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ALGEBRAIC RELATIONS AND ASYMPTOTIC FORMULAS FOR FIBONACCI RECIPROCAL SUMS

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1. Introduction

Let $\alpha,\ \beta \in \mathbb{C}$ satisfy $|\beta| < 1$ and $\alpha\beta = -1$. We put

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad (n \geq 0), \tag{1}$$

$$V_n = \alpha^n + \beta^n \qquad (n \geq 0). \tag{2}$$

If $\alpha + \beta = a \in \mathbb{Z}$, then $\{U_n\}_{n\geq 0}$ (respectively, $\{V_n\}_{n\geq 0}$) is a sequence of generalized Fibonacci numbers (respectively, Lucas numbers), which satisfies

$$X_{n+2} = aX_{n+1} + X_n \quad (n \geq 0)$$

with initial values $(X_0, X_1) = (0, 1)$ (respectively, $(X_0, X_1) = (2, a)$). Indeed, if $\beta = (1-\sqrt{5})/2$, we have the Fibonacci and Lucas numbers: $U_n = F_n, V_n = L_n$ $(n \geq 0)$. If $\beta = 1 - \sqrt{2}$, then $\{U_n\}_{n\geq 0}$ is a sequence of the *Pell numbers* defined by $P_0 = 0, P_1 = 1$, $P_{n+2} = 2P_{n+1} + P_n$ $(n \geq 0)$ (cf. [9]). Duverney, Ke. Nishioka, Ku. Nishioka, and the last named author [2] (see also [1]) proved the transcendence of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{U_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{V_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{U_{2n-1}^{s}}, \quad \sum_{n=1}^{\infty} \frac{1}{V_{2n}^{s}} \quad (s = 1, 2, 3, \ldots)$$

by using Nesterenko's theorem on the Ramanujan functions $P(q)$, $Q(q)$, and $R(q)$ (see Section 5).

In this article, we discuss algebraic independence and algebraic relations for reciprocal sums of generalized Fibonacci numbers (respectively, Lucas numbers). Moreover we present asymptotic formulas of them as β tends to a critical value.

2. Algebraic relations for reciprocal sums

In what follows s always denotes a nonnegative integer. Set $\sigma_0(s) = 1$, and for $s \geq 2$ let $\sigma_1(s), \ldots, \sigma_{s-1}(s)$ be the elementary symmetric functions of the $s-1$ numbers $-1, -2^2, \ldots, -(s-1)^2$ defined by

$$\sigma_i(s) = (-1)^i \sum_{1\leq r_1 < \ldots < r_i \leq s-1} r_1^2 \cdots r_i^2 \quad (1 \leq i \leq s-1).$$

The coefficients of the following expansions

$$\operatorname{cosec}^2 x = \frac{1}{x^2} + \sum_{j=0}^{\infty} a_j x^{2j}, \quad \sec^2 x = \sum_{j=0}^{\infty} b_j x^{2j}$$

are given by

$$a_{j-1}=\frac{(-1)^{j-1}(2j-1)2^{2j}B_{2j}}{(2j)!},\quad b_{j-1}=\frac{(-1)^{j-1}(2j-1)2^{2j}(2^{2j}-1)B_{2j}}{(2j)!}$$

$(j\geq 1)$, where $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$, ... are the Bernoulli numbers.

For the sequences defined by (1) and (2), and for $s\geq 1$, set

$$\Phi_{2s}:=(\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_n^{2s}},\qquad \Psi_{2s}:=\sum_{n=1}^{\infty}\frac{1}{V_n^{2s}},$$

$$\Phi_{2s}^*:=(\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{U_n^{2s}},\qquad \Psi_{2s}^*:=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{V_n^{2s}}.$$

For these sums, we have the following ([3]):

Theorem 2.1. *Suppose that $\alpha,\ \beta\in\overline{\mathbb{Q}}$. Then the numbers Φ_2, Φ_4, Φ_6 are algebraically independent, and for any integer $s\geq 4$ the number Φ_{2s} is written as*

$$\Phi_{2s}=\frac{1}{(2s-1)!}\left(\sigma_{s-1}(s)\mu_s-\sum_{j=1}^{s-1}\frac{(-1)^j(2j)!}{2^{2j+3}}\sigma_{s-j-1}(s)(\varphi_j-(-1)^s\psi_j-a_j)\right)$$

with

$$\mu_s=\Phi_2\quad(s\ odd),\quad =\frac{1}{3}\left(4\Phi_2^2+2\Phi_2-18\Phi_4+\omega-\frac{5}{4}\right)\quad(s\ even),$$

$$\varphi_1=\frac{4}{3}\left(32\Phi_2^2-5\Phi_2-\omega+\frac{13}{10}\right),\quad \varphi_2=-\frac{4}{63}(24\Phi_2-1)\left(112\Phi_2^2-21\Phi_2-5\omega+\frac{77}{12}\right),$$

$$\varphi_j=\frac{3}{(j-2)(2j+3)}\sum_{i=1}^{j-2}\varphi_i\varphi_{j-i-1}\quad(j\geq 3),$$

$$\psi_1=\frac{4}{3}\left(16\Phi_2^2-13\Phi_2-5\omega+\frac{25}{4}\right),\quad \psi_2=\frac{4}{9}(24\Phi_2-1)\left(16\Phi_2^2-13\Phi_2-5\omega+\frac{25}{4}\right),$$

$$\psi_j=\frac{1}{j(2j-1)}\left(2(24\Phi_2-1)\psi_{j-1}-3\sum_{i=1}^{j-2}\psi_i\psi_{j-i-1}\right)\quad(j\geq 3),$$

where $\omega=(56\Phi_6+5/4)/(4\Phi_2+1)$.

Remark 2.1. If $s\geq 4$, then $(1+4\Phi_2)^{[s/2]}(\Phi_{2s}-r_s\Phi_4)\in\mathbb{Q}[\Phi_2,\Phi_6]$, and the total degree of this does not exceed $s+[s/2]$, where $r_s\in\mathbb{Q}$ ($r_s=0$ if and only if s is odd). Some of the algebraic relations are given by the following table:

		$x=\Phi_2\quad y=\Phi_4\quad z=\Phi_6$
$s=4$	$\Phi_8=$	$\frac{3}{70}y+\frac{1}{1890(4x+1)^2}\Big(1280x^6-3456x^5+576x^4+8960x^3z$
		$-444x^3+20160x^2z-81x^2+1512xz+15680z^2-42z\Big)$
$s=5$	$\Phi_{10}=$	$\frac{1}{297(4x+1)^2}\Big(512x^7-704x^6+162x^5-1600x^4z-30x^4$
		$+2560x^3z-15x^3+450x^2z+4760xz^2+75xz+700z^2+15z\Big)$

Theorem 2.2. *Suppose that α, $\beta \in \overline{\mathbb{Q}}$. Then the numbers Φ_2^*, Φ_4^*, Φ_6^* are algebraically independent, and for any integer $s \geq 4$ the number Φ_{2s}^* is written as*

$$\Phi_{2s}^* = \frac{1}{(2s-1)!}\left(\sigma_{s-1}(s)\mu_s + \sum_{j=1}^{s-1} \frac{(-1)^j(2j)!}{2^{2j+3}}\sigma_{s-j-1}(s)\big(\varphi_j + (-1)^s\psi_j - a_j\big)\right)$$

with

$$\mu_s = \Phi_2^* \quad (s \text{ odd}), \quad = \frac{1}{24}(4\xi - 1) \quad (s \text{ even}),$$

$$\varphi_1 = -\frac{4}{45}\Big(180\Phi_4^* - 10\xi^2 + 5\xi - \frac{11}{8}\Big), \quad \varphi_2 = -\frac{16}{189}\xi\Big(180\Phi_4^* - 6\xi^2 + 5\xi - \frac{11}{8}\Big),$$

$$\varphi_j = \frac{3}{(j-2)(2j+3)}\sum_{i=1}^{j-2}\varphi_i\varphi_{j-i-1} \quad (j \geq 3),$$

$$\psi_1 = -\frac{4}{9}\Big(180\Phi_4^* + 2\xi^2 + 5\xi - \frac{11}{8}\Big), \quad \psi_2 = \frac{16}{27}\xi\Big(180\Phi_4^* + 2\xi^2 + 5\xi - \frac{11}{8}\Big),$$

$$\psi_j = -\frac{1}{j(2j-1)}\left(8\xi\psi_{j-1} + 3\sum_{i=1}^{j-2}\psi_i\psi_{j-i-1}\right) \quad (j \geq 3),$$

where $\xi = \xi(\Phi_2^*, \Phi_4^*, \Phi_6^*)$ is a number satisfying

$$8\xi^3 + 5\xi^2 + (1440\Phi_4^* - 46)\xi - \Big(252\Phi_2^* + 1260\Phi_4^* - 7560\Phi_6^* - \frac{177}{16}\Big) = 0.$$

Theorem 2.3. *Suppose that α, $\beta \in \overline{\mathbb{Q}}$. Then the numbers Ψ_2, Ψ_4, Ψ_6 are algebraically independent, and for any integer $s \geq 4$ the number Ψ_{2s} is written as*

$$\Psi_{2s} = \frac{1}{(2s-1)!}\left(\sigma_{s-1}(s)\mu_s + \sum_{j=1}^{s-1} \frac{(-1)^j(2j)!}{2^{2j+3}}\sigma_{s-j-1}(s)\big(\psi_j - (-1)^s(\varphi_j - b_j)\big)\right)$$

with

$$\mu_s = \Psi_2 \quad (s \text{ odd}), \quad = 4\Psi_2^2 + \Psi_2 - 6\Psi_4 \quad (s \text{ even}),$$

$$\varphi_1 = \frac{1}{2}(8\Psi_2 + 1)(8\Psi_2 + \eta + 1), \quad \varphi_2 = \frac{1}{12}(8\Psi_2 + 1)(8\Psi_2 + \eta + 1)(24\Psi_2 + \eta + 3),$$

$$\varphi_j = \frac{1}{j(2j-1)}\left((24\Psi_2 + \eta + 3)\varphi_{j-1} + 3\sum_{i=1}^{j-2}\varphi_i\varphi_{j-i-1}\right) \quad (j \geq 3),$$

$$\psi_1 = -\frac{1}{2}(8\Psi_2 + 1)(8\Psi_2 - \eta + 1), \quad \psi_2 = \frac{1}{12}(8\Psi_2 + 1)(8\Psi_2 - \eta + 1)(24\Psi_2 - \eta + 3),$$

$$\psi_j = -\frac{1}{j(2j-1)}\left((24\Psi_2 - \eta + 3)\psi_{j-1} + 3\sum_{i=1}^{j-2}\psi_i\psi_{j-i-1}\right) \quad (j \geq 3),$$

where $\eta = \eta(\Psi_2, \Psi_6)$ is a number satisfying

$$(\eta + 5)^2 = -192\Psi_2^2 - 48\Psi_2 + 6 + \frac{3840\Psi_6 + 30}{8\Psi_2 + 1}.$$

Theorem 2.4. *Suppose that $\alpha, \beta \in \overline{\mathbb{Q}}$. Then the numbers $\Psi_2^*, \Psi_4^*, \Psi_6^*$ are algebraically independent, and for any integer $s \geq 4$ the number Ψ_{2s}^* is written as*

$$\Psi_{2s}^* = \frac{1}{(2s-1)!}\left(\sigma_{s-1}(s)\mu_s + \sum_{j=1}^{s-1} \frac{(-1)^j(2j)!}{2^{2j+3}}\sigma_{s-j-1}(s)\big(\psi_j + (-1)^s(\varphi_j - b_j)\big)\right)$$

with

$$\mu_s = \Psi_2^* \quad (s \text{ odd}), \quad = \frac{1}{8}(\theta - 1) \quad (s \text{ even}),$$

$$\varphi_1 = -\frac{1}{2}(96\Psi_4^* - \theta^2 + 2\theta - 3), \quad \varphi_2 = \frac{1}{12\theta}(96\Psi_4^* - \theta^2 + 2\theta - 3)(96\Psi_4^* - 3\theta^2 + 2\theta - 3),$$

$$\varphi_j = -\frac{1}{j(2j-1)}\left((96\Psi_4^* - 3\theta^2 + 2\theta - 3)\frac{\varphi_{j-1}}{\theta} - 3\sum_{i=1}^{j-2}\varphi_i\varphi_{j-i-1}\right) \quad (j \geq 3),$$

$$\psi_1 = -\frac{1}{2}(96\Psi_4^* + \theta^2 + 2\theta - 3), \quad \psi_2 = \frac{1}{12\theta}(96\Psi_4^* + \theta^2 + 2\theta - 3)(96\Psi_4^* + 3\theta^2 + 2\theta - 3),$$

$$\psi_j = -\frac{1}{j(2j-1)}\left((96\Psi_4^* + 3\theta^2 + 2\theta - 3)\frac{\psi_{j-1}}{\theta} + 3\sum_{i=1}^{j-2}\psi_i\psi_{j-i-1}\right) \quad (j \geq 3),$$

where $\theta = \theta(\Psi_2^*, \Psi_4^*, \Psi_6^*)$ is a number satisfying

$$\theta^2 - (192\Psi_4^* - 6)\theta + 1920\Psi_6^* - 64\Psi_2^* - 7 = 0.$$

For algebraic independence of general sums we have the following ([7]):

Theorem 2.5. *Let s_1, s_2, s_3 be distinct positive integers. Then the numbers Φ_{2s_1}, Φ_{2s_2}, Φ_{2s_3} are algebraically independent over $\mathbb{Q}$ if and only if at least one of s_1, s_2, s_3 is even.*

The quantities ξ, η, and θ in Theorems 2.2, 2.3, and 2.4, respectively, are algebraic functions of the corresponding sums for $1 \leq s \leq 3$. In these theorems, we have to know the branches of ξ, η, and θ depending on the parameter a (or β). Since $a = \alpha + \beta \in \mathbb{C}$, we write $\beta = (a/2)(1 - \sqrt{1 + 4a^{-2}})$, which satisfies $\beta(a) = O(a^{-1})$ as $a \to \infty$. Then each reciprocal sum is a function of a or β. For example the branch of η is given below (for ξ and θ see [3]):

Theorem 2.6. *Under the same suppositions as in Theorem 2.3, we have the following:*

(i) *The function $\eta = \eta(a)$ is holomorphic for $|a| > 5.431$, and is expressible in the form*

$$\eta(a) = -5 + \sqrt{\chi(a)}$$

with

$$\chi(a) = -192\Psi_2^2 - 48\Psi_2 + 6 + (3840\Psi_6 + 30)/(8\Psi_2 + 1)$$

satisfying $\chi(a) = 36 + O(a^{-2})$ as $a \to \infty$. Here the branch is taken so that $\sqrt{\chi(\infty)} = 6$.

(ii) *For $a = 1$ corresponding to the Lucas numbers,*

$$\eta(1) = -5 - \sqrt{\chi(1)} \quad (< -5),$$

and for any real number $a \geq 2.4$

$$\eta(a) = -5 + \sqrt{\chi(a)} \quad (> -5).$$

3. Reciprocal sums of odd terms

In addition to the notation $\sigma_i(s)$ defined in Section 2, let $\tau_1(s), \dots, \tau_s(s)$ $(s \geq 1)$ be the elementary symmetric functions of the s numbers $-1, -3^2, \dots, -(2s-1)^2$ given by

$$\tau_i(s) = (-1)^i \sum_{1\leq r_1<\cdots<r_i\leq s} (2r_1-1)^2 \cdots (2r_i-1)^2 \qquad (1 \leq i \leq s),$$

and for $s \geq 0$ set $\tau_0(s) = 1$.

For $p \geq 1$ and for $s \geq 1$, consider the reciprocal sums of odd terms:

$$f_p := (\alpha-\beta)^{-p} \sum_{n=1}^{\infty} \frac{1}{U_{2n-1}^p}, \quad g_{2s} := \sum_{n=1}^{\infty} \frac{1}{V_{2n-1}^{2s}}, \quad g_{2s-1}^* := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{V_{2n-1}^{2s-1}},$$

where $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ are the sequences given by (1) and (2), respectively. For these sums we have the following ([4]):

Theorem 3.1. *Suppose that $\alpha, \beta \in \overline{\mathbb{Q}}$. Then the numbers f_1, f_2, f_3 are algebraically independent, and for any integer $s \geq 2$*

$$f_{2s} = \frac{(-1)^{s-1}}{(2s-1)!} \left(\sigma_{s-1}(s) f_2 - \sum_{j=1}^{s-1} \frac{(-1)^j (2j)!}{2^{2j+3}} \sigma_{s-j-1}(s) \varphi_j \right)$$

and

$$f_{2s+1} = \frac{(-1)^s}{2^{2s+2}(2s)!} \sum_{j=0}^{s} (-1)^j (2j)! \tau_{s-j}(s) \psi_j,$$

where

$$\varphi_1 = 16 f_1 (f_1 - 8 f_3), \quad \varphi_2 = -\frac{\varphi_1}{3f_1}(f_1 - 8f_3 - 16 f_1^3), \quad \psi_0 = 4f_1, \quad \psi_1 = 16 f_3 - 2 f_1,$$

$$\varphi_j = \frac{1}{j(2j-1)} \left(6 \frac{\varphi_2 \varphi_{j-1}}{\varphi_1} - 3 \sum_{i=1}^{j-2} \varphi_i \varphi_{j-i-1} \right) \quad (j \geq 3),$$

$$\psi_j = \frac{1}{j(2j-1)} \left(\frac{(2\psi_0^3 + \psi_1)\psi_{j-1}}{\psi_0} + 3\psi_0 \sum_{i=1}^{j-2} \psi_i \psi_{j-i-1} + \sum_{m=1}^{j-3} \psi_m \sum_{i=1}^{j-m-2} \psi_i \psi_{j-m-i-1} \right) \quad (j \geq 2).$$

Theorem 3.2. *Suppose that $\alpha, \beta \in \overline{\mathbb{Q}}$. Then the numbers g_2, g_4, g_6 are algebraically independent, and for any integer $s \geq 4$ the number g_{2s} is written as*

$$g_{2s} = \frac{1}{(2s-1)!} \left(\sigma_{s-1}(s) g_2 + \sum_{j=1}^{s-1} \frac{(-1)^j (2j)!}{2^{2j+3}} \sigma_{s-j-1}(s) \varphi_j \right),$$

where

$$\varphi_1 = -16(g_2 + 6g_4), \quad \varphi_2 = \frac{16}{3}(g_2 + 30 g_4 + 120 g_6),$$

$$\varphi_j = \frac{1}{j(2j-1)} \left(6 \frac{\varphi_2 \varphi_{j-1}}{\varphi_1} - 3 \sum_{i=1}^{j-2} \varphi_i \varphi_{j-i-1} \right) \quad (j \geq 3).$$

Theorem 3.3. *Suppose that $\alpha,\ \beta \in \overline{\mathbb{Q}}$. Then the numbers g_1^*, g_3^* are algebraically independent, and for any integer $s \geq 2$ the number g_{2s+1}^* is written as*

$$g_{2s+1}^* = \frac{1}{2^{2s}(2s)!}\sum_{j=0}^{s}\frac{(-1)^j(2j)!}{4}\tau_{s-j}(s)\varphi_j,$$

where

$$\varphi_0 = 4g_1^*, \quad \varphi_1 = -2g_1^* - 16g_3^*,$$

$$\varphi_j = \frac{1}{j(2j-1)}\left(\frac{(\varphi_1 - 2\varphi_0^3)\varphi_{j-1}}{\varphi_0} - 3\varphi_0\sum_{i=1}^{j-2}\varphi_i\varphi_{j-i-1} - \sum_{m=1}^{j-3}\varphi_m\sum_{i=1}^{j-m-2}\varphi_i\varphi_{j-m-i-1}\right) \quad (j \geq 2).$$

4. Reciprocal sums of even terms

Let E_{2j} denote the Euler numbers $E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ \ldots$, which satisfy

$$\sec x = \sum_{j=0}^{\infty}\frac{(-1)^j E_{2j}}{(2j)!}x^{2j}.$$

For $s \geq 1$ and for $p \geq 1$, consider the reciprocal sums of even terms:

$$h_{2s} := (\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_{2n}^{2s}}, \qquad l_p := \sum_{n=1}^{\infty}\frac{1}{V_{2n}^{p}},$$

$$h_{2s}^* := (\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{U_{2n}^{2s}}, \qquad l_p^* := \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{V_{2n}^{p}},$$

where $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ are the sequences given by (1) and (2), respectively. For these sums we have the following ([5]):

Theorem 4.1. *Suppose that $\alpha,\ \beta \in \overline{\mathbb{Q}}$. Then the numbers h_2, h_4, h_6 are algebraically independent, and for any integer $s \geq 4$*

$$h_{2s} = \frac{1}{(2s-1)!}\left(\sigma_{s-1}(s)h_2 - \sum_{j=1}^{s-1}\frac{(-1)^j(2j)!}{2^{2j+3}}\sigma_{s-j-1}(s)(\varphi_j - a_j)\right),$$

where

$$\varphi_1 = \frac{1}{15}(1 + 240h_2 + 1440h_4), \quad \varphi_2 = \frac{2}{189}(1 - 504h_2 - 15120h_4 - 60480h_6),$$

$$\varphi_j = \frac{3}{(j-2)(2j+3)}\sum_{i=1}^{j-2}\varphi_i\varphi_{j-i-1} \quad (j \geq 3). \tag{3}$$

Theorem 4.2. *Suppose that $\alpha,\ \beta \in \overline{\mathbb{Q}}$. Then the numbers h_2^* and h_4^* are algebraically independent, and for any integer $s \geq 3$*

$$h_{2s}^* = \frac{1}{(2s-1)!}\sum_{j=0}^{s-1}\frac{(-1)^{j-1}(2j+1)!}{2^{2j+3}}\sigma_{s-j-1}(s)\left(\psi_{j+1} + \frac{a_j}{2j+1}\right),$$

where

$$\psi_1 = -\frac{1}{3} - 8h_2^*, \quad \psi_2 = -\frac{1}{45} + \frac{16}{3}h_2^* + 32h_4^*,$$

(4)

$$\psi_j = \frac{1}{(2j+1)(j-2)}\left(-3\psi_1\psi_{j-1} + 3\sum_{i=1}^{j-1}\psi_i\psi_{j-i} + \sum_{m=1}^{j-2}\psi_m\sum_{i=1}^{j-m-1}\psi_i\psi_{j-m-i}\right) \quad (j \geq 3).$$

Theorem 4.3. *Suppose that $\alpha,\ \beta \in \overline{\mathbb{Q}}$. Then the numbers $l_1,\ l_2,\ l_3$ are algebraically independent, and for any integer $s \geq 2$*

$$l_{2s} = \frac{(-1)^{s-1}}{(2s-1)!}\left(\sigma_{s-1}(s)l_2 + \sum_{j=1}^{s-1}\frac{(-1)^j(2j)!}{2^{2j+3}}\sigma_{s-j-1}(s)(\varphi_j - b_j)\right),$$

$$l_{2s+1} = \frac{(-1)^s}{2^{2s}(2s)!}\left(\tau_s(s)l_1 + \frac{1}{4}\sum_{j=1}^{s}\tau_{s-j}(s)\big((-1)^j(2j)!\,\psi_j - E_{2j}\big)\right),$$

where

$$\varphi_1 = (1+4l_1)(1-4l_1+32l_3), \quad \varphi_2 = \frac{2\varphi_1}{3(1+4l_1)}(1+4l_1+24l_1^2+32l_1^3+16l_3),$$

$$\psi_0 = 1+4l_1, \quad \psi_1 = \frac{1}{2}(1-4l_1+32l_3),$$

$$\varphi_j = \frac{1}{j(2j-1)}\left(\frac{6\varphi_2\varphi_{j-1}}{\varphi_1} + 3\sum_{i=1}^{j-2}\varphi_i\varphi_{j-i-1}\right) \quad (j \geq 3),$$

$$\psi_j = \frac{1}{j(2j-1)}\left(\frac{(2\psi_0^3+\psi_1)\psi_{j-1}}{\psi_0} + 3\psi_0\sum_{i=1}^{j-2}\psi_i\psi_{j-i-1} + \sum_{m=1}^{j-3}\psi_m\sum_{i=1}^{j-m-2}\psi_i\psi_{j-m-i-1}\right) \quad (j \geq 2).$$

Theorem 4.4. *Suppose that $\alpha,\ \beta \in \overline{\mathbb{Q}}$. Then the numbers l_1^* and l_2^* are algebraically independent, and for any integer $s \geq 2$*

$$l_{2s-1}^* = \frac{(-1)^{s+1}}{2^{2s}(2s-2)!}\sum_{j=0}^{s-1}\tau_{s-j-1}(s-1)\big(E_{2j} - (-1)^j(2j)!\,\psi_j\big),$$

$$l_{2s}^* = \frac{(-1)^s}{(2s-1)!}\sum_{j=0}^{s-1}\frac{(-1)^j(2j+1)!}{2^{2j+3}}\sigma_{s-j-1}(s)\Big(\varphi_j - \frac{b_j}{2j+1}\Big),$$

where

$$\varphi_0 = 1-8l_2^*, \quad \varphi_1 = \frac{\varphi_0}{6\psi_0^2}(\varphi_0^2+\psi_0^4), \quad \varphi_2 = \frac{\varphi_0}{120\psi_0^4}(\varphi_0^4+14\varphi_0^2\psi_0^4+\psi_0^8),$$

$$\psi_0 = 1-4l_1^*, \quad \psi_1 = \frac{\varphi_0^2}{2\psi_0},$$

$$\varphi_j = \frac{1}{j(2j+1)}\left(\frac{3\varphi_1\varphi_{j-1}}{\varphi_0} + 3\varphi_0^2\varphi_{j-2} + 3\varphi_0\sum_{i=1}^{j-3}\varphi_i\varphi_{j-i-2} + \sum_{m=1}^{j-4}\varphi_m\sum_{i=1}^{j-m-3}\varphi_i\varphi_{j-m-i-2}\right) \quad (j \geq 3).$$

$$\psi_j = \frac{1}{j(2j-1)}\left(\frac{(2\psi_0^3 + \psi_1)\psi_{j-1}}{\psi_0} + 3\psi_0\sum_{i=1}^{j-2}\psi_i\psi_{j-i-1} + \sum_{m=1}^{j-3}\psi_m\sum_{i=1}^{j-m-2}\psi_i\psi_{j-m-i-1}\right) \quad (j \geq 2).$$

For reciprocal sums of evenly even terms and of unevenly even terms, we also obtain algebraic relations ([5]). For the Fibonacci reciprocal sums

$$\chi_{2s} := \frac{1}{2}(h_{2s} + h_{2s}^*) = (\alpha - \beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_{4n}^{2s}}, \quad \chi_{2s}^\sharp := \frac{1}{2}(h_{2s} - h_{2s}^*) = (\alpha - \beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_{4n-2}^{2s}},$$

we have the following:

Theorem 4.5. *Suppose that α, $\beta \in \overline{\mathbb{Q}}$. Then the numbers χ_2, χ_4, χ_6 are algebraically independent, and for any integer $s \geq 4$*

$$\chi_{2s} = \frac{1}{(2s-1)!}\left(\sigma_{s-1}(s)\chi_2 - \sum_{j=1}^{s-1}\frac{(-1)^j(2j)!}{2^{2j+4}}\sigma_{s-j-1}(s)\big(\varphi_j - 2a_j - (2j+1)\psi_{j+1}\big)\right),$$

with

$$\varphi_1 = \frac{z^2}{12} - 64(\chi_2 + 6\chi_4) - \frac{4}{15}, \quad \varphi_2 = \frac{z}{189}\Big(-\frac{11}{4}z^2 + 2880(\chi_2 + 6\chi_4) + 12\Big),$$

$$\psi_1 = -\frac{z}{6}, \quad \psi_2 = \frac{z^2}{36} - 32(\chi_2 + 6\chi_4) - \frac{2}{15},$$

and φ_j, ψ_j $(j \geq 3)$ defined by the same recurrence formulas as (3) *and* (4). *Here z is a number satisfying the cubic equation*

$$z^3 - \big(2880(\chi_2 + 6\chi_4) + 12\big)z - 8064(\chi_2 + 30\chi_4 + 120\chi_6) + 16 = 0.$$

Theorem 4.6. *Suppose that α, $\beta \in \overline{\mathbb{Q}}$. Then the numbers $\chi_2^\sharp$, $\chi_4^\sharp$, $\chi_6^\sharp$ are algebraically independent, and for any integer $s \geq 4$*

$$\chi_{2s}^\sharp = \frac{1}{(2s-1)!}\left(\sigma_{s-1}(s)\chi_2^\sharp - \sum_{j=1}^{s-1}\frac{(-1)^j(2j)!}{2^{2j+4}}\sigma_{s-j-1}(s)\big(\varphi_j + (2j+1)\psi_{j+1}\big)\right),$$

with

$$\varphi_1 = \frac{z^2}{60} + \frac{64}{5}(\chi_2^\sharp + 6\chi_4^\sharp), \quad \varphi_2 = \frac{z}{189}\Big(\frac{z^2}{4} - 576(\chi_2^\sharp + 6\chi_4^\sharp)\Big),$$

$$\psi_1 = -\frac{z}{6}, \quad \psi_2 = -\frac{z^2}{180} + \frac{32}{5}(\chi_2^\sharp + 6\chi_4^\sharp),$$

and φ_j, ψ_j $(j \geq 3)$ defined by the same recurrence formulas as (3) *and* (4). *Here $z = 2(\chi_2^\sharp + 30\chi_4^\sharp + 120\chi_6^\sharp)/(\chi_2^\sharp + 6\chi_4^\sharp)$.*

5. Proof of Theorem 4.1

Consider the complete elliptic integrals of the first and second kind with the modulus $k \neq 0, \pm 1$ defined by

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad E = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt$$

for $k^2 \in \mathbb{C} \setminus (\{0\} \cup \{z \mid z \geq 1\})$. The branch of each integrand is chosen so that it tends to 1 as $t \to 0$. Set

$$q = e^{-\pi c}, \quad c = K'/K, \quad K' = K(k'), \quad k^2 + (k')^2 = 1.$$

Choose $c = c(\beta)$ (or $q = q(\beta)$) so that $q = e^{-\pi c} = \beta^2$, $\beta = -e^{-\pi c/2}$. By [13, Tables 1(i)]

$$\begin{aligned}(5) \qquad h_{2s} &= (\alpha-\beta)^{-2s} \sum_{\nu=1}^{\infty} \frac{1}{U_{2\nu}^{2s}} \\ &= 2^{-2s} \sum_{\nu=1}^{\infty} \operatorname{cosech}^{2s}(\nu\pi c) = \frac{1}{(2s-1)!} \sum_{j=0}^{s-1} \sigma_{s-j-1}(s) A_{2j+1}(\beta^2)\end{aligned}$$

(note that $\sigma_i(s)$ denotes $\alpha_i(s)$ in [13]), where

$$A_{2j+1}(q) := \sum_{n=1}^{\infty} \frac{n^{2j+1} q^{2n}}{1-q^{2n}}.$$

The q-series A_{2j+1} is generated from the Fourier expansion of $\operatorname{ns}^2 z$:

$$(6) \qquad \left(\frac{2K}{\pi}\right)^2 \operatorname{ns}^2\left(\frac{2Kx}{\pi}\right) = \frac{4K(K-E)}{\pi^2} + \operatorname{cosec}^2 x - 8 \sum_{j=0}^{\infty} (-1)^j A_{2j+1} \frac{(2x)^{2j}}{(2j)!}$$

(cf. [13, Tables 1(i)], [8], [12, p. 535]), where $\operatorname{ns} z = 1/\operatorname{sn} z$ with $w = \operatorname{sn} z$ defined by

$$z = \int_0^w \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}.$$

The power series expansion of $\operatorname{ns}^2 z$ gives the expressions (cf. [11], [13, Table 1(i)]):

$$(7) \qquad \begin{cases} P(q^2) := 1 - 24A_1(q) = \left(\frac{2K}{\pi}\right)^2 \left(\frac{3E}{K} - 2 + k^2\right), \\ Q(q^2) := 1 + 240A_3(q) = \left(\frac{2K}{\pi}\right)^4 (1 - k^2 + k^4), \\ R(q^2) := 1 - 504A_5(q) = \left(\frac{2K}{\pi}\right)^6 \frac{1}{2}(1+k^2)(1-2k^2)(2-k^2) \end{cases}$$

with $q = e^{-\pi c}$, $c = K'/K$. We refer here the theorem of Nesterenko ([10]).

Nesterenko's Theorem. *If $\rho \in \mathbb{C}$ with $0 < |\rho| < 1$, then*

$$\text{trans.deg}_{\mathbb{Q}}\, \mathbb{Q}(\rho, P(\rho), Q(\rho), R(\rho)) \geq 3.$$

This theorem with (7) implies the following:

Lemma 5.1. *If $q = e^{-\pi c} \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$, then K/π, E/π, and k are algebraically independent.*

The following lemma is proved by using the fact that $u = \mathrm{ns}^2 z$ satisfies the differential equation $(u')^2 = 4u(u-1)(u-k^2)$.

Lemma 5.2. *The coefficients of the expansion* $\mathrm{ns}^2 z = z^{-2} + \sum_{j=0}^{\infty} c_j z^{2j}$ *are given by*

$$c_0 = \frac{1}{3}(1+k^2), \quad c_1 = \frac{1}{15}(1-k^2+k^4), \quad c_2 = \frac{1}{189}(1+k^2)(1-2k^2)(2-k^2),$$

$$(j-2)(2j+3)c_j = 3\sum_{i=1}^{j-2} c_i c_{j-i-1} \quad (j \geq 3).$$

Now we are ready to prove Theorem 4.1. It follows from (5) that

$$h_2 = A_1, \quad 6h_4 = A_3 - A_1, \quad 120h_6 = A_5 - 5A_3 + 4A_1,$$

or equivalently,

$$A_1 = h_2, \quad A_3 = h_2 + 6h_4, \quad A_5 = h_2 + 30h_4 + 120h_6. \tag{8}$$

By (7) and Lemma 5.1, the numbers h_2, h_4, h_6 are algebraically independent. The formula (6) yields

$$\varphi_j := \left(\frac{2K}{\pi}\right)^{2j+2} c_j = a_j - (-1)^j \frac{2^{2j+3}}{(2j)!} A_{2j+1} \quad (j \geq 1). \tag{9}$$

Here c_j are the coefficients given by Lemma 5.2. In particular,

$$\varphi_1 = \frac{1}{15}(1+240A_3), \quad \varphi_2 = \frac{2}{189}(1-504A_5), \tag{10}$$

which together with (8) imply the expressions of φ_1 and φ_2 in terms of h_2, h_4 and h_6. Combining (5) and (9) we get the expression of h_{2s} in terms of $\{\varphi_j\}_{j\geq 1}$. Multiplying both sides of the formula in Lemma 5.2 by $(2K/\pi)^{2j+2}$, and using the relation $\varphi_j = (2K/\pi)^{2j+2}c_j$, we obtain the recurrence relation for φ_j $(j \geq 3)$.

6. Asymptotic formulas

Let $\alpha, \beta \in \mathbb{C}$ satisfy $\alpha\beta = -1$, $|\beta| < 1$. Then the reciprocal sums

$$h_{2s} = (\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_{2n}^{2s}}, \quad g_{2s-1}^{*} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{V_{2n-1}^{2s-1}}$$

$(s \in \mathbb{N})$ are holomorphic for $|\beta| < 1$. These sums may also be regarded as functions of the modulus k of Jacobian elliptic functions (see Section 5, also [3], [4], [5]). Asymptotic expressions these sums as $\beta \to -1+0$ (or $k \to 1-0$) are presented as follows ([6]):

Theorem 6.1. *For* $-1 < \beta < -1 + \delta_0$, *we have*

$$(\alpha^2 - \beta^2)^{2s} h_{2s} = \left(1 + O(e^{-\pi^2/(2\eta)})\right) \sum_{j=0}^{\infty} \lambda_j^{(s)} \eta^j,$$

$$\eta := -\log(-\beta) = (1+\beta)(1 + O(1+\beta)),$$

where δ_0 *is a sufficiently small positive number. The sum on the right-hand side is a convergent series in* η, *whose coefficients* $\lambda_j^{(s)} \in \mathbb{Q}[\pi]$ *are given by*

$$\lambda_0^{(s)} = \frac{2^{2s-1}(-1)^{s-1}B_{2s}}{(2s)!}\pi^{2s},$$

and (i) *for* $s = 1, 2,$

$$\lambda_1^{(1)} = -2, \quad \lambda_2^{(1)} = \frac{2}{9}\pi^2 + \frac{2}{3}, \quad \lambda_3^{(1)} = -\frac{8}{3}, \quad \lambda_4^{(1)} = \frac{16}{135}\pi^2 + \frac{8}{9}, \quad \ldots,$$

$$\lambda_1^{(2)} = 0, \quad \lambda_2^{(2)} = \frac{4}{135}\pi^4 - \frac{4}{9}\pi^2, \quad \lambda_3^{(2)} = \frac{16}{3}, \quad \ldots,$$

(ii) *for* $s \geq 3$,

$$\lambda_1^{(s)} = 0, \quad \lambda_2^{(s)} = \frac{2^{2s-1}(-1)^{s-1}}{3 \cdot (2s-1)!}\pi^{2s-2}\big(2B_{2s}\pi^2 + s(2s-1)B_{2s-2}\big), \quad \lambda_3^{(s)} = 0, \quad \ldots$$

Theorem 6.2. *For* $-1 < \beta < -1 + \delta_0$, *we have*

$$(\alpha+\beta)^{2s-1} g_{2s-1}^* = \big(1 + O(e^{-\pi^2/(2\eta)})\big) \sum_{j=0}^{\infty} \mu_j^{(s)} \eta^{2j},$$

where δ_0 *and* η *are as in Theorem 1. The sum on the right-hand side is a convergent series in* η, *whose coefficients* $\mu_j^{(s)} \in \mathbb{Q}[\pi]$ *are given by*

$$\mu_0^{(s)} = \frac{(-1)^{s-1}E_{2s-2}}{2^{2s}(2s-2)!}\pi^{2s-1},$$

and (i) *for* $s = 1$,

$$\mu_1^{(1)} = \frac{\pi}{24}, \quad \mu_2^{(1)} = \frac{\pi}{480}, \quad \ldots,$$

(ii) *for* $s \geq 2$,

$$\mu_1^{(s)} = \frac{(-1)^{s-1}(2s-1)}{2^{2s+1} \cdot 3 \cdot (2s-2)!}\pi^{2s-3}\big(E_{2s-2}\pi^2 + 8(s-1)(2s-3)E_{2s-4}\big), \quad \ldots$$

Degenerate cases of our expressions coincide with Euler's formulas for $\zeta(2s) = \sum_{n=1}^{\infty} n^{-2s}$ and $L(2s-1) = \sum_{n=1}^{\infty}(-1)^{n+1}(2n-1)^{-(2s-1)}$, respectively. For $-1 < \beta < -1 + \delta_0$ and for $n \geq 1$, observing that

$$\left|\frac{\alpha^{2n} - \beta^{2n}}{\alpha^2 - \beta^2}\right| = \left|\sum_{\nu=0}^{n-1} \alpha^{2n-2-2\nu}\beta^{2\nu}\right| = \left|\sum_{\nu=0}^{n-1} \beta^{4\nu-2n+2}\right| = \left|\frac{1}{2}\sum_{\nu=0}^{n-1}\big(\beta^{4\nu-2n+2} + \beta^{-(4\nu-2n+2)}\big)\right| \geq n,$$

we have for $s \in \mathbb{N}$

$$\lim_{\beta \to -1+0} (\alpha^2 - \beta^2)^{2s} h_{2s} = \sum_{n=1}^{\infty} \frac{1}{n^{2s}}.$$

Therefore, letting β tend to $-1+0$ in Theorem 6.1, we obtain

$$\zeta(2s) = \frac{2^{2s-1}(-1)^{s-1}B_{2s}}{(2s)!}\pi^{2s} \quad (s \in \mathbb{N}).$$

For each $s \in \mathbb{N}$ a similar argument concerning Theorem 6.2 leads us

$$\lim_{\beta \to -1+0} (\alpha+\beta)^{2s-1} g_{2s-1}^* = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^{2s-1}} = \frac{(-1)^{s-1}E_{2s-2}}{2^{2s}(2s-2)!}\pi^{2s-1}.$$

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