$ are positive integers. The sequence of partial quotients $a_0,a_1,a_2,\dots$ can be determined uniquely by the algorithm:

$$\begin{aligned}\alpha&=a_0+1/\alpha_1, & a_0&=\lfloor\alpha\rfloor,\\ \alpha_n&=a_n+1/\alpha_{n+1}, & a_n&=\lfloor\alpha_n\rfloor\quad(n\ge 1).\end{aligned}$$

Such expansions are well characterized by truncating the expansion:

$$\frac{p_n}{q_n}=[a_0;a_1,\dots,a_n]=a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}.$$

They are the best rational approximations to α and are called *convergents* (see [2, Module 5]). It is well-known that p_n's and q_n's satisfy the recurrence relations:

$$\begin{aligned}p_n&=a_np_{n-1}+p_{n-2}\quad(n\ge 0),\quad p_{-1}=1,\quad p_{-2}=0,\\ q_n&=a_nq_{n-1}+q_{n-2}\quad(n\ge 0),\quad q_{-1}=1,\quad q_{-2}=0.\end{aligned}$$

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Given integers r and i with $r \geq 2$, $0 \leq i \leq r-1$, we denote the *leaping convergents* by

$$\frac{p_{rn+i}}{q_{rn+i}} \quad (n = 0, 1, 2, \dots).$$

This concept was hinted by Elsner ([4]) and has been developed in [9, 10, 11, 13, 14]. Bumby and Flahive ([1]) called them *leapers* in a slightly different meaning.

2 Diophantine approximations

It is known that

$$\frac{1}{q_{n+1} + q_n} < |p_n - q_n\alpha| < \frac{1}{q_{n+1}} \quad (n \geq 0)$$

([8, p. 20]). More precisely, by using the notation above,

$$\begin{aligned} p_n - q_n\alpha &= \frac{(-1)^{n+1}}{\alpha_{n+1}q_n + q_{n-1}} \\ &= \frac{(-1)^{n+1}}{\alpha_1\alpha_2\dots\alpha_{n+1}} \quad (n \geq 0) \end{aligned}$$

(e.g. see [2, Lemma 5.4]). Since $\alpha_n > 1$ $(n \geq 1)$, we have $p_n - q_n\alpha \to 0$ $(n \to \infty)$. Hence, p_n/q_n $(n = 0, 1, 2, \dots)$ are the best rational approximations to α.

When α is a real quadratic irrational, this error can be well characterized. Suppose that

$$\alpha = \sqrt{a^2+1} = [a; \overline{2a}\,] = [a; 2a, 2a, \dots],$$

where a is a positive integer. Then by

$$\alpha_1 = \alpha_2 = \dots = \sqrt{a^2+1} + a,$$

we obtain

$$\begin{aligned} p_n - q_n\alpha &= \frac{(-1)^{n+1}}{(\sqrt{a^2+1} + a)^{n+1}} \\ &= (-\sqrt{a^2+1} + a)^{n+1} \\ &= e^{-(n+1)\sinh^{-1} a}. \end{aligned}$$

3 Leaping convergents

In [11] we obtained the explicit forms of the leaping convergents of the continued fraction expansion $e^{1/s} = [1; \overline{s(2k-1)-1, 1, 1}\,]_{k=1}^{\infty}$ $(s \geq 2)$. Let p_n/q_n be the nth convergent of the continued fraction expansion of $e^{1/s}$ $(s \geq 2)$ and p_n^*/q_n^* be that of $e = [2; \overline{1, 2k, 1}\,]_{k=1}^{\infty}$. p_n/q_n itself does not have any explicit form, but we can see something in view of leaping convergents. For $n \geq 1$ we have

$$
\begin{aligned}
p_{3n} &= \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} s^k, \\
p_{3n+1} &= \sum_{k=0}^{n} \frac{(n+k+1)!}{k!(n-k)!} s^{k+1}, \\
p_{3n+2} &= (n+1) \sum_{k=0}^{n+1} \frac{(n+k)!}{k!(n-k+1)!} s^k, \\
q_{3n} &= \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k)!}{k!(n-k)!} s^k, \\
q_{3n+1} &= (n+1) \sum_{k=0}^{n+1} (-1)^{n-k+1} \frac{(n+k)!}{k!(n-k+1)!} s^k, \\
q_{3n+2} &= \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k+1)!}{k!(n-k)!} s^{k+1}.
\end{aligned}
$$

Obtaining such explicit forms and proving the results are elementary and omitted. However, there are several interesting applications by using such expressions. We introduce one application in this article. Other applications can be seen in e.g. [1, 5, 6].

4 Diophantine approximations of $e^{1/s}$ and $e^{2/s}$ in terms of integrals

If α is not a quadratic irrational, it becomes complicated to express the error function $p_n - q_n\alpha$. Cohn ([3]) got an idea to express this error function in terms of integrals when $\alpha = e$. This idea was immediately extended by Osler ([15]), who expressed this error explicitly in terms of integrals. Namely, when

p_n/q_n is the n-th convergent of the continued fraction of $e^{1/s}$, he showed that for $n \ge 0$

$$p_{3n} - q_{3n}e^{1/s} = -\frac{1}{s^{n+1}}\int_0^1 \frac{x^n(x-1)^n}{n!}e^{x/s}dx\,, \tag{1}$$

$$p_{3n+1} - q_{3n+1}e^{1/s} = \frac{1}{s^{n+1}}\int_0^1 \frac{x^{n+1}(x-1)^n}{n!}e^{x/s}dx \tag{2}$$

and

$$p_{3n+2} - q_{3n+2}e^{1/s} = \frac{1}{s^{n+1}}\int_0^1 \frac{x^n(x-1)^{n+1}}{n!}e^{x/s}dx\,. \tag{3}$$

This result explains that each left-hand side tends to 0 because each right-hand side tends to 0 as n tends to infinity. Hence, it is demonstrated that the simple continued fraction expansion of $e^{1/s}$ $(s \ge 2)$ is given by $e^{1/s} = [1; \overline{(2k-1)s-1, 1, 1}]_{k=1}^{\infty}$.

The result itself may be interesting independently, but using the concept of leaping convergents, we can obtain similar results concerning the values other than $e^{1/s}$. If we substitute combinatorial expressions of leaping convergents of $e^{1/s}$ in the previous section, we have the following.

Theorem 1. *For* $n \ge 0$

$$\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}s^k - e^{1/s}\sum_{k=0}^{n}(-1)^{n-k}\frac{(n+k)!}{k!(n-k)!}s^k$$
$$= -\frac{1}{s^{n+1}}\int_0^1 \frac{x^n(x-1)^n}{n!}e^{x/s}dx\,, \tag{4}$$

$$\sum_{k=0}^{n}\frac{(n+k+1)!}{k!(n-k)!}s^{k+1} - e^{1/s}(n+1)\sum_{k=0}^{n+1}(-1)^{n-k+1}\frac{(n+k)!}{k!(n-k+1)!}s^k$$
$$= \frac{1}{s^{n+1}}\int_0^1 \frac{x^{n+1}(x-1)^n}{n!}e^{x/s}dx\,, \tag{5}$$

$$(n+1)\sum_{k=0}^{n+1}\frac{(n+k)!}{k!(n-k+1)!}s^k - e^{1/s}\sum_{k=0}^{n}(-1)^{n-k}\frac{(n+k+1)!}{k!(n-k)!}s^{k+1}$$
$$= \frac{1}{s^{n+1}}\int_0^1 \frac{x^n(x-1)^{n+1}}{n!}e^{x/s}dx\,. \tag{6}$$

The identities (4), (5), (6) yield the similar results concerning other kinds of real numbers related to $e^{1/s}$.

It is known that the continued fraction expansion of $e^{2/s}$ is given by

$$e^{2/s} = \left[1; \overline{\frac{(6k-5)s-1}{2}, (12k-6)s, \frac{(6k-1)s-1}{2}, 1, 1}\right]_{k=1}^{\infty},$$

where $s > 1$ is odd (See [16], §32, (2)). In [12] the author gave a proof of the continued fraction expansion of $e^{2/s}$ by showing similar errors explicitly.

Theorem 2. *Let p_n/q_n be the n-th convergent of the continued fraction of $e^{2/s}$. Then, for $n \ge 0$*

$$p_{5n} - q_{5n}e^{2/s} = -\left(\frac{2}{s}\right)^{3n+1} \int_0^1 \frac{x^{3n}(x-1)^{3n}}{(3n)!} e^{2x/s} dx\,, \tag{7}$$

$$p_{5n+1} - q_{5n+1}e^{2/s} = -\frac{2^{3n+1}}{s^{3n+2}} \int_0^1 \frac{x^{3n+1}(x-1)^{3n+1}}{(3n+1)!} e^{2x/s} dx\,, \tag{8}$$

$$p_{5n+2} - q_{5n+2}e^{2/s} = -\left(\frac{2}{s}\right)^{3n+3} \int_0^1 \frac{x^{3n+2}(x-1)^{3n+2}}{(3n+2)!} e^{2x/s} dx\,, \tag{9}$$

$$p_{5n+3} - q_{5n+3}e^{2/s} = \left(\frac{2}{s}\right)^{3n+3} \int_0^1 \frac{x^{3n+3}(x-1)^{3n+2}}{(3n+2)!} e^{2x/s} dx\,, \tag{10}$$

and

$$p_{5n+4} - q_{5n+4}e^{2/s} = \left(\frac{2}{s}\right)^{3n+3} \int_0^1 \frac{x^{3n+2}(x-1)^{3n+3}}{(3n+2)!} e^{2x/s} dx\,. \tag{11}$$

The proof in [12] was done term by term calculations by using the basic relations $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$. The proof here is based upon the explicit combinatorial expressions of the leaping convergents of $e^{2/s}$ in [14]. Let p_n/q_n be the n-th convergent of $e^{2/s}$.

Proposition 1. *For $n = 0, 1, 2, \dots$ we have*

$$p_{5n} = \sum_{k=0}^{3n} \frac{(3n+k)!}{k!(3n-k)!} \left(\frac{s}{2}\right)^k ,$$

$$p_{5n+1} = \sum_{k=0}^{3n+1} \frac{(3n+k+1)!}{k!(3n-k+1)!} \frac{s^k}{2^{k+1}} ,$$

$$p_{5n+2} = \sum_{k=0}^{3n+2} \frac{(3n+k+2)!}{k!(3n-k+2)!} \left(\frac{s}{2}\right)^k ,$$

$$p_{5n+3} = \sum_{k=0}^{3n+2} \frac{(3n+k+3)!}{k!(3n-k+2)!} \left(\frac{s}{2}\right)^{k+1} ,$$

$$p_{5n+4} = 3(n+1) \sum_{k=0}^{3n+3} \frac{(3n+k+2)!}{k!(3n-k+3)!} \left(\frac{s}{2}\right)^k$$

and

$$q_{5n} = \sum_{k=0}^{3n} (-1)^{3n-k} \frac{(3n+k)!}{k!(3n-k)!} \left(\frac{s}{2}\right)^k ,$$

$$q_{5n+1} = \sum_{k=0}^{3n+1} (-1)^{3n-k+1} \frac{(3n+k+1)!}{k!(3n-k+1)!} \frac{s^k}{2^{k+1}} ,$$

$$q_{5n+2} = \sum_{k=0}^{3n+2} (-1)^{3n-k+2} \frac{(3n+k+2)!}{k!(3n-k+2)!} \left(\frac{s}{2}\right)^k ,$$

$$q_{5n+3} = 3(n+1) \sum_{k=0}^{3n+3} (-1)^{3n-k+3} \frac{(3n+k+2)!}{k!(3n-k+3)!} \left(\frac{s}{2}\right)^k ,$$

$$q_{5n+4} = \sum_{k=0}^{3n+2} (-1)^{3n-k+2} \frac{(3n+k+3)!}{k!(3n-k+2)!} \left(\frac{s}{2}\right)^{k+1} .$$

By using these combinatorial expressions of leaping convergents, we can prove Theorem 2 very easily. If we replace s by $s/2$ and n by $3n$ in (4), then we get (7). If we replace s by $s/2$ and n by $3n+1$ in (4) and divide both sides by 2, then we get (8). If we replace s by $s/2$ and n by $3n+2$ in (4), then we get (9). If we replace s by $s/2$ and n by $3n+2$ in (5), then we get (10). If we replace s by $s/2$ and n by $3n+2$ in (6), then we get (11).

5 Diophantine approximations of linear forms of e in terms of integrals

The method mentioned in the previous section is applicable to any linear form of $e^{1/s}$ or $e^{2/s}$ if the explicit forms of the corresponding leaping convergents are explicitly written. For example, it is known that

$$\frac{e+1}{3} = [\,1;4,\overline{5,4k-3,1,1,36k-16,1,1,4k-2,1,1,36k-4,}$$
$$\overline{1,1,4k-1,1,5,4k,1}\,]_{k=1}^{\infty}$$

(e.g. see [7, p.294, (19)]). In fact, this is a special case of

$$\frac{e^{1/(3s+1)}+1}{3} = [\,0;1,2,\overline{(12k-11)s+(4k-5),1,5,(12k-9)s+(4k-4),1,5,}$$
$$\overline{(12k-7)s+(4k-3),1,1,9(12k-5)s+4(9k-4),1,1,}$$
$$\overline{(12k-3)s+(4k-2),1,1,9(12k-1)s+4(9k-1),1,1}\,]_{k=1}^{\infty}\,.$$

If $s=0$, the rule $[\dots,a,-b,\gamma]=[\dots,a-1,1,b-1,-\gamma]$ is applied for

$$[\,0;1,2,-1,1,5,0,1,\overline{5,4k-3,1,1,36k-16,}$$
$$\overline{1,1,4k-2,1,1,36k-4,1,1,4k-1,1,5,4k,1}\,]_{k=1}^{\infty}\,.$$

Let p_n/q_n be the n-th convergent of the continued fraction expansion of $(e^{1/(3s+1)}+1)/3$. Then by induction on n it is shown that

$$p_{18n+2} = 2\sum_{i=0}^{3n}\frac{(6n+2i)!}{(2i)!(6n-2i)!}(3s+1)^{2i}\,,\quad q_{18n+2} = 3\sum_{k=0}^{6n}\frac{(-1)^k(6n+k)!}{k!(6n-k)!}(3s+1)^k\,,$$
$$p_{18n+3} = -\frac{5}{3}\sum_{i=0}^{3n}\frac{(6n+2i)!}{(2i)!(6n-2i)!}(3s+1)^{2i} + \frac{1}{3}\sum_{i=0}^{3n}\frac{(6n+2i+2)!}{(2i+1)!(6n-2i)!}(3s+1)^{2i+1}\,,$$
$$q_{18n+3} = \sum_{k=0}^{6n+1}(-1)^{k-1}\frac{(18n-2k+3)(6n+k)!}{k!(6n-k+1)!}(3s+1)^k\,,$$
$$p_{18n+4} = \frac{1}{3}\sum_{i=0}^{3n}\frac{(6n+2i)!}{(2i)!(6n-2i)!}(3s+1)^{2i} + \frac{1}{3}\sum_{i=0}^{3n}\frac{(6n+2i+2)!}{(2i+1)!(6n-2i)!}(3s+1)^{2i+1}\,,$$
$$q_{18n+4} = \sum_{k=0}^{6n}(-1)^k\frac{(6n+k+1)!}{k!(6n-k)!}(3s+1)^{k+1}\,,$$
$$p_{18n+5} = 2\sum_{i=0}^{3n}\frac{(6n+2i+2)!}{(2i+1)!(6n-2i)!}(3s+1)^{2i+1}\,,$$
$$q_{18n+5} = 3\sum_{k=0}^{6n+1}\frac{(-1)^{k-1}(6n+k+1)!}{k!(6n-k+1)!}(3s+1)^k\,,$$
$$p_{18n+6} = \frac{1}{3}\sum_{i=0}^{3n+1}\frac{(6n+2i+2)!}{(2i)!(6n-2i+2)!}(3s+1)^{2i} - \frac{5}{3}\sum_{i=0}^{3n}\frac{(6n+2i+2)!}{(2i+1)!(6n-2i)!}(3s+1)^{2i+1}\,,$$
$$q_{18n+6} = 2\sum_{k=0}^{6n+2}\frac{(-1)^k(9n-k+3)(6n+k+1)!}{k!(6n-k+2)!}(3s+1)^k\,,$$

$$p_{18n+7} = \frac{1}{3}\sum_{i=0}^{3n+1} \frac{(6n+2i+2)!}{(2i)!(6n-2i+2)!}(3s+1)^{2i} + \frac{1}{3}\sum_{i=0}^{3n} \frac{(6n+2i+2)!}{(2i+1)!(6n-2i)!}(3s+1)^{2i+1},$$
$$q_{18n+7} = \sum_{k=0}^{6n+1} \frac{(-1)^{k-1}(6n+k+2)!}{k!(6n-k+1)!}(3s+1)^k,$$
$$p_{18n+8} = 2\sum_{i=0}^{3n+1} \frac{(6n+2i+2)!}{(2i)!(6n-2i+2)!}(3s+1)^{2i+1},$$
$$q_{18n+8} = 3\sum_{k=0}^{6n+2} \frac{(-1)^{k}(6n+k+2)!}{k!(6n-k+2)!}(3s+1)^k,$$
$$p_{18n+9} = -\sum_{i=0}^{3n+1} \frac{(6n+2i+2)!}{(2i)!(6n-2i+2)!}(3s+1)^{2i} + \frac{1}{3}\sum_{i=0}^{3n+1} \frac{(6n+2i+4)!}{(2i+1)!(6n-2i+2)!}(3s+1)^{2i+1},$$
$$q_{18n+9} = \sum_{k=0}^{6n+3} \frac{(-1)^{k-1}(12n-k+6)(6n+k+2)!}{k!(6n-k+3)!}(3s+1)^k,$$
$$p_{18n+10} = \sum_{i=0}^{3n+1} \frac{(6n+2i+2)!}{(2i)!(6n-2i+2)!}(3s+1)^{2i} + \frac{1}{3}\sum_{i=0}^{3n+1} \frac{(6n+2i+4)!}{(2i+1)!(6n-2i+2)!}(3s+1)^{2i+1},$$
$$q_{18n+10} = \sum_{k=0}^{6n+3} \frac{(-1)^{k}(6n-2k+3)(6n+k+2)!}{k!(6n-k+3)!}(3s+1)^k,$$
$$p_{18n+11} = \frac{2}{3}\sum_{i=0}^{3n+1} \frac{(6n+2i+4)!}{(2i+1)!(6n-2i+2)!}(3s+1)^{2i+1},$$
$$q_{18n+11} = \sum_{k=0}^{6n+3} \frac{(-1)^{k-1}(6n+k+3)!}{k!(6n-k+3)!}(3s+1)^k,$$
$$p_{18n+12} = \sum_{i=0}^{3n+2} \frac{(6n+2i+4)!}{(2i)!(6n-2i+4)!}(3s+1)^{2i} - \frac{1}{3}\sum_{i=0}^{3n+1} \frac{(6n+2i+4)!}{(2i+1)!(6n-2i+2)!}(3s+1)^{2i+1},$$
$$q_{18n+12} = \sum_{k=0}^{6n+4} \frac{(-1)^{k}(12n+k+8)(6n+k+3)!}{k!(6n-k+4)!}(3s+1)^k,$$
$$p_{18n+13} = \sum_{i=0}^{3n+2} \frac{(6n+2i+4)!}{(2i)!(6n-2i+4)!}(3s+1)^{2i} + \frac{1}{3}\sum_{i=0}^{3n+1} \frac{(6n+2i+4)!}{(2i+1)!(6n-2i+2)!}(3s+1)^{2i+1},$$
$$q_{18n+13} = \sum_{k=0}^{6n+4} \frac{(-1)^{k}(6n+2k+4)(6n+k+3)!}{k!(6n-k+4)!}(3s+1)^k,$$
$$p_{18n+14} = 2\sum_{i=0}^{3n+2} \frac{(6n+2i+4)!}{(2i)!(6n-2i+4)!}(3s+1)^{2i}, \quad q_{18n+14} = 3\sum_{k=0}^{6n+4} \frac{(-1)^{k}(6n+k+4)!}{k!(6n-k+4)!}(3s+1)^k,$$
$$p_{18n+15} = -\sum_{i=0}^{3n+2} \frac{(6n+2i+4)!}{(2i)!(6n-2i+4)!}(3s+1)^{2i} + \frac{1}{3}\sum_{i=0}^{3n+2} \frac{(6n+2i+6)!}{(2i+1)!(6n-2i+4)!}(3s+1)^{2i+1},$$
$$q_{18n+15} = \sum_{k=0}^{6n+5} \frac{(-1)^{k-1}(12n-k+10)(6n+k+4)!}{k!(6n-k+5)!}(3s+1)^k,$$
$$p_{18n+16} = \sum_{i=0}^{3n+2} \frac{(6n+2i+4)!}{(2i)!(6n-2i+4)!}(3s+1)^{2i} + \frac{1}{3}\sum_{i=0}^{3n+2} \frac{(6n+2i+6)!}{(2i+1)!(6n-2i+4)!}(3s+1)^{2i+1},$$
$$q_{18n+16} = \sum_{k=0}^{6n+5} \frac{(-1)^{k}(6n-2k+5)(6n+k+4)!}{k!(6n-k+5)!}(3s+1)^k,$$
$$p_{18n+17} = \frac{2}{3}\sum_{i=0}^{3n+2} \frac{(6n+2i+6)!}{(2i+1)!(6n-2i+4)!}(3s+1)^{2i+1},$$
$$q_{18n+17} = \sum_{k=0}^{6n+5} \frac{(-1)^{k-1}(6n+k+5)!}{k!(6n-k+5)!}(3s+1)^k,$$
$$p_{18n+18} = \sum_{i=0}^{3n+3} \frac{(6n+2i+6)!}{(2i)!(6n-2i+6)!}(3s+1)^{2i} - \frac{1}{3}\sum_{i=0}^{3n+2} \frac{(6n+2i+6)!}{(2i+1)!(6n-2i+4)!}(3s+1)^{2i+1},$$
$$q_{18n+18} = \sum_{k=0}^{6n+6} \frac{(-1)^{k}(12n+k+12)(6n+k+5)!}{k!(6n-k+6)!}(3s+1)^k,$$
$$p_{18n+19} = \sum_{i=0}^{3n+3} \frac{(6n+2i+6)!}{(2i)!(6n-2i+6)!}(3s+1)^{2i} + \frac{1}{3}\sum_{i=0}^{3n+2} \frac{(6n+2i+6)!}{(2i+1)!(6n-2i+4)!}(3s+1)^{2i+1},$$
$$q_{18n+19} = \sum_{k=0}^{6n+6} \frac{(-1)^{k}(6n+2k+6)(6n+k+5)!}{k!(6n-k+6)!}(3s+1)^k.$$

Hence, for $n \geq 0$ we obtain the following.

Theorem 3.

$$
\begin{aligned}
p_{18n+2} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+2} &= -\int_0^1 \frac{x^{6n}(x-1)^{6n}}{(3s+1)^{6n+1}(6n)!} e^{x/(3s+1)} dx \,, \\
p_{18n+3} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+3} &= \int_0^1 \frac{(x+2)x^{6n}(x-1)^{6n}}{3(3s+1)^{6n+1}(6n)!} e^{x/(3s+1)} dx \,, \\
p_{18n+4} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+4} &= \int_0^1 \frac{x^{6n}(x-1)^{6n+1}}{3(3s+1)^{6n+1}(6n)!} e^{x/(3s+1)} dx \,, \\
p_{18n+5} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+5} &= -\int_0^1 \frac{x^{6n+1}(x-1)^{6n+1}}{(3s+1)^{6n+2}(6n+1)!} e^{x/(3s+1)} dx \,, \\
p_{18n+6} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+6} &= \int_0^1 \frac{(x+2)x^{6n+1}(x-1)^{6n+1}}{3(3s+1)^{6n+2}(6n+1)!} e^{x/(3s+1)} dx \,, \\
p_{18n+7} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+7} &= \int_0^1 \frac{x^{6n+1}(x-1)^{6n+2}}{3(3s+1)^{6n+2}(6n+1)!} e^{x/(3s+1)} dx \,, \\
p_{18n+8} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+8} &= -\int_0^1 \frac{x^{6n+2}(x-1)^{6n+2}}{(3s+1)^{6n+3}(6n+2)!} e^{x/(3s+1)} dx \,, \\
p_{18n+9} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+9} &= \int_0^1 \frac{(x+1)x^{6n+2}(x-1)^{6n+2}}{3(3s+1)^{6n+3}(6n+2)!} e^{x/(3s+1)} dx \,, \\
p_{18n+10} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+10} &= \int_0^1 \frac{(x-2)x^{6n+2}(x-1)^{6n+2}}{3(3s+1)^{6n+3}(6n+2)!} e^{x/(3s+1)} dx \,, \\
p_{18n+11} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+11} &= -\int_0^1 \frac{x^{6n+3}(x-1)^{6n+3}}{3(3s+1)^{6n+4}(6n+3)!} e^{x/(3s+1)} dx \,,
\end{aligned}
$$

$$
\begin{aligned}
p_{18n+12} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+12} &= \int_0^1 \frac{(3x-1)x^{6n+3}(x-1)^{6n+3}}{3(3s+1)^{6n+4}(6n+3)!} e^{x/(3s+1)} dx\,, \\
p_{18n+13} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+13} &= \int_0^1 \frac{(3x-2)x^{6n+3}(x-1)^{6n+3}}{3(3s+1)^{6n+4}(6n+3)!} e^{x/(3s+1)} dx\,, \\
p_{18n+14} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+14} &= -\int_0^1 \frac{x^{6n+4}(x-1)^{6n+4}}{(3s+1)^{6n+5}(6n+4)!} e^{x/(3s+1)} dx\,, \\
p_{18n+15} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+15} &= \int_0^1 \frac{(x+1)x^{6n+4}(x-1)^{6n+4}}{3(3s+1)^{6n+5}(6n+4)!} e^{x/(3s+1)} dx\,, \\
p_{18n+16} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+16} &= \int_0^1 \frac{(x-2)x^{6n+4}(x-1)^{6n+4}}{3(3s+1)^{6n+5}(6n+4)!} e^{x/(3s+1)} dx\,, \\
p_{18n+17} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+17} &= -\int_0^1 \frac{x^{6n+5}(x-1)^{6n+5}}{3(3s+1)^{6n+6}(6n+5)!} e^{x/(3s+1)} dx\,, \\
p_{18n+18} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+18} &= \int_0^1 \frac{(3x-1)x^{6n+5}(x-1)^{6n+5}}{3(3s+1)^{6n+6}(6n+5)!} e^{x/(3s+1)} dx\,, \\
p_{18n+19} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+19} &= \int_0^1 \frac{(3x-2)x^{6n+5}(x-1)^{6n+5}}{3(3s+1)^{6n+6}(6n+5)!} e^{x/(3s+1)} dx\,.
\end{aligned}
$$

Remark. When $s = 1$, if we denote the n-th convergent of the continued fraction expansion of $(e^{1/4}+1)/3$ by p_n^*/q_n^*, then the relation

$$
\frac{p_n^*}{q_n^*} = \frac{p_{n+2}}{q_{n+2}} \quad (n \geq 0)
$$

is applied to the above Theorem.
When $s = 0$, if we denote the n-th convergent of the continued fraction expansion of $(e+1)/3$ by p_n^{**}/q_n^{**}, then the relation

$$
\frac{p_n^{**}}{q_n^{**}} = \frac{p_{n+2}}{q_{n+2}} \quad (n \geq 0)
$$

is applied to the above Theorem. For example, for $n \geq 0$ we have

$$
\begin{aligned}
p_{18n+3}^{**} - \frac{e+1}{3} q_{18n+3}^{**} &= p_{18n+9} - \frac{e^{1/(3s+1)}+1}{3} q_{18n+9} \\
&= \frac{1}{3} \int_0^1 \frac{(x+1)x^{6n+2}(x-1)^{6n+2}}{(6n+2)!} e^x dx\,.
\end{aligned}
$$

6 Quadratic irrational revisited

Let $\alpha = \sqrt{a^2+1} = [a; \overline{2a}]$ and p_n/q_n be its nth convergent. Then it is proved that

$$
\begin{aligned}
p_{2n-1} &= n \sum_{k=0}^{n} \frac{(n+k-1)!}{(2k)!(n-k)!}(2a)^{2k} \\
&= \cosh(2n \sinh^{-1} a) = \frac{(\sqrt{a^2+1}+a)^{2n} + (\sqrt{a^2+1}-a)^{2n}}{2}, \\
q_{2n-1} &= \sum_{k=0}^{n-1} \frac{(n+k)!}{(2k+1)!(n-k-1)!}(2a)^{2k+1} = \sum_{k=0}^{n-1} \binom{n+k}{2k+1}(2a)^{2k+1} \\
&= \frac{\sinh(2n \sinh^{-1} a)}{\sqrt{a^2+1}} = \frac{(\sqrt{a^2+1}+a)^{2n} - (\sqrt{a^2+1}-a)^{2n}}{2\sqrt{a^2+1}}, \\
p_{2n} &= (2n+1) \sum_{k=0}^{n} \frac{(n+k)!}{(2k+1)!(n-k)!} \frac{(2a)^{2k+1}}{2} \\
&= \sinh((2n+1) \sinh^{-1} a) = \frac{(\sqrt{a^2+1}+a)^{2n+1} - (\sqrt{a^2+1}-a)^{2n+1}}{2}, \\
q_{2n} &= \sum_{k=0}^{n} \frac{(n+k)!}{(2k)!(n-k)!}(2a)^{2k} = \sum_{k=0}^{n} \binom{n+k}{2k}(2a)^{2k} \\
&= \frac{\cosh((2n+1) \sinh^{-1} a)}{\sqrt{a^2+1}} = \frac{(\sqrt{a^2+1}+a)^{2n+1} + (\sqrt{a^2+1}-a)^{2n+1}}{2\sqrt{a^2+1}}.
\end{aligned}
$$

However, it has not known whether the identity $p_n - q_n\alpha = e^{-(n+1)\sinh^{-1} a}$ plays a basic role in quadratic irrationals, corresponding to Theorem 1 in the case of $e^{1/s}$.

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