数理解析研究所講究録
第 1667 巻 2009 年 66-70

Matrix monotone functions and matrix convex functions as truncated completely monotone functions

Jun Tomiyama, EP of Tokyo Metropolitan University

May, 2009

1 Introduction

We recall first that a real valued continuous C^{∞}-function f defined on an open interval $I=(\alpha,\beta)$ is said to be completely monotone if it satisfies the following property

$$(-1)^{n}f^{(n)}(t)\geq 0 \qquad \text{for all integers } n\geq 0.$$

The function is also said to be completely monotone if it is defined as a continuous function on the closed interval $[\alpha,\beta]$.

This class of functions (when I is the half positive line) has been known since the time of S.N.Bernstein by his characterization theorems of this class known as 'Little' and 'Big' Bernstain theorem (cf.[1, Chap.1.5]). In this lecture we shall discuss relationship between matrix monotone functions (resp. matrix convex functions) and this class of functions in the truncated form. Relation between operator monotone functions and completely monotone functions is known before whereas the relation between operator convex functions and this class has been discussed only recentry. Furthermore, we discuss special aspects of 2-monotonicity and 2-convexity in the theory of matrix monotone functions and matrix convex functions.

In the following we refer most of those related results from the book [1] except our works [2] and [3].

2 Discussion and results

Let I be a nontrivial open interval of the real line R. A real valued continuous function f on I is said to be n-monotone if for any pair of selfadjoint matrices

a, b of M_n (n by n matrix algebra) whose spectra are in I we have that $a \leq b$ implies $f(a) \leq f(b)$. Here the functional calculus $f(a)$ means the selfadjoint matrix (an operator on C^n) defined as

$$f(a) = (f(\lambda_i)) \quad \text{for a diagonalized matrix } a = (\lambda_i).$$

We denote the set of all n-monotone functions for I by $P_n(I)$. On the other hand we call f on I n-convex if for a pair of selfadjoint matrices $\{a, b\}$ satisfying the condition for spectra we have

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

When the inequality becomes the other way around we say that the function is n-concave. Write as $K_n(I)$ the set of all n-convex functions for I. If we have a continuous function having similar propertiess on the algebra of all bounded linear algebras on an infinite dimensional Hilbert space we call such function operator monotone and operator convex respectively. Denote them as $P_\infty(I)$ and $K_\infty(I)$. It is then not so difficult to see that the intersection of $P_n(I)$ coincides with $P_\infty(I)$. Similarly, the intersection of $K_n(I)$ coincides with $K_\infty(I)$. The class of completely monotone functions then appears in the proof of Loewner's most important result of the characterization of an operator monotone function f on the interval $(-1, \infty)$ having an analytic extension to the upper half plain as a Pick function (analytic function defined in the upper half plain whose range remains the same domain) in such a way that $f'(t)$ is completely monotone in this interval.

Now actually we can see that the above result is not concerned with such a particular interval but holds for an open interval in general. Moreover we can obtain the following truncated forms for n-monotone functions as well as for n-convex functions , which imply the results for operator monotone functions and operator convex functions.

We first remark that if f is two monotone and f' vanishes at some point then f becomes a constant. Similarly if f is two convex and its second derivative vanishes at some point it becomes linear. Therefore, in both cases we may assume that f' and f" are strictly positive on I in general.

Theorem 2.1 *(Hansen-Tomiyama) Let f be a function defined in an interval of the form (α, ∞) for some real α.*

(i) If f is n-monotone and $2n - 1$ times continuously differentiable, then

$$(-1)^k f^{(k+1)}(t) \geq 0 \quad k = 0, 1, \ldots, 2n - 2.$$

Therefore,the function f and its even derivatives up to order $2n - 4$ are concave functions, and the odd derivatives up to order $2n - 3$ are convex functions.

(ii) If f is n-convex and $2n$ times continuously differentiable, then

$$(-1)^k f^{(k+2)}(t) \geq 0 \quad k = 0, 1, \ldots, 2n-2.$$

Therefore, the function f and its even derivatives up to order $2n-2$ are convex functions, and the odd derivatives up to order $2n-3$ are concave functions.

As an immeadiate consequence we have, as in the case of an operator monotone function,the following

Corollary 2.2 *If f is operator convex, then its second derivative f" becomes a completely monotone function.*

We leave details of this fact to the reference [3]. A key point of the proof of this theorem is a geometrical observation of the the following situation. Namely, if f is 2-monotone and in the class $C^3(I)$ f is written as

$$f(t) = \frac{1}{c(t)^2} \quad \text{for a positive concave function } c(t).$$

Moreover, if f is 2-convex and in $C^4(I)$ f is written as

$$f(t) = \frac{1}{d(t)^3} \quad \text{for a positive concave function } d(t).$$

Notice that as mentioned above we may assume here that f' and f" are srictly positive according to each case.

The difference between the upper half plain for a Pick function and the right half plain appeared in the Bernsein's theorem seems to stem from the difference between f' and the function itself.

It should be also worthwhile to mention the degree of differentiability of relevant matrix functions. In fact, in the above arguments we have put the conditions such as f belongs to the class $C^3(I)$ and so on. There are results about automatic differentiability for n-monotone functions and n-convex functions, but in general we can not ask for a two monotone function three times continuous differentiability. There is however another argument called regularization explained below, by which we may freely assume enough differentiabily of a relevant function (cf.[1, Section 1..4]).

Let $\varphi(t)$ be a C^∞-function on the real line, vanishing out side the closed interval $[-1, 1]$. We also assume that $\varphi(t)$ is nonnegative and even and normalized as

$$\int_{-1}^{1} \varphi(t)dt = 1.$$

This is a molifier used often in the theory of partial differential equations. Now for a given positive ε we consider the ε -regularized function f_ε defined as

$$f_\varepsilon(t) = \frac{1}{\varepsilon}\int \varphi\left(\frac{t-s}{\varepsilon}\right) f(s)ds = \int \varphi(s)f(t-\varepsilon s)ds.$$

When a continuous function f is defined on an open interval (α, β) this regularization f_ε makes sense on the interval $(\alpha + \varepsilon, \beta - \varepsilon)$. It is a C^∞-function and moreover becomes n-monotone and n-convex whenever f is n-monotone and n-convex respectively. Since f_ε converges to f uniformly on any subinterval of $(\alpha + \varepsilon, \beta - \varepsilon)$ we may replace f by the C^∞-function f_ε in our arguments . Furthermore, it is known that when f is operator monotone it becomes automatically a C^∞-function. This is also true for an operator convex function.

We next consider the paticularlity of two monotonicity and two convexity in the theory. We regard theory of these kinds of matrix functions as non-commutative calculus meaning that we use matrix algebras as our basic scaling. In case of usual calculus, we use scaling of numbers as the base in the theory. Thus the class $P_1(I)$ and $K_1(I)$ are simply the classes of numerical monotone functions and of numerical convex functions. In this sense, the step from $P_1(I)$ and $K_1(I)$ to the classes of two monotone and two convex functions is a big jump in the theory. A typical example to show this jump is the pair of the functions, $logt$ and $expt$ on the positive half line. In calculus, they make a good combination of mutually inverse monotone functions but once non-commitativity comes in although $logt$ becomes operator monotone , that is, n-monotone for all (positive) integer n, the exponential function can not be even two monotone.

Now in the arguments of matrix functions we assume that a relevent interval should be non-trivial. The reason of this assumption for operator monotone and operator convex functions is usually explained by representations by integrals of those functions, very deep results. We can see however that this is simply because of the change of aspects into non-commutative setting. In fact, considering suitable differentiability if f is two monotone or two convex we can write f' or f" by means of positive concave functions $c(t)$ and $d(t)$. Therefore, if f is defined on the whole real line $c(t)$, as well as $d(t)$, be positive concave functions on R. A geometric aspect of a positive concave function on the real line then easily tells us that $c(t)$ and $d(t)$ have to be constant. Notice that if $c(t)$ (also $d(t)$) is considered on the positive half line it can be an increasing function, but if it must be considered on the another half line as positive concave function it has to be constant. This also shows that the degree two is a turning point of the theory.

Thus, we obtain the following

Proposition 2.3 *A two monotone function defined on the whole real line is linear, and a two convex function on the real line becomes (at most) quadratic.*

From the degree two change of aspects of the theory towards more big degrees stem mainly from meanings of the order. We however face quite often real difficulty of non-commutativity from the step of the degree two to three. The most typical example of such aspect is the problem of local property for monotone and convex functions.

Theorem 2.4 *(Local property theorem of n-monotone functions). Let* (α, β) *and* (γ, δ) *be two overlaping open intervals in this order. Suppose a function* f *be n-monotone both on* (α, β) *and on* (γ, δ)*, then* f *is n-monotone on the (connected) open interval* (α, δ).

This most deep theorem, contrary to its simple formulation, was rather easily proved in case of a two monotone function, but it took almost forty years to obtain an exact whole proof ([1]), which is long enough. Moreover, the corresponding (suspected) local property theorem (whose formulation will be easily figured out) has been proved only recently for two convex functions ([3]). We believe to have the local property theorem for arbitrary n-convex functions, but even for a three convex function the theorem is still out of our ideas.

References

[1] W.Donoghue, Monotone matrix functions and analytic continuation, Springer 1974

[2] F.Hansen and J.Tomiyama, Differential analysis of matrix convex functions, Linear Algebra and its Appl.,420(2007),102-116.

[3] F.Hansen and J.Tomiyama, Differential analysis of matrix convex functions, to appear in J. Inequality in Pure and Applied Math.,