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THE HARDY-LITTLEWOOD MAXIMAL FUNCTION, A_∞, AND THE BELLMAN FUNCTION TECHNIQUE

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1. Background: Maximal function and A_∞ weights

Today I would like to talk a little bit about some research involving Muckenhoupt A_∞ weights and the Bellman function technique. In recent years, the Bellman technique has evolved a great deal and, excitingly, has been used to re-prove, with sharp constants, several classical results from the theory of A_∞; so I believe the topic should be of interest. For those of the audience who aren't familiar with this area, I will first present some recent ([9]) work to give you a sense of things; then I'll talk about some work-in-progress with T. Wall (that has benefited much from discussions with L. Slavin).

1.1. **Basic definitions.**

First recall the definition of the *Hardy-Littlewood maximal operator* M, defined on $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes Q (or balls) centered at x with sides parallel to the axes. M is bounded on $L^p(\mathbb{R}^n), 1 < p \leq \infty$; this is usually demonstrated by Marcinkiewicz interpolation between its weak (1,1) boundedness, i.e., the fact that for any $\alpha > 0$,

$$|\{x \in \mathbb{R}^n \mid Mf(x) > \alpha\}| \leq \frac{C\|f\|_1}{\alpha},$$

and the operator's obvious L^∞ boundedness. The weak (1,1) boundedness is in turn usually proven by covering lemma argument; the canonical references for these results are the tomes of Stein [14, 15] and [4].

A non-negative function $w \in L^1_{loc}(\mathbb{R}^n)$ is called a *Muckenhoupt* $A_p(\mathbb{R}^n)$ *weight* if its A_p *characteristic* (or "constant") is finite, i.e.,

$$A_p(w) := \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes (with sides parallel to the axes again, as they will be for the rest of the paper). This class arises naturally in the study of the Hardy-Littlewood maximal operator; for example, for $1 < p < \infty$, M is bounded on $L^p(w\,dx)$ if and only if $w \in A_p$. The union $\cup_{p>1} A_p$ of all A_p classes is denoted A_∞.

One characteristic fact about Muckenhoupt weights is that any A_p weight must satisfy some *reverse Hölder inequality*, i.e., for some $s > 1$, there exists a $C > 0$ such that over all cubes $Q \subset \mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q w^s\right)^{1/s} \leq C\frac{1}{|Q|}\int_Q w;$$

briefly, we say that w lies in the reverse Hölder class RH_s, with the infimum of the C called the reverse Hölder characteristic or constant $RH_s(w)$. That is, every A_∞ weight lies in some RH_s (and, in fact, vice-versa). The dependence of s and of $RH_s(w)$ on p and $A_p(w)$ has been long known; however, the precise understanding of this relation was only achieved recently by Vasyunin [16] and Dindoš and Wall [3].

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It will be useful later to note that

$$w \in A_p \cap RH_s \iff w^s \in A_{s(p-1)+1}, \tag{1}$$

a consequence of Hölder and reverse Hölder inequalities ([7]).

1.2. Factorization of A_p.

In 1980, P. Jones showed the celebrated "Jones Factorization Theorem," factorizing A_p weights into A_1 weights (i.e., w such that $Mw(x) \le cw(x)$ a.e.):

Theorem. $w \in A_p \iff w = w_0 w_1^{1-p}$ for some $w_0, w_1 \in A_1$.

In 1995, Cruz-Uribe and Neugebauer extended the factorization to simultaneously include the RH_s data as well as the A_p, using the analogous tools: a *minimal operator* m defined using the infimum of averages rather than the supremum, and a limiting reverse Hölder class RH_∞ defined as all w such that $cmw(x) \ge w(x)$ a.e.

Theorem. $w \in A_p \cap RH_s \iff w = w_0 w_1$, where $w_0 \in A_1 \cap RH_s$ and $w_1 \in A^p \cap RH_\infty$.

We will give an elementary explanation of their result, using the symmetries between the A_p and RH_s structures on A_∞.

We will need the following characterizations of A_1 and RH_∞:

$$w \in A_1 \iff w \in A_\infty \cap e^{BLO}$$

(where $BLO \subset BMO$ is the set of functions of *bounded lower oscillation* i.e., all ϕ such that $\sup_Q \frac{1}{|Q|} \int_Q \phi - \inf_Q \phi < \infty$), and

$$w \in RH_\infty \iff w \in e^{BUO} (= e^{-BLO}).$$

Let me show a little of how these characterizations work: the reverse direction is the interesting one.

Proof. Suppose $w \in A_\infty \cap e^{BLO}$, and we want to show $Mw(x) \le cw(x)$ a.e.. Let us introduce $M^\natural$, defined by $M^\natural f(x) = \sup_{Q \ni x} \fint_Q f$ (at one point I ignorantly believed I had introduced this operator myself; later I found it had been studied by C. Bennett two decades earlier). In fact,

$$\phi \in BLO \iff M^\natural \phi(x) \le \phi(x) + C,$$

so for us $M^\natural \log w \le \log w + C_1$. Further, $M^\natural$ commutes with log on A_∞, since any $w \in A_\infty$ satisfies a *reverse Jensen inequality*

$$\frac{1}{|Q|} \int_Q w \le C e^{\frac{1}{|Q|} \int_Q \log w}$$

(where the infimum of such C is again denoted $A_\infty(w)$); so $\log M^\natural w - M^\natural \log w \le \log C_2$. Putting those two together, $\log M^\natural w \le \log w + C_1 + \log C_2$, so $Mw(x) \le Cw(x)$ a.e., and we're done. The proof for RH_∞ is similar. □

Remember that we are trying to give a simple explanation of the refined Jones factorization $w \in A_p \cap RH_s \iff w = w_0 w_1$, where $w_0 \in A_1 \cap RH_s$, $w_1 \in A_p \cap RH_\infty$. The crucial lemma in Cruz-Uribe–Neugebauer's proof is the fact that $w \in A_1 \iff w^{1-p} \in A_p \cap RH_\infty$. But this is trivial:

Proof. $w \in A_1 \iff w \in A_\infty \cap e^{BLO} \iff \left\{ \begin{array}{c} w \in A_{p'} \\ w^{1-p} \in e^{BUO} = RH_\infty \end{array} \right\}$
$\iff \left\{ \begin{array}{c} w^{1-p} \in A_p \\ w^{1-p} \in RH_\infty \end{array} \right\}$. □

Now we can give the simplified proof of the refined Jones factorization theorem.

Proof. By (1), $w \in A_p \cap RH_s \iff w^s \in A_{s(p-1)+1}$ which, by the original Jones factorization, means exactly that $w^s = v_0 v_1^{1-[s(p-1)+1]} = v_0 v_1^{-s(p-1)}$ for some $v_0, v_1 \in A_1$. Taking roots of both sides, we get $w = v_0^{1/s} v_1^{1-p}$; $v_0, v_1 \in A_1$. Take $w_0 = v_0^{1/s}$ and $w_1 = v_1^{1-p}$: by (1) again, $w_0 \in A_1 \cap RH_s$; and by the crucial lemma, $w_1 \in A_p \cap RH_\infty$. □

2. Bellman approach around Coifman-Rochberg

Now that you have a better feel for A_∞, I'd like to talk about some current work, and how I believe the Bellman function approach might enable us to resolve it.

2.1. The problem: $M_s : A_{\infty,s} \to A_{1,s}$?

One useful fact about the maximal function is that it "improves" weights; i.e., given any weight $w \in A_\infty$, Mw actually lies in A_1. A bit over ten years ago, my advisor asked me: what happens in the multiparameter case? That is, suppose we consider the *strong maximal function* M_s, defined by

$$M_s(f)(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f|,$$

where R denotes *rectangles* (i.e., parallelepipeds) with sides parallel to the axes; similarly, we consider weights $A_{p,s}$ defined with respect to rectangles, etc. The classical proof that $M : A_\infty \to A_1$ relies (essentially) on the a result of Coifman-Rochberg [1]) that given any $f \in L^1_{loc}$, and any $\delta \in (0,1)$, $(Mf)^\delta$ lies in A_1. However, this fact is false for M_s: a counterexample was given by Soria ([13]).

I haven't yet answered my advisor's question, though much of the work I gave in the background section was developed in response to it: the mapping of A_∞ into A_1 by M is, thanks to the John-Nirenberg inequality, equivalent to the boundedness of $M^\natural : BMO \to BLO$ ([8]); both are equivalent to the analogous statement for the minimal operator ([9]). However, last year, while attending the Conference for Harmonic Analysis and Partial Differential Equations in El Escorial, Spain, I happened to meet Treven Wall (then at Edinburgh), who had just completed some work with Martin Dindoš using the Bellman function technique to get *sharp* results on the relation between the A_p class and characteristic of a weight and its corresponding reverse Hölder class and characteristic. To me, this was a startling result, one which I would never have dreamed possible. Seeing my interest, Wall kindly sat with me for some forty minutes and walked me through their proof. It occurred to me that the problem my advisor had asked might be amenable to attack by this sort of approach; so I asked Wall if he were willing to try it on the known one-parameter case (hoping naively that it might extend to the multiparameter case afterwards), and he agreed.

Actually, I had heard about the Bellman function technique also somewhat over ten years ago, when a fellow graduate student (Janine Wittwer) suggested that I read the seminal paper by Nazarov and Treil ([6]). At that time, I refused to read it: the paper was long and the technique (to me) mysterious. In hindsight, I regret not having listened to her advice!

The approach was first used by Burkholder in the mid-'80s, and then used with great effect by Nazarov, Treil, and Volberg, starting in the mid '90s. In 2003, Vasyunin [16] recognized certain Bellman problems (in particular, the determination of the sharp reverse Hölder class and characteristic from the A_p class and characteristic) to be related to the solution of Monge-Ampère boundary value problems. This combination of methods was subsequently used by Dindoš and Wall [3] to obtain the inverse result; by Slavin, Stokolos, and Vasyunin [10] to get sharp bounds for dyadic M on $L^p(\mathbb{R})$; and by Slavin and Volberg ([11, 12]) to obtain other sharp versions of classical results. Recently, Vasyunin and Volberg ([17]) have gone more deeply into the method of characteristics used to solve the Monge-Ampère PDE, simplifying further some of the previous results. I had the good fortune of meeting Slavin at a workshop at the Centre de Recerca Matemàtica in Bellaterra, Spain; he magnanimously spent long hours with me answering questions and working on problems.

It seems to me that the "new Bellman Philosophy" is based on the following meta-observation. Many of the constructs ("B", say) of interest in classical harmonic analysis depend on (or are relations between) "martingale variables," i.e., constructs V which satisfy a relation of the form "$V = \frac{V_- + V_+}{2}$"; for example, the average $< f >_I$ of a function f over an interval I is in turn the average $\frac{<f>_{I_-} + <f>_{I_+}}{2}$ of the function over the left and right subintervals $I_-, I_+ \subset I$. The constructs B themselves often also satisfy a "pseudo-concavity" condition

$$B(V) \gtrapprox \frac{B(V_-) + B(V_+)}{2}$$

These two facts together force the objects (and thus many objects of interest in harmonic analysis) to be solutions of Monge-Ampère PDEs; as such, they can be solved for *explicitly*, yielding *sharp* results.

2.2. The Bellman Approach to $M : A_\infty \to A_1$.

2.2.1. *The basic set-up.*

Recall the dyadic maximal operator M is defined (for $\phi \in L^1_{loc}(\mathbb{R})$, say)

$$M\phi(x) = \sup_{I \ni x} < |\phi| >_I,$$

i.e., by taking the supremum of averages of $|\phi|$ (note that we've switched notation to the one commonly used by Bellman people) over all dyadic intervals I containing x. For $\delta > 0$, let $w \in A^{\delta}_{\infty}$ mean that for all dyadic intervals I,

$$< w >_I \leq \delta e^{< \log w >_I},$$

i.e., that w is an A_∞ weight with characteristic $A_\infty(w) \leq \delta$.

Our hope is to show $Mw \in A_1$, so we define the relevant Bellman function by

$$B(x, y, z; \delta) = \sup_{\substack{w \in A^{\delta}_{\infty} \\ < w >_I = x \\ < \log w >_I = y \\ \inf_I Mw = z}} \left\{ \frac{1}{|I|} \int_I Mw \right\}$$

(we will usually suppress the δ); it has domain $\Omega = \{(x, y, z) : e^y \leq x \leq \delta e^y; x \leq z\}$. Showing that $B(x, y, z) \leq Cz$ would imply $Mw \in A_1$, but of course the problem is: *How do we figure out what B is?*

2.2.2. *Observations about B.*

The following observations are trivial:

Homogeneity: for any $\tau > 0$,

$$B(x, y, z) = \frac{1}{\tau} B(\tau x, y + \log \tau, \tau z), \tag{2}$$

i.e.

$$B(x, y, z) = z\, B(\frac{x}{z}, y - \log z, 1). \tag{3}$$

Boundary values: on the left boundary of Ω, $y = \log x$ implies the weight must be constant ($\equiv \inf_Q Mw = z$); so

$$B(e^y, y, z) = B(x, \log x, z) = z. \tag{4}$$

Pseudoconcavity: Let I_- and I_+ be the left and right halves of the dyadic interval I; $x_\pm, y_\pm$ be the averages, of w and $\log w$ over the corresponding halves, and $z_\pm = inf_{I_\pm} Mw$. Then $x_I = \frac{x_- + x_+}{2}$ and similarly $y_I = \frac{y_- + y_+}{2}$; the last variable satisfies

$$z = \min(z_-, z_+).$$

Now $< Mw >_I = \frac{1}{2}[< Mw >_{I} + < Mw >_{I_+}]$, so taking supremums, we get the following *pseudoconcavity* condition:

$$B(x, y, \min(z_-, z_+)) \geq \frac{[B(x_-, y_-, z_-) + B(x_+, y_+, z_+)]}{2} \tag{5}$$

2.2.3. *Consequence of pseudoconcavity.* Without any loss of generality, let us assume $z_- \leq z_+$; we notate $p = (x, y)$ for convenience.

Then

$$B(p, z_-) \geq \frac{B(p_-, z_-) + B(p_+, z_+)}{2}$$

$$B(p, z_-) \geq \frac{B(p_-, z_-) + B(p_+, z_-) + B(p_+, z_+) - B(p_+, z_-)}{2}$$

i.e.,

$$\left[B(p, z_-) - \frac{B(p_-, z_-) + B(p_+, z_-)}{2} \right] - \frac{B(p_+, z_+) - B(p_+, z_-)}{2} \geq 0.$$

Using Taylor expansion (plus the "martingale variable property"), the above becomes

$$-\frac{1}{4}(\Delta p)^t \left[\frac{\partial^2 B}{\partial p^2}(\gamma) \right] (\Delta p) - \frac{1}{2}[B(p_+, z_+) - B(p_+, z_-)] \geq 0; \tag{6}$$

by the Mean Value Theorem, we get (for some $\beta \in (z_-, z_+)$)

$$-\frac{1}{4}(\Delta p)^t \left[\frac{\partial^2 B}{\partial p^2}(\gamma)\right](\Delta p) - \frac{1}{2}B_z(p_+, \beta)(\Delta z) \geq 0. \tag{7}$$

It seems probable that $B_z \geq 0$; in that case, the above implies that $\frac{\partial^2 B}{\partial p^2}$ should be negative semidefinite.

2.2.4. *Monge-Ampère boundary value problem.*

One hopes to use the homogeneity to reduce the order, and the fact that the Bellman function is an extremal condition, to get a Monge-Ampère type boundary value problem. Let

$$G(a, b) = B(\frac{x}{z}, y - \log z, 1), \tag{8}$$

with $a = \frac{x}{z}, b = y - \log z$; by (3) $B(x, y, z) = zG(a, b)$. Then, by the above negative semidefiniteness,

$$G_{aa}G_{bb} - G_{ab}^2 \leq 0, G_{aa} \leq 0 \tag{9}$$

on the domain $\Omega = \{0 < a \leq 1; \frac{\log a}{c} \leq b \leq \log a\}$; further, $G(a, \log a) = 1$ (by the left boundary value condition (4)).

We sharpen our conditions (changing the first inequality in (9) to an equality, and further assuming $B_z \equiv 0$ on the right boundary of Ω) to get the following Monge-Ampère problem:

$$\begin{cases} G_{aa}G_{bb} - G_{ab}^2 = 0 \\ G(a, \log a) = 1 \\ G(1, b) = G_a(1, b) + G_b(1, b) \end{cases} \tag{10}$$

For such PDEs, the domain Ω is foliated by lines along which the function G is itself linear, and the PDE can be solved using a certain method of characteristics (see Vasyunin-Volberg [17], pp. 10ff).

2.2.5. *Using the Method of Characteristics.*

Let us parametrize the foliation lines by t; let $(u(t), \log u(t))$ and $(1, v(t))$ denote the coordinates of that line's intersection with $\partial\Omega^-$ and $\partial\Omega^+$. On that line, G can be expressed ([10, 17]) as

$$G(a, b) = ta + f(t)b + g(t). \tag{11}$$

Now, on $\partial\Omega^-$ we have (the left boundary value condition)

$$G(u(t), \log u(t)) = tu(t) + f(t)\log u(t) + g(t) = 1$$

which implies $f(t) = -tu(t)$ and thus

$$g(t) = 1 - tu(t) + tu(t)\log u(t). \tag{12}$$

Similarly, on $\partial\Omega^+$, one has

$$G(1, v) = G_a(1, v) + G_b(1, v) = t + f(t)$$

which implies

$$g(t) = 1 - v(t). \tag{13}$$

Equating the expressions (12) and (13) for g yields

$$t = \frac{1}{u[v - \log u]}$$

and thus

$$G(a, b) = \frac{1}{v - \log u}\left[\frac{a}{u} - b + v - 1\right].$$

In terms of the original function, the above says

$$B(x, y, z) = \frac{z}{v - \log u}\left[\frac{x}{uz} - y + \log z + v - 1\right]. \tag{14}$$

However, it's not clear what one might do from here. How do we figure out what u and v are in terms of (x, y, z)? Further, is it even true that $B_z \geq 0$ (and thus that $\frac{\partial^2 B}{\partial p^2}$ need be negative semi-definite)? We are currently stuck....

3. Sharp RH_s Constants in the Jones Factorization

I would like to close my talk by proposing one open question. Recall the original Jones factorization: given an A_p weight w, one can factorize it as $w = uv^{1-p}$, where $u, v \in A_1$. That factorization is not unique, so there's an obvious question: is it possible to factorize in such a way that one obtains sharp constants in the factorization (possibly via a Bellman approach)? In fact, a result along these lines (pointed out by C. Pèrez at the CRM workshop) is already known, though not via a Bellman proof: a paper of E. Hernández ([5]) shows, via a "Rubio algorithm" (i.e., the iteration method used by Rubio de Francia to give a simple proof of the Jones factorization), that it is possible to factorize so that

$$[w]_{A_p} \approx [u]_{A_1} [v]_{A_1}^{p-1}.$$

So let me propose a follow-up question: is it possible to analogously control the RH_s constants in Cruz-Uribe–Neugebauer's *refined* Jones factorization, either by Bellman or perhaps by doing a "Rubio algorithm" with the *minimal* operator? (I think so.) Is it possible to control both sets of constants simultaneously? (I don't think so.)

Thank you all for your kind attention.

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