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On multiple Bernoulli polynomials and multiple L-functions of root systems

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§1. Introduction: Review of Classical Theory

In this article we propose generalizations of Bernoulli polynomials and L-functions associated with root systems. To state our results, first we recall the classical theory for the Riemann zeta-function and Bernoulli numbers.

The following is a well-known formula for the Riemann zeta-function and Bernoulli numbers.

For $k \in \mathbb{Z}_{\geq 1}$,

$$2\zeta(2k) = -B_{2k}\frac{(2\pi i)^{2k}}{(2k)!},$$

where

$$\frac{te^t}{e^t - 1} = -\sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

By using this formula, we obtain for $k \in \mathbb{Z}_{\geq 1}$,

$$\zeta(2k) + (-1)^{2k}\zeta(2k) = -B_{2k}\frac{(2\pi i)^{2k}}{(2k)!},$$

$$\zeta(2k+1) + (-1)^{2k+1}\zeta(2k+1) = -B_{2k+1}\frac{(2\pi i)^{2k+1}}{(2k+1)!} = 0.$$

Hence we have important relations:

For $k \in \mathbb{Z}_{\geq 2}$,

$$\zeta(k) + (-1)^k\zeta(k) \quad = \quad -B_k\frac{(2\pi i)^k}{k!},$$

value-relations = Bernoulli numbers.

This procedure can be applied to Lerch zeta-functions and periodic Bernoulli functions. Let $\varphi(s,y)$ be the Lerch zeta-function defined by

$$\varphi(s,y) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n y}}{n^s}.$$

Then a formula for Lerch zeta-functions implies

For $k \in \mathbb{Z}_{\geq 2}$ and $y \in \mathbb{R}$,

$$\begin{aligned} \varphi(k,y) + (-1)^k \varphi(k,-y) &= -B_k(\{y\}) \frac{(2\pi i)^k}{k!}, \\ \text{functional relations} &= \text{periodic Bernoulli functions.} \end{aligned}$$

Here

$$\frac{te^{t\{y\}}}{e^t - 1} = -\sum_{k=0}^{\infty} B_k(\{y\}) \frac{t^k}{k!},$$

and $\{y\} = y - [y]$ (i.e. fractional part).

Once we obtain periodic Bernoulli functions, we can calculate special values of L-functions.

For a primitive character χ of conductor f and $k \in \mathbb{Z}_{\geq 2}$ satisfying $(-1)^k \chi(-1) = 1$, we have

$$\begin{aligned} L(k,\chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^k} \\ &= \frac{(-1)^{k+1}}{2} \frac{(2\pi i)^k}{k! f^k} g(\chi) B_{k,\bar{\chi}}, \end{aligned}$$

where $g(\chi)$ is the Gauss sum and

$$B_{k,\chi} = f^{k-1} \sum_{a=1}^{f} \chi(a) B_k(a/f).$$

Our aim is to find a good class of multiple zeta-functions which generalize the theory above.

§2. Overview of Our Results

Based on the observation given in the previous section, we will construct multiple generalizations of Bernoulli polynomials and multiple L-functions associated with arbitrary root systems. Before introducing the general theory, we give two simple theorems by using the explicit form of the root system of type A_2.

For $s_1, s_2, s_3 \in \mathbb{C}$ and $y_1, y_2 \in \mathbb{R}$, we consider the convergent series

$$\zeta_2(s_1, s_2, s_3, y_1, y_2; A_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i(my_1 + ny_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3}}.$$

Theorem A. *For $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 2}$,*

$$\begin{aligned}
&\zeta_2(k_1,k_2,k_3,y_1,y_2;A_2) + (-1)^{k_1}\zeta_2(k_1,k_3,k_2,-y_1+y_2,y_2;A_2)\\
&\quad + (-1)^{k_2}\zeta_2(k_3,k_2,k_1,y_1,y_1-y_2;A_2) + (-1)^{k_2+k_3}\zeta_2(k_3,k_1,k_2,-y_1+y_2,-y_1;A_2)\\
&\quad + (-1)^{k_1+k_3}\zeta_2(k_2,k_3,k_1,-y_2,y_1-y_2;A_2) + (-1)^{k_1+k_2+k_3}\zeta_2(k_2,k_1,k_3,-y_2,-y_1;A_2)\\
&\qquad = (-1)^3 P(k_1,k_2,k_3,y_1,y_2;A_2)\frac{(2\pi i)^{k_1+k_2+k_3}}{k_1!k_2!k_3!},
\end{aligned}$$

where $P(k_1,k_2,k_3,y_1,y_2;A_2)$ is a multiple periodic Bernoulli function (defined later). In particular, we have

$$\zeta_2(2,2,2,0,0;A_2) = \frac{1}{6}(-1)^3\frac{1}{3780}\frac{(2\pi i)^{2+2+2}}{2!2!2!} = \frac{\pi^6}{2835}.$$

cf.

$$\varphi(k,y) + (-1)^k\varphi(k,-y) = -B_k(\{y\})\frac{(2\pi i)^k}{k!}, \qquad \zeta(2) = \frac{1}{2}(-1)\frac{1}{6}\frac{(2\pi i)^2}{2!} = \frac{\pi^2}{6}.$$

For $s_1, s_2, s_3 \in \mathbb{C}$ and primitive Dirichlet characters χ_1, χ_2, χ_3, consider the convergent series

$$L_2(s_1,s_2,s_3,\chi_1,\chi_2,\chi_3;A_2) = \sum_{m,n=1}^{\infty}\frac{\chi_1(m)\chi_2(n)\chi_3(m+n)}{m^{s_1}n^{s_2}(m+n)^{s_3}}.$$

Theorem B. *For $k \in \mathbb{Z}_{\geq 2}$ and a primitive Dirichlet character χ of conductor f such that $(-1)^k\chi(-1) = 1$,*

$$L_2(k,k,k,\chi,\chi,\chi;A_2) = \frac{(-1)^{3k+3}}{6}\left(\frac{(2\pi i)^k}{k!f^k}g(\chi)\right)^3 B_{k,k,k,\bar\chi,\bar\chi,\bar\chi}(A_2),$$

where $B_{k_1,k_2,k_3,\chi_1,\chi_2,\chi_3}(A_2)$ is a multiple generalized Bernoulli number (defined later). In particular, for $\rho_5 : \rho_5(1) = \rho_5(4) = 1, \rho_5(2) = \rho_5(3) = -1$, we have

$$L_2(2,2,2,\rho_5,\rho_5,\rho_5;A_2) = \frac{(-1)^{6+3}}{6}\left(\frac{(2\pi i)^2}{2!5^2}\sqrt{5}\right)^3\left(-\frac{28}{125}\right) = -\frac{112\sqrt{5}}{1171875}\pi^6.$$

cf.

$$L(k,\chi) = \frac{(-1)^{k+1}}{2}\frac{(2\pi i)^k}{k!f^k}g(\chi)B_{k,\bar\chi}, \qquad L(2,\rho_5) = \frac{(-1)^{2+1}}{2}\frac{(2\pi i)^2}{2!5^2}\sqrt{5}\,\frac{4}{5} = \frac{4\sqrt{5}}{125}\pi^2.$$

Theorems A and B are special cases of our main theorems. In the following sections, we will formulate these facts.

§3. Root Systems

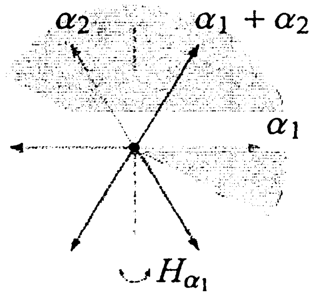
For reader's convenience, we give the definition and several examples of root systems.

§§3.1. Definitions

Let V be an r dimensional real vector space equipped with inner product $\langle \cdot, \cdot \rangle$.

A root system $\Delta \subset V$ is a set of vectors (roots):

(1) $|\Delta| < \infty$ and $0 \notin \Delta$,
(2) $\sigma_\alpha \Delta = \Delta$ for all $\alpha \in \Delta$,
(3) $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$,
(4) $\alpha, c\alpha \in \Delta \Longrightarrow c = \pm 1$,



where σ_α denotes the reflection with respect to the hyperplane H_α orthogonal to α and $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ (coroot).

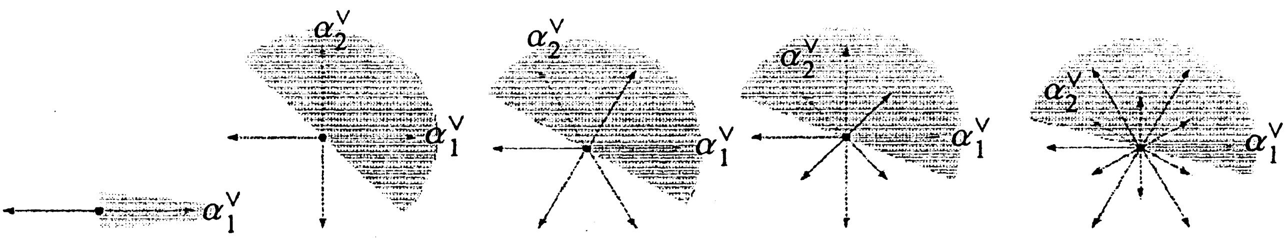
Let W be the Weyl group (the group generated by all σ_α). Let $\{\alpha_1, \ldots, \alpha_r\}$ be fundamental roots (a basis s.t. $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta$ with all $c_i \geq 0$ or $c_i \leq 0$). Let Δ_+ be positive roots (all roots $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta$ with all $c_i \geq 0$) and P_{++}, strictly dominant weights ($= \bigoplus \mathbb{Z}_{\geq 1}\lambda_i$, $\{\lambda_1, \ldots, \lambda_r\}$ dual basis of $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$). The key fact which plays an essential role is that the nice group W acts on Δ.

§§3.2. Examples

Since we mainly treat coroots, we give examples of root systems in terms of coroots. Note that if Δ is a root system, then $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$ is also a root system.

There is only one root system of rank 1 and there are four root systems of rank 2:

A_1 | $A_1 \times A_1$ | A_2 | B_2 (or C_2) | G_2



$$\Delta_+^\vee = \{\alpha_1^\vee\} \quad \{\alpha_1^\vee, \alpha_2^\vee\} \quad \begin{Bmatrix} \alpha_1^\vee, \alpha_2^\vee \\ \alpha_1^\vee + \alpha_2^\vee \end{Bmatrix} \quad \begin{Bmatrix} \alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee \\ \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee \end{Bmatrix} \quad \begin{Bmatrix} \alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee \\ \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee \\ \alpha_1^\vee + 3\alpha_2^\vee \\ 2\alpha_1^\vee + 3\alpha_2^\vee \end{Bmatrix}$$

In this article, we use these root systems in examples for simplicity. It should be noted that root systems are classified as $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ and our theory can be applied to all these root systems.

§4. Zeta-Functions of Root Systems

§§4.1. Witten Zeta-Functions

As prototypes of zeta-functions of root systems, we give the definition of Witten zeta-functions, which were originally introduced to calculate the volumes of certain moduli spaces.

Witten zeta-functions ([13, 14]): For a complex simple Lie algebra $\mathfrak{g}$ of type X_r,

$$\zeta_W(s;X_r) = \sum_{\varphi}(\dim\varphi)^{-s} = K(X_r)^s \sum_{\lambda\in P_{++}} \prod_{\alpha\in\Delta_+} \frac{1}{\langle \alpha^\vee, \lambda\rangle^s},$$

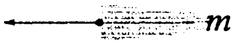
where the summation runs over all finite dimensional irreducible representations φ and $K(X_r) \in \mathbb{Z}_{\geq 1}$ is a constant.

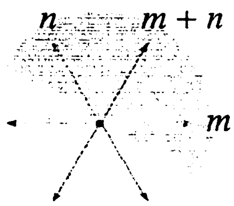
From the second expression of the definition, we see that the explicit forms of Witten zeta-functions are obtained by formally replacing $\alpha_1^\vee$ and $\alpha_2^\vee$ by m and n respectively:

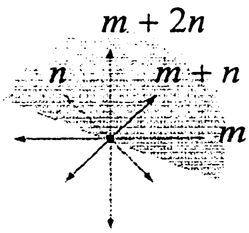
$$\zeta_W(s;A_1) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \zeta(s),$$

$$\zeta_W(s;A_2) = 2^s \sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s},$$

$$\zeta_W(s;B_2) = 6^s \sum_{m,n=1}^{\infty} \frac{1}{m^s n^s (m+n)^s (m+2n)^s}.$$







§§4.2. Zeta-Functions of Root Systems

Definition 1 ([6, 7, 8, 12]). Zeta-functions of root systems: For a root system Δ of type X_r, define

$$\zeta_r(\mathbf{s},\mathbf{y};X_r) = \sum_{\lambda\in P_{++}} e^{2\pi i\langle \mathbf{y},\lambda\rangle} \prod_{\alpha\in\Delta_+} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_\alpha}},$$

where $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^{|\Delta_+|}$ and $\mathbf{y} \in V$.

To define an action of the Weyl group, we extend $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+}$ to $(s_\alpha)_{\alpha\in\Delta}$ by $s_\alpha = s_{-\alpha}$ and define $(w\mathbf{s})_\alpha = s_{w^{-1}\alpha}$. Then we have our first theorem.

Theorem 1 ([8]). *For* $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in \mathbb{Z}_{\geq 2}^{|\Delta_+|}$, *we have*

$$\sum_{w\in W}\Big(\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-} (-1)^{k_\alpha}\Big)\zeta_r(w^{-1}\mathbf{k}, w^{-1}\mathbf{y}; X_r) = (-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};X_r)\Big(\prod_{\alpha\in\Delta_+} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Big),$$

where $P(\mathbf{k},\mathbf{y};X_r)$ *is a multiple periodic Bernoulli function (defined later).*

cf. $(X_r = A_1)$

$$\varphi(k,y) + (-1)^k\varphi(k,-y) = -B_k(\{y\})\frac{(2\pi i)^k}{k!} \qquad (W = \{\mathrm{id}, \sigma_\alpha\}).$$

§5. Special Zeta-Values

Theorem 1 directly implies the following theorem:

Theorem 2 ([8]). *For* $\mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in (2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ *satisfying* $w^{-1}\mathbf{k} = \mathbf{k}$ *for all* $w \in W$,

$$\zeta_r(\mathbf{k}, \mathbf{0}; X_r) = \frac{(-1)^{|\Delta_+|}}{|W|}P(\mathbf{k}, \mathbf{0}; X_r)\Big(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Big) \in \mathbb{Q}\pi^{\sum_{\alpha\in\Delta_+}k_\alpha}.$$

cf. $(X_r = A_1)$

$$\zeta(k) = \frac{-1}{2}B_k\frac{(2\pi i)^k}{k!} \in \mathbb{Q}\pi^k \qquad (k \in 2\mathbb{Z}_{\geq 1}).$$

In particular, $\mathbf{k} = (k)_{\alpha\in\Delta_+}$ with $k \in 2\mathbb{Z}_{\geq 1}$ (that is, all $k_\alpha = k$) satisfies the condition in Theorem 2. In this case, $\zeta_r(\mathbf{k}, \mathbf{0}; X_r) \in \mathbb{Q}\pi^{|\Delta_+|k}$ was shown by Witten and Zagier. Our statement is a true generalization of their results since we also have for example,

$$\begin{aligned}\zeta_2((2,4,4,2), \mathbf{0}; B_2) &= \sum_{m,n=1}^{\infty}\frac{1}{m^2n^4(m+n)^4(m+2n)^2} \\ &= \frac{(-1)^4}{2^2 2!}\frac{53}{1513512000}\Big(\frac{(2\pi i)^2}{2!}\Big)^2\Big(\frac{(2\pi i)^4}{4!}\Big)^2 \\ &= \frac{53\pi^{12}}{6810804000}.\end{aligned}$$

§6. Multiple Periodic Bernoulli Functions

In this section, we give the definitions of generating functions of multiple periodic Bernoulli functions. Let $\mathscr{V}$ be the set of all bases $\mathbf{V} \subset \Delta_+$, $\mathbf{V}^* = \{\mu_\beta^{\mathbf{V}}\}_{\beta\in\mathbf{V}}$, the dual basis of $\mathbf{V}^\vee = \{\beta^\vee\}_{\beta\in\mathbf{V}}$. Let $Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$ be the coroot lattice and $L(\mathbf{V}^\vee) = \bigoplus_{\beta\in\mathbf{V}}\mathbb{Z}\beta^\vee$, which is a sublattice of $Q^\vee$ with finite index ($|Q^\vee/L(\mathbf{V}^\vee)| < \infty$).

Fix a certain $\phi \in V$ and define a multiple generalization of fractional part as

$$\{\mathbf{y}\}_{\mathbf{V},\beta} = \begin{cases}\{\langle\mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle\phi, \mu_\beta^{\mathbf{V}}\rangle > 0), \\ 1 - \{-\langle\mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle\phi, \mu_\beta^{\mathbf{V}}\rangle < 0).\end{cases}$$

By using these definitions, we have

Definition 2 (generating function [8, 9, 10]). For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+}$,

$$\begin{aligned}F(\mathbf{t}, \mathbf{y}; X_r) = &\sum_{\mathbf{V}\in\mathscr{V}}\Big(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}}\frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}}t_\beta\langle\gamma^\vee, \mu_\beta^{\mathbf{V}}\rangle}\Big) \\ &\times\frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|}\sum_{q\in Q^\vee/L(\mathbf{V}^\vee)}\Big(\prod_{\beta\in\mathbf{V}}\frac{t_\beta\exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1}\Big).\end{aligned}$$

Definition 3 (multiple periodic Bernoulli functions [8, 9, 10]).

$$F(\mathbf{t},\mathbf{y};X_r) = \sum_{\mathbf{k}\in\mathbf{Z}_{\geq 0}^{|\Delta_+|}} P(\mathbf{k},\mathbf{y};X_r) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}.$$

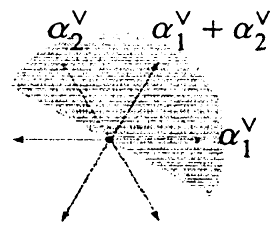
cf. $(X_r = A_1)$

$$F(t,y) = \frac{te^{t\{y\}}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(\{y\})\frac{t^k}{k!}.$$

§7. Example: A_2 Case

We calculate a multiple periodic Bernoulli function and its generating function in the case of the root system of type A_2.

We have the basic data as follows:



$$\Delta_+^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\},\ \mathscr{V} = \{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\},$$
$$\mathbf{t} = (t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_1+\alpha_2}) = (t_1, t_2, t_3),$$
$$\mathbf{y} = y_1\alpha_1^\vee + y_2\alpha_2^\vee.$$

Fix a sufficiently small $\varepsilon > 0$ and $\phi = \alpha_1^\vee + \varepsilon\alpha_2^\vee$. Then by using these data, we have the generating function and a multiple periodic Bernoulli function as

$F(\mathbf{t},\mathbf{y};A_2) =$ basis $\mathbf{V} \subset \Delta_+$, dual basis $\mathbf{V}^*$

$$\frac{t_3}{t_3 - t_1 - t_2}\frac{t_1e^{t_1\{y_1\}}}{e^{t_1} - 1}\frac{t_2e^{t_2\{y_2\}}}{e^{t_2} - 1} \qquad (\mathbf{V}_1^\vee = \{\alpha_1^\vee, \alpha_2^\vee\}, \mathbf{V}_1^* = \{\lambda_1, \lambda_2\})$$
$$+ \frac{t_2}{t_2 + t_1 - t_3}\frac{t_1e^{t_1\{y_1-y_2\}}}{e^{t_1} - 1}\frac{t_3e^{t_3\{y_2\}}}{e^{t_3} - 1} \qquad (\mathbf{V}_2^\vee = \{\alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee\}, \mathbf{V}_2^* = \{\lambda_1 - \lambda_2, \lambda_2\})$$
$$+ \frac{t_1}{t_1 + t_2 - t_3}\frac{t_2e^{t_2(1-\{y_1-y_2\})}}{e^{t_2} - 1}\frac{t_3e^{t_3\{y_1\}}}{e^{t_3} - 1} \qquad (\mathbf{V}_3^\vee = \{\alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\}, \mathbf{V}_3^* = \{\lambda_2 - \lambda_1, \lambda_1\})$$

For $\mathbf{k} = \mathbf{2} = (2,2,2)$,

$$P(\mathbf{2},(y_1,y_2);A_2) = \frac{1}{3780} + \frac{1}{90}(\{y_1\} - \{y_1 - y_2\} - \{y_2\})$$
$$\cdots$$
$$+ \frac{1}{30}(-\{y_1\}^6 + 4\{y_1 - y_2\}\{y_1\}^5 - 5\{y_1 - y_2\}^2\{y_1\}^4$$
$$- \{y_2\}^6 - 4\{y_1 - y_2\}\{y_2\}^5 - 5\{y_1 - y_2\}^2\{y_2\}^4).$$

We have a functional relation corresponding to this multiple periodic Bernoulli function:

$$\zeta_2(\mathbf{2},(y_1,y_2);A_2)+\zeta_2(\mathbf{2},(-y_1+y_2,y_2);A_2)+\zeta_2(\mathbf{2},(y_1,y_1-y_2);A_2)$$
$$+\zeta_2(\mathbf{2},(-y_2,y_1-y_2);A_2)+\zeta_2(\mathbf{2},(-y_1+y_2,-y_1);A_2)+\zeta_2(\mathbf{2},(-y_2,-y_1);A_2)$$
$$=(-1)^3 P(\mathbf{2},(y_1,y_2);A_2)\frac{(2\pi i)^6}{(2!)^3}.$$

In particular if $(y_1,y_2)=(0,0)$, then

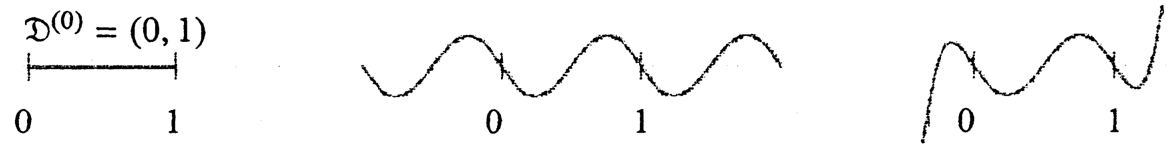
$$\zeta_2(\mathbf{2},(0,0);A_2)=\frac{1}{6}(-1)^3\frac{1}{3780}\frac{(2\pi i)^6}{(2!)^3}=\frac{\pi^6}{2835}.$$

cf. $(X_r = A_1)$

$$\zeta(2)=\frac{1}{2}(-1)\frac{1}{6}\frac{(2\pi i)^2}{2!}=\frac{\pi^2}{6},\qquad B_2(\{y\})=\frac{1}{6}-\{y\}+\{y\}^2.$$

§8. Multiple Bernoulli Polynomials

In the classical theory, Bernoulli polynomials can be derived by the analytic continuation of periodic Bernoulli functions. We explain this fact. Let $\mathfrak{H}=\{y\in\mathbb{R}\mid\{y\}\in\mathbb{Z}\}=\mathbb{Z}$ (discontinuous points of $\{y\}$). Let $\mathbb{R}\setminus\mathfrak{H}=\coprod_{\nu\in\mathbb{Z}}\mathfrak{D}^{(\nu)}$, where $\mathfrak{D}^{(\nu)}=(\nu,\nu+1)$. From each $\mathfrak{D}^{(\nu)}$ to $\mathbb{C}$, the function $B(\{y\})$ is analytically continued to a polynomial function $B_k^{(\nu)}(y)=B_k(y-\nu)\in\mathbb{Q}[y]$.



$\mathbb{R}\setminus\mathfrak{H}=\coprod_{\nu\in\mathfrak{J}}\mathfrak{D}^{(\nu)}$ $\qquad B_k(\{y\})$ $\qquad B_k^{(0)}(y)=B_k(y)$

A similar procedure works well in general cases and we can define multiple generalizations of Bernoulli polynomials.

Let

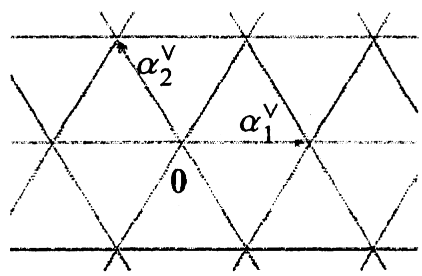
$$\mathfrak{H}=\bigcup_{\mathrm{V}\in\mathscr{V}}\bigcup_{q\in Q^\vee}\bigcup_{\beta\in\mathrm{V}}\{\mathbf{y}\in V\mid\{\mathbf{y}+q\}_{\mathrm{V},\beta}\in\mathbb{Z}\}$$

(discontinuous points of $\{\mathbf{y}+q\}_{\mathrm{V},\beta}$ appearing in the generating function).

Let

$$V\setminus\mathfrak{H}=\coprod_{\nu\in\mathfrak{J}}\mathfrak{D}^{(\nu)},$$

where $\mathfrak{D}^{(\nu)}$ is an open connected component, $\mathfrak{J}$ is a set of indices.

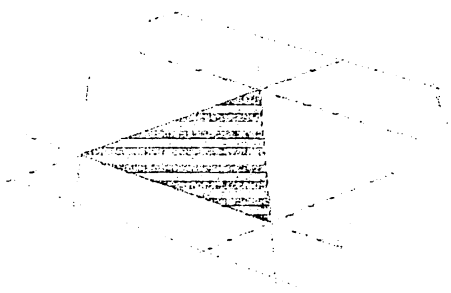


A_2 case

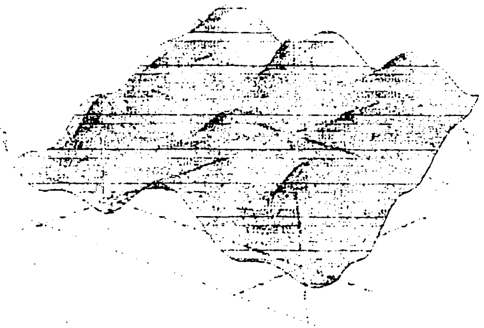
Theorem 3 ([8, 9, 10]). *From each region $\mathfrak{D}^{(\nu)}$ to the whole space $\mathbb{C}\otimes V$, $P(\mathbf{k},\mathbf{y};X_r)$ is analytically continued in $\mathbf{y}$ to a polynomial function $B_{\mathbf{k}}^{(\nu)}(\mathbf{y};X_r)\in\mathbb{Q}[\mathbf{y}]$ of total degree at most $|\mathbf{k}|=\sum_{\alpha\in\Delta_+}k_\alpha$, where $\mathbf{y}=\sum_{n=1}^{r}y_n\alpha_n^\vee$.*

§§8.1. Example: A_2 Case

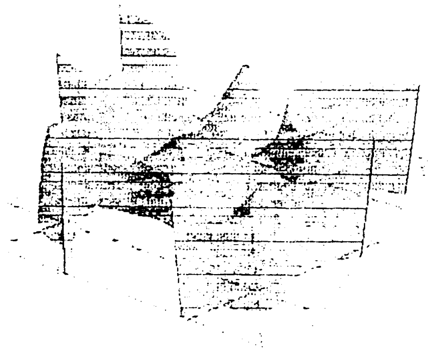
The Bernoulli polynomial $B_{\mathbf{2}}^{(0)}(\mathbf{y}; A_2)$ is obtained by the analytic continuation of the periodic Bernoulli function $P(\mathbf{2}, \mathbf{y}; A_2)$ from the region $\mathfrak{D}^{(0)}$.



$V \setminus \mathfrak{H} = \coprod_{\nu\in\mathfrak{Z}} \mathfrak{D}^{(\nu)}$



$P(\mathbf{2}, \mathbf{y}; A_2)$
(Periodic Bernoulli function)



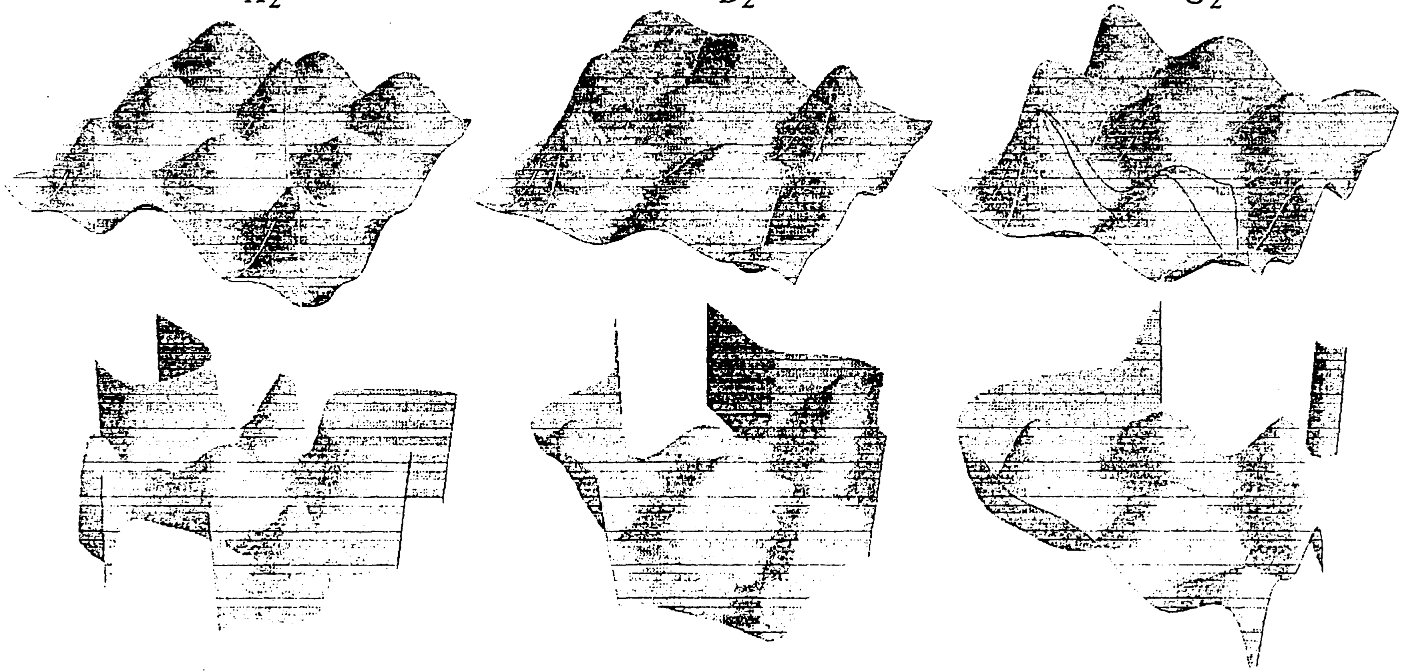
$B_{\mathbf{2}}^{(0)}(\mathbf{y}; A_2)$
(Bernoulli polynomial)

The explicit form of the Bernoulli polynomial $B_{\mathbf{2}}^{(0)}(\mathbf{y}; A_2)$ is given as follows:

$$\begin{aligned} B_{\mathbf{2}}^{(0)}(\mathbf{y}; A_2) = {} & \frac{1}{3780} + \frac{1}{45}(y_1y_2 - y_1^2 - y_2^2) + \frac{1}{18}(3y_1y_2^2 - 3y_1^2y_2 + 2y_1^3) \\ & + \frac{1}{9}(-2y_1y_2^3 - 3y_1^2y_2^2 + 4y_1^3y_2 - 2y_1^4 + y_2^4) \\ & + \frac{1}{30}(-5y_1y_2^4 + 10y_1^2y_2^3 + 10y_1^3y_2^2 - 15y_1^4y_2 + 6y_1^5) \\ & + \frac{1}{30}(6y_1y_2^5 - 5y_1^2y_2^4 - 5y_1^4y_2^2 + 6y_1^5y_2 - 2y_1^6 - 2y_2^6) \in \mathbb{Q}[\mathbf{y}]. \end{aligned}$$

§§8.2. Further Examples: A_2, B_2, G_2 Cases

A_2 B_2 G_2



The graphs in the upper (resp. lower) row are those of periodic Bernoulli functions (resp. Bernoulli polynomials).

We summarize what we have obtained: we have constructed periodic Bernoulli functions so that they describe functional-relations of multiple zeta-functions of root systems, which can be calculated by using the generating function; Bernoulli polynomials are obtained by the analytic continuation of periodic Bernoulli functions.

$$\sum_{w\in W}\Big(\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-}(-1)^{k_\alpha}\Big)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y};X_r)=(-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};X_r)\Big(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Big),$$

$$F(\mathbf{t},\mathbf{y};X_r)=\sum_{\mathbf{k}\in\mathbb{Z}_{\geq0}^{|\Delta_+|}}P(\mathbf{k},\mathbf{y};X_r)\prod_{\alpha\in\Delta_+}\frac{t_\alpha^{k_\alpha}}{k_\alpha!},$$

$$P(\mathbf{k},\mathbf{y};X_r)\Longleftrightarrow B_{\mathbf{k}}^{(\mathbf{v})}(\mathbf{y};X_r)\in\mathbb{Q}[\mathbf{y}].$$

§9. L-Functions of Root Systems

We give an application of periodic Bernoulli functions or equivalently Bernoulli polynomials. For this purpose, we define an L-analogue of zeta-functions of root systems.

Definition 4 ([9, 10]). L-functions of root systems: For a root system Δ of type X_r, define

$$L_r(\mathbf{s},\chi;X_r)=\sum_{\lambda\in P_{++}}\prod_{\alpha\in\Delta_+}\frac{\chi_\alpha(\langle\alpha^\vee,\lambda\rangle)}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}},$$

where $\chi=(\chi_\alpha)_{\alpha\in\Delta_+}$ is a set of primitive Dirichlet characters of conductors $f_\alpha\in\mathbb{Z}_{\geq1}$.

We extend $\chi=(\chi_\alpha)_{\alpha\in\Delta_+}$ to $(\chi_\alpha)_{\alpha\in\Delta}$ by $\chi_\alpha=\chi_{-\alpha}$ and define $(w\chi)_\alpha=\chi_{w^{-1}\alpha}$. Then we have value-relations of L-functions.

Theorem 4 ([9, 10]). *For* $\mathbf{s}=\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{Z}_{\geq2}^{|\Delta_+|}$,

$$\sum_{w\in W}\Big(\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-}(-1)^{k_\alpha}\chi_\alpha(-1)\Big)L_r(w^{-1}\mathbf{k},w^{-1}\chi;X_r)$$

$$=(-1)^{|\Delta_+|}\Big(\prod_{\alpha\in\Delta_+}\chi_\alpha(-1)g(\chi_\alpha)\frac{(2\pi i)^{k_\alpha}}{k_\alpha!f^{k_\alpha}}\Big)B_{\mathbf{k},\bar{\chi}}(X_r),$$

where $B_{\mathbf{k},\chi}(X_r)$ *is a multiple generalized Bernoulli number (defined later).*

cf. $(X_r=A_1)$

$$L(k,\chi)+(-1)^k\chi(-1)L(k,\chi)=-\chi(-1)g(\chi)\frac{(2\pi i)^k}{k!f^k}B_{k,\bar{\chi}}.$$

§10. Special L-Values

Theorem 4 directly implies a formula for special values of L-functions:

Theorem 5 ([9, 10]). *For* $\mathbf{k} \in (\mathbb{Z}_{\geq 2})^{|\Delta_+|}$ *and* χ *s.t.* $w^{-1}\mathbf{k} = \mathbf{k}$, $w^{-1}\chi = \chi$ *for all* $w \in W$ *and* $(-1)^{k_\alpha}\chi_\alpha(-1) = 1$ *for all* $\alpha \in \Delta_+$,

$$L_r(\mathbf{k}, \chi; X_r) = \frac{(-1)^{|\mathbf{k}|+|\Delta_+|}}{|W|}\Big(\prod_{\alpha\in\Delta_+} \frac{(2\pi i)^{k_\alpha}}{k_\alpha! f_\alpha^{k_\alpha}} g(\chi_\alpha)\Big) B_{\mathbf{k},\overline{\chi}}(X_r).$$

cf. $(X_r = A_1)$

$$L(k,\chi) = \frac{(-1)^{k+1}}{2}\frac{(2\pi i)^k}{k! f^k} g(\chi) B_{k,\overline{\chi}}.$$

As an example, let ρ_7 be the Dirichlet character of conductor 7 defined by $\rho_7(1) = \rho_7(6) = 1$, $\rho_7(2) = \rho_7(5) = e^{2\pi i/3}$, $\rho_7(3) = \rho_7(4) = e^{4\pi i/3}$. Then the Gauss sum is $g(\rho_7) = 2(\cos(2\pi/7) + e^{2\pi i/3}\cos(4\pi/7) + e^{4\pi i/3}\cos(6\pi/7))$ and we have

$$\begin{aligned} L_2((2,4,4,2),(1,\rho_7,\rho_7,1);B_2) &= \sum_{m,n=1}^{\infty} \frac{\rho_7(n)\rho_7(m+n)}{m^2 n^4 (m+n)^4 (m+2n)^2} \\ &= \frac{(-1)^{12+4}}{2^2 2!}\Big(\frac{(2\pi i)^2}{2!}\Big)^2\Big(\frac{(2\pi i)^4}{4! 7^4} g(\rho_7)\Big)^2\Big(\frac{69967019}{6988350600} + \frac{102810289\sqrt{-3}}{6988350600}\Big) \\ &= g(\rho_7)^2 \pi^{12}\Big(\frac{69967019}{1812890273725377 00} + \frac{102810289\sqrt{-3}}{181289027372537700}\Big). \end{aligned}$$

We give two more examples. Let ρ_5 be the quadratic character of conductor 5. Then we have

$$L_2((2,2,2,2),(\rho_5,\rho_5,\rho_5,\rho_5);B_2) = \frac{92}{29296875}\pi^8;$$

$$L_3((2,2,2,2,2,2),(\rho_5,\rho_5,\rho_5,\rho_5,\rho_5,\rho_5);A_3) = -\frac{1856}{213623046875}\pi^{12}.$$

The latter can be regarded as a character analogue of the formula in [1, Prop. 8.5].

§11. Multiple Generalized Bernoulli Numbers

The generating function of multiple generalized Bernoulli numbers is given in terms of that of multiple Bernoulli polynomials as in the classical theory.

Definition 5 (generating function [9, 10]). For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+}$,

$$G(\mathbf{t},\chi;X_r) = \sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha}\Big(\prod_{\alpha\in\Delta_+}\frac{\chi_\alpha(a_\alpha)}{f_\alpha}\Big)F(\mathbf{f}\,\mathbf{t},\mathbf{y}(\mathbf{a};\mathbf{f});X_r),$$

where $F(\mathbf{t},\mathbf{y};X_r)$ is the generating function of multiple periodic Bernoulli functions and $\mathbf{f}\,\mathbf{t} = (f_\alpha t_\alpha)_{\alpha\in\Delta_+}$, $\mathbf{y}(\mathbf{a};\mathbf{f}) = \sum_{\alpha\in\Delta_+} a_\alpha \alpha^\vee / f_\alpha$.

Definition 6 (multiple generalized Bernoulli numbers [9, 10]).

$$G(\mathbf{t},\chi;X_r)=\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}} B_{\mathbf{k},\chi}(X_r)\prod_{\alpha\in\Delta_+}\frac{t_\alpha^{k_\alpha}}{k_\alpha!},$$

$$B_{\mathbf{k},\chi}(X_r)=\Bigl(\prod_{\alpha\in\Delta_+} f_\alpha^{k_\alpha-1}\Bigr)\sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha}\Bigl(\prod_{\alpha\in\Delta_+}\chi_\alpha(a_\alpha)\Bigr)P(\mathbf{k},\mathbf{y}(\mathbf{a};\mathbf{f});X_r).$$

cf. $(X_r=A_1)$

$$G(t,\chi)=\sum_{a=1}^{f}\frac{\chi(a)}{f}F(ft,a/f)=\sum_{a=1}^{f}\frac{\chi(a)}{f}\frac{fte^{ft\{a/f\}}}{e^{ft}-1}=\sum_{k=0}^{\infty}B_{k,\chi}\frac{t^k}{k!}.$$

$$B_{k,\chi}=f^{k-1}\sum_{a=1}^{f}\chi(a)B_k(\{a/f\}).$$

§§11.1. Properties

Theorem 6 ([9, 10]). *Assume that $f_\alpha>1$ if Δ is of type A_1. Then for $w\in W$,*

$$B_{w^{-1}\mathbf{k},w^{-1}\chi}(X_r)=\Bigl(\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-}(-1)^{k_\alpha}\chi_\alpha(-1)\Bigr)B_{\mathbf{k},\chi}(X_r).$$

Hence $B_{\mathbf{k},\chi}(X_r)=0$ if there exists an element $w\in W_{\mathbf{k}}\cap W_\chi$ such that

$$\prod_{\alpha\in\Delta_+\cap w^{-1}\Delta_-}(-1)^{k_\alpha}\chi_\alpha(-1)\neq 1,$$

where $W_{\mathbf{k}}$ and W_χ are the stabilizers of $\mathbf{k}$ and χ respectively.

cf. $(X_r=A_1)$

$$B_{k,\chi}=0\quad\text{if}\ (-1)^k\chi(-1)\neq 1.$$

Several other properties in the classical theory such as

$$F(t,y)=F(-t,-y)\ \text{for}\ y\in\mathbb{R}\setminus\mathbb{Z},\qquad B_k(1-y)=(-1)^kB_k(y),\qquad \frac{1}{t}\frac{\partial}{\partial y}F(t,y)=F(t,y)$$

can be reinterpreted in terms of root systems and Weyl groups.

§12. Appendix: Integral Representation

The analytic continuations of multiple zeta-functions were already obtained by Matsumoto [11], Essouabri [3], de Crisenoy [2], etc. However we give yet another method which is a generalization of the formula

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{\Gamma(s)(e^{2\pi is}-1)}\int_C\frac{z^{s-1}}{e^z-1}dz\qquad (C\text{: Hankel contour}).$$

For $\xi \in \mathbb{C}^R$, $\boldsymbol{a}, \boldsymbol{s} \in \mathbb{C}^N$ and $\boldsymbol{b} \in \mathbb{C}^{N\times R}$, consider the multiple series

$$\zeta(\xi, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}) =$$
$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_R=0}^{\infty} \frac{e^{\xi_1 m_1} \cdots e^{\xi_R m_R}}{(a_1 + b_{11}m_1 + \cdots + b_{1R}m_R)^{s_1} \cdots (a_N + b_{N1}m_1 + \cdots + b_{NR}m_R)^{s_N}}.$$

Theorem 7 ([4, 5]).
$$\zeta(\xi, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{s}) = \frac{1}{\Gamma(s_1)\cdots\Gamma(s_N)} \prod_{t\in S} \frac{1}{e^{2\pi i t(s)} - 1} \times$$
$$\int_{\Sigma} \frac{e^{(b_{11}+\cdots+b_{1R}-a_1)z_1} \cdots e^{(b_{N1}+\cdots+b_{NR}-a_N)z_N} z_1^{s_1-1} \cdots z_N^{s_N-1}}{(e^{z_1 b_{11}+\cdots+z_N b_{N1}} - e^{\xi_1}) \cdots (e^{z_1 b_{1R}+\cdots+z_N b_{NR}} - e^{\xi_R})} dz_1 \wedge \cdots \wedge dz_N,$$
where Σ is essentially a union of surfaces and S is a set of linear functionals on $\mathbb{C}^N$.

From the integrand, we can construct generating functions of Bernoulli numbers for nonpositive domain.

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